Implications of large-cardinal principles in homotopical localization

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Abstract

The existence of arbitrary cohomological localizations on the homotopy category of spaces has remained unproved since Bousfield settled the same problem for homology theories in the decade of 1970. This is related with another open question, namely whether or not every homotopy idempotent functor on spaces is an f-localization for some map f. We prove that both questions have an affirmative answer assuming the validity of a suitable large-cardinal axiom from set theory (Vopěnka's principle). We also show that it is impossible to prove that all homotopy idempotent functors are f-localizations using the ordinary ZFC axioms of set theory (Zermelo–Fraenkel axioms with the axiom of choice), since a counterexample can be displayed under the assumption that all cardinals are nonmeasurable, which is consistent with ZFC.

Introduction

Homotopy idempotent functors appear frequently in algebraic topology. A homotopy idempotent functor is a functor E from some model category [27] to itself that carries weak equivalences to weak equivalences and is equipped with a natural transformation $\eta\colon \mathrm{Id} \to E$ such that both ηE and $E\eta$ induce weak equivalences $EX \simeq EEX$ for all X. The first nontrivial instances of homotopy idempotent functors that were recognized as such were localizations at sets of primes and, more generally, homological localizations [2], [5].

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In [16], Dror Farjoun developed a theory of localization with respect to any map $f\colon A \to B$. His construction associates functorially with each space X a map $X \to L_f X$ which is universal, up to homotopy, among maps from X into (fibrant) spaces Y such that the map of function complexes

$$\mathrm{map}(B, Y) \longrightarrow \mathrm{map}(A, Y)$$

induced by f is a weak equivalence. (In this article, spaces will be simplicial sets, and maps and function complexes will be unbased.) For each map f, the functor L_f is homotopy idempotent and continuous, that is, it induces a natural map of function complexes

$$\mathrm{map}(X, Y) \longrightarrow \mathrm{map}(L_f X,\ L_f Y)$$

for all X and Y, preserving composition and identity.

All examples of homotopy idempotent functors known until now on the model category of simplicial sets are special cases of f-localizations for suitable choices of the map f; see [16, 1.E]. This led Dror Farjoun to ask in [14] and [16, 1.A] if every homotopy idempotent functor on simplicial sets is equivalent to some f-localization.

In this article we show that it is impossible to answer this question affirmatively using the Zermelo–Fraenkel axioms of set theory and the axiom of choice (briefly, ZFC axioms). Moreover, a negative answer to this question in ZFC is not to be expected, as it would imply the inconsistency of certain large-cardinal axioms that are believed to be consistent with ZFC after many years of related developments in set theory.

On one hand, we define a homotopy idempotent functor E on simplicial sets by associating with every X the nerve $NP_{\mathcal{A}}\,\pi X$, where πX is the fundamental groupoid of X and $P_{\mathcal{A}}$ is a homotopy idempotent functor on the model category of groupoids [9] which annihilates the class $\mathcal{A}$ of groups of the form $\mathbf{Z}^{\kappa}/\mathbf{Z}^{<\kappa}$ for all cardinals κ (see Section 6 for further explanation of this terminology). If there exists a map f such that $E \simeq L_f$, then we infer that $\mathrm{Hom}(\mathbf{Z}^{\kappa}/\mathbf{Z}^{<\kappa},\ \mathbf{Z}) \neq 0$ for some cardinal κ, and it is known that the smallest such κ is then measurable. However, the existence of measurable cardinals cannot be proved in ZFC; see [1] or [24]. Therefore, the assertion that $E \not\simeq L_f$ for any f is consistent with ZFC; in fact it can be derived from Gödel's axiom of constructibility, which implies that measurable cardinals do not exist.

On the other hand, if we assume Vopěnka's principle [1], [24], then we can prove that every homotopy idempotent functor E on simplicial sets is equivalent to f-localization for some map f. A preliminary study in the category of groups was carried out in [13].

Our results rely on work of Adámek–Rosický [1] and Dugas–Göbel [17]. Vopěnka's principle has also been used in a recent article by Rosický and Tholen [30] in order to ensure the existence of certain model structures on locally presentable categories.

The crucial existence result is the following. Given any class $\mathcal{D}$ of fibrant simplicial sets, consider the class $\mathcal{S}$ of maps $X \to Y$ such that the induced map

$$\mathrm{map}(Y, D) \longrightarrow \mathrm{map}(X, D)$$

is a weak equivalence for all D in $\mathcal{D}$. Then, assuming Vopěnka's principle, there is a map f such that the f-localization functor L_f renders invertible precisely the maps in $\mathcal{S}$.

This is a strong result, since it tells us that Vopěnka's principle implies the existence of localization with respect to any generalized cohomology theory; see Corollary 5.4 below. Whether cohomological localizations can be shown to exist in ZFC or not is a long standing open question; see [7, 2.6] and [26, Ch. 7].

In order to infer that, if Vopěnka's principle is true, then every homotopy idempotent functor on simplicial sets is equivalent to L_f for some map f, we need to show that if E is homotopy idempotent (and takes fibrant values), then, for all X and Y, the map of function complexes

$$\mathrm{map}(EX, EY) \longrightarrow \mathrm{map}(X, EY)$$

induced by the natural map $X \to EX$ is a weak equivalence. This was proved by Dror Farjoun in [15] under the additional assumption that E be continuous. We show that it is still true if the continuity assumption is omitted. To achieve this, we replace the function complex $\mathrm{map}(X, Y)$ by the nerve of the category $\mathcal{L}(X, Y)$ whose objects are diagrams $X \to C \leftarrow Y$ where the backwards arrow is a weak equivalence, and morphisms are commutative diagrams with X and Y fixed, as in [18], [19], [20]. Thus, every functor E that carries weak equivalences to weak equivalences induces a functor

$$\mathcal{L}(X, Y) \longrightarrow \mathcal{L}(EX, EY)$$

for all X and Y, and this is what is needed for the argument. Another possible approach to avoid imposing continuity could be to use a result in [28, § 6], according to which every functor on simplicial sets that carries weak equivalences to weak equivalences is naturally equivalent to a continuous functor.

We conclude by showing that the existence of a universal acyclic space, which was proved by Bousfield in [7] for f-localizations, cannot be proved in ZFC for homotopy idempotent functors in general. A universal acyclic space for a homotopy idempotent functor E is a simplicial set U such that localization with respect to the map $U \to *$ kills the same simplicial sets as E does. (We say that a simplicial set X is killed by E if EX is contractible.)

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1 Locally presentable and accessible categories

Our main sources for this section were [1] and [22]. A cardinal λ is *regular* if it is infinite and cannot be expressed as a sum of cardinals $\Sigma_{i<\alpha}\lambda_i$ where $\alpha < \lambda$ and $\lambda_i < \lambda$ for all i. Otherwise, λ is called *singular*. The first infinite cardinal $\aleph_0$ is regular and so is every successor cardinal. Many limit cardinals are singular; in fact, the statement that $\aleph_0$ is the only regular limit cardinal is consistent with ZFC.

A partially ordered set is called *directed* if every pair of elements has an upper bound. More generally, for any regular cardinal λ, a partially ordered set is called λ-*directed* if every subset of cardinality smaller than λ has an upper bound. Note that λ-directed implies μ-directed if $\mu < \lambda$, and, if λ is any infinite cardinal, then λ-directed implies *filtered*, as in [8, XII] or in [1, 1.A]. (The latter concept is defined for more general categories, not necessarily partially ordered sets.)

An object X of a category $\mathcal{C}$ is called λ-*presentable*, where λ is a regular cardinal, if the functor $\mathcal{C}(X,-)$ preserves λ-directed colimits, that is, colimits of diagrams $D\colon I \to \mathcal{C}$ where I is a λ-directed partially ordered set. In other words, X is λ-presentable if whenever I is λ-directed and a morphism $f\colon X \to \operatorname{colim}_I D$ is given, there is an object $i \in I$ such that f factorizes through $D(i)$ and, if g and g' are any two such factorizations $X \to D(i)$, then there exists an object $j \geq i$ such that g and g' coincide in $D(j)$. Every μ-presentable object is λ-presentable if $\mu < \lambda$.

A category $\mathcal{C}$ is *locally presentable* if it is cocomplete and there is a regular cardinal λ and a set $\mathcal{X}$ of λ-presentable objects such that every object of $\mathcal{C}$ is a λ-directed colimit of objects from $\mathcal{X}$. This concept was originally introduced by Gabriel and Ulmer in [22]; see also [1, 1.B]. If the assumption of cocompleteness is weakened by imposing instead that λ-directed colimits exist in $\mathcal{C}$, then $\mathcal{C}$ is called λ-*accessible*. The category $\mathcal{C}$ is called *accessible* if it is λ-accessible for some regular cardinal λ. If $\mathcal{C}$ is accessible, then there are arbitrarily large regular cardinals λ such that $\mathcal{C}$ is λ-accessible; see [1, 2.14].

The category of groups is locally presentable, with $\lambda = \aleph_0$ and $\mathcal{X}$ a set of representatives of all isomorphism classes of finitely presented groups (in the usual sense of admitting a presentation with a finite number of generators and a finite number of relations). The category of topological spaces is not locally presentable; see [22, p. 64]. However, the category of simplicial sets is locally presentable, with $\lambda = \aleph_0$ and $\mathcal{X}$ a set of representatives of all isomorphism classes of finite simplicial sets. The category of CW-complexes and cellular maps is not locally presentable, since it is not cocomplete.

2 Idempotent functors

An *idempotent monad* on a category $\mathcal{C}$ is a pair (E, η) consisting of a functor E and a natural transformation $\eta\colon \mathrm{Id} \to E$ such that $\eta_{EX}\colon EX \to EEX$ is an isomorphism for every object X, and $\eta_{EX} = E\eta_X$ for all X; cf. [2]. For simplicity, we say that a functor E is idempotent if it is part of an idempotent monad. Then we also call it a *reflection* or a *localization* (although other authors use the term localization in a more restrictive sense). The natural transformation η was called a *coaugmentation* in [16].

An object X and a morphism $f\colon A \to B$ in a category $\mathcal{C}$ are *orthogonal* if the map

$$\mathcal{C}(f, X)\colon \mathcal{C}(B, X) \longrightarrow \mathcal{C}(A, X)$$

is bijective; that is, X and f are orthogonal if for every morphism $g\colon A \to X$ there is a unique morphism $h\colon B \to X$ such that $h \circ f = g$. The class of objects that are orthogonal to a given class $\mathcal{S}$ of morphisms is denoted by $\mathcal{S}^\perp$ and called the *orthogonal complement* of $\mathcal{S}$. The same notation is used by exchanging the role of objects and morphisms.

Every idempotent functor E gives rise to a class of morphisms $\mathcal{S}$ and a class of objects $\mathcal{D}$ such that $\mathcal{S}^\perp = \mathcal{D}$ and $\mathcal{D}^\perp = \mathcal{S}$, namely the class $\mathcal{S}$ of morphisms $f\colon X \to Y$ such that $Ef\colon EX \to EY$ is an isomorphism (such morphisms are called *E-equivalences*), and the class $\mathcal{D}$ of objects Z such that $Z \cong EY$ for some Y (objects in this class are said to be *E-local*); see [2] or [10] for further details.

A class $\mathcal{D}$ of objects in a category $\mathcal{C}$ is called *reflective* if it is the class of E-local objects for some idempotent functor E. (No distinction is made here between a class of objects and the full subcategory with those objects.) A class of objects $\mathcal{D}$ is called a *small-orthogonality class* if there is a set $\mathcal{M}$ of morphisms (not a proper class) such that $\mathcal{M}^{\perp} = \mathcal{D}$. If the category $\mathcal{C}$ has coproducts and all hom-sets $\mathcal{C}(X, Y)$ are nonempty, then a class of objects $\mathcal{D}$ is a small-orthogonality class if and only if $\mathcal{D} = \{\varphi\}^{\perp}$ for some morphism φ, since a set of morphisms and their coproduct have the same orthogonal complement.

A detailed proof of the following fundamental theorem can be found in [1, 1.36]. Various versions of this fact have been described in the literature. It goes back essentially to Gabriel–Ulmer [22]; see also Bousfield's article [6].

Theorem 2.1 *Every small-orthogonality class of objects in a locally presentable category is reflective.* $\square$

In Section 6 we will use this result in the category of groups. For a group homomorphism φ, we use the symbol L_{φ} and the term *φ-localization* to denote the reflection onto the class $\{\varphi\}^{\perp}$, whose members are called *φ-local* groups.

3 Homotopy idempotent functors

The homotopy-theoretical version of Theorem 2.1 is the following; cf. [6], [16]. Given any map of simplicial sets $f\colon A \to B$, a simplicial set X is called *f-local* if X is fibrant and the map of function complexes

$$\mathrm{map}(f, X)\colon \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X) \tag{3.1}$$

is a weak equivalence. As shown in [16] or [23], for every map f there is a coaugmented functor L_f on simplicial sets, called *f-localization*, which carries weak equivalences to weak equivalences and is a reflection onto the class of f-local simplicial sets in the homotopy category. When the map f has the form $A \to *$ for some A, the notation P_A is commonly used instead of L_f. (The letter P was chosen since Postnikov sections are the basic examples.)

We extend this terminology as follows. Given any class $\mathcal{S}$ of maps, a simplicial set X will be called *$\mathcal{S}$-local* if it is f-local for all maps f in $\mathcal{S}$. Similarly, for a class $\mathcal{D}$ of fibrant simplicial sets, a map $f\colon A \to B$ will be called a *$\mathcal{D}$-equivalence* if

$$\mathrm{map}(f, D)\colon \mathrm{map}(B, D) \longrightarrow \mathrm{map}(A, D)$$

is a weak equivalence for every D in $\mathcal{D}$. This is a stronger form of orthogonality that we will call *simplicially enriched orthogonality*, or, more shortly, *simplicial orthogonality*. Thus, the class of $\mathcal{D}$-equivalences is the *simplicial orthogonal complement* of $\mathcal{D}$, and analogously with the role of spaces and maps reversed. This is consistent with the term *f-local equivalences*, which was used in [7], [23], and in other places, to denote the simplicial orthogonal complement of the class of f-local spaces. For shortness, f-local equivalences are also called *f-equivalences* in many places, as we will do in this article.

For each map f, the functor L_f is idempotent up to homotopy. We next make precise the definition of a homotopy idempotent functor, with the aim of discussing, later in the article, whether or not this notion is more general than f-localization.

Definition 3.1 A functor E on the category of simplicial sets is *homotopy idempotent* if it carries weak equivalences to weak equivalences and is equipped with a natural transformation $\eta\colon \mathrm{Id} \to E$ such that $\eta_{EX} \simeq E\eta_X$ and $\eta_{EX}\colon EX \to EEX$ is a weak equivalence for all X.

We will impose, in addition, that EX be fibrant for all X. This allows to shorten many statements and does not cause any loss of generality, since, if E has the properties imposed in the definition, then so does any fibrant approximation to E.

Note that η is asked to be a natural transformation on the category of simplicial sets, not only up to homotopy. We also recall from [15, Lemma 2.1.3] that, if we impose that both η_{EX} and $E\eta_X$ be weak equivalences, then it follows automatically that $\eta_{EX} \simeq E\eta_X$.

A functor on simplicial sets that carries weak equivalences to weak equivalences is called a *homotopy functor*. Thus, every homotopy functor defines a functor on the homotopy category. Definition 3.1 says that a homotopy idempotent functor is a homotopy functor with a coaugmentation η such that (E, η) is an idempotent monad on the homotopy category. Accordingly, a map $g\colon X \to Y$ of simplicial sets will be called an *E-equivalence* if $Eg\colon EX \to EY$ is a weak equivalence, and a simplicial set Z will be called *E-local* if it is fibrant and $Z \simeq EY$ for some Y. Then the classes of E-equivalences and E-local simplicial sets are orthogonal in the homotopy category, that is, if $g\colon X \to Y$ is an E-equivalence and Z is E-local, then g induces a bijection of (unbased) homotopy classes of maps

$$[Y, Z] \cong [X, Z].$$

But it is crucial to emphasize that, if E is homotopy idempotent, then E-equivalences and E-local simplicial sets are also simplicially orthogonal, that is, if $g\colon X \to Y$ is an

E-equivalence and Z is E-local, then g induces a weak equivalence of function complexes

$$\mathrm{map}(Y,Z) \simeq \mathrm{map}(X,Z).$$

This fact will be proved in the next section. (All this can be formulated in the pointed homotopy category as well; cf. [16, 1.A.7].)

4 Homotopy idempotence and function complexes

The fact that, for each homotopy idempotent functor E, the simplicial orthogonal complement of the class of E-local simplicial sets coincides with the class of E-equivalences was first proved by Dror Farjoun in [15] under the assumption that the functor E be continuous. (A functor E is called *continuous* or *simplicial* if there is a natural map of simplicial sets

$$\mathrm{map}(X,Y) \longrightarrow \mathrm{map}(EX,EY)$$

for all X and Y, preserving composition and identity; cf. [27, II.1].) Thus, our argument in this section shows that the continuity assumption in [15, Theorem 2.1] is not essential, provided that the functor E carries weak equivalences to weak equivalences.

The next two lemmas are important ingredients in the proof. Let $\mathcal{L}(X,Y)$ be the category whose objects are the diagrams $X \to C \leftarrow Y$ of simplicial sets in which the map $C \leftarrow Y$ is a weak equivalence, and whose morphisms are commutative diagrams

$$\begin{array}{ccccc} X & \to & C_1 & \leftarrow & Y \\ {\scriptstyle =}\downarrow & & \downarrow & & {\scriptstyle =}\downarrow \\ X & \to & C_2 & \leftarrow & Y. \end{array}$$

We denote by $\mathcal{L}^{\mathrm{c}}(X,Y)$ the full subcategory of $\mathcal{L}(X,Y)$ whose objects are the diagrams $X \to C \leftarrow Y$ where the backwards map $C \leftarrow Y$ is a trivial cofibration.

Lemma 4.1 *For any two simplicial sets X and Y, the inclusion $J\colon \mathcal{L}^{\mathrm{c}}(X,Y) \hookrightarrow \mathcal{L}(X,Y)$ induces a homotopy equivalence of nerves.*

Proof. Define a functor $F\colon \mathcal{L}(X,Y) \to \mathcal{L}^{\mathrm{c}}(X,Y)$ in the following way. Given a diagram $X \xrightarrow{\alpha} C \xleftarrow{\beta} Y$ in which β is a weak equivalence, choose a functorial factorization of β

into a trivial cofibration followed by a trivial fibration with a right inverse, as in Brown's lemma [23, 7.7.1]. Specifically, factor the map $\beta \amalg \mathrm{id}\colon Y \coprod C \to C$ as

$$Y \coprod C \xrightarrow{\delta \amalg \sigma} M \xrightarrow{\gamma} C,$$

where M is the mapping cylinder of $\beta \amalg \mathrm{id}$. Then the map $\gamma\colon M \to C$ comes with a canonical section $\sigma\colon C \to M$. The value of F on $X \to C \leftarrow Y$ is defined to be the diagram

$$X \xrightarrow{\sigma\circ\alpha} M \xleftarrow{\delta} Y,$$

where the backwards map δ is now a trivial cofibration. The commutativity of the diagram

$$\begin{array}{ccccc} X & \rightarrow & M & \leftarrow & Y \\ {\scriptstyle =}\downarrow & & \downarrow & & {\scriptstyle =}\downarrow \\ X & \rightarrow & C & \leftarrow & Y \end{array}$$

yields a natural transformation from $F \circ J$ to the identity, and also from $J \circ F$ to the identity, as needed. $\square$

This result requires a set-theoretical comment. Since the categories $\mathcal{L}^{\mathrm{c}}(X, Y)$ and $\mathcal{L}(X, Y)$ have a proper class of objects, their nerves are "large simplicial sets", according to the following standard convention. In this section, we adopt as an axiom the statement that every set belongs to some *universe*, as defined e.g. in [4, Ch. 1]. We then fix a universe $\mathcal{U}$, in which we tacitly work. The *successor universe* $\mathcal{U}^+$ is the unique smallest universe such that $\mathcal{U} \in \mathcal{U}^+$. Elements of $\mathcal{U}$ are called *small sets* and elements of $\mathcal{U}^+$ are called *large sets* or *classes*. A category is *small* if its sets of objects and morphisms are small. The nerve of an arbitrary category $\mathcal{C}$ is a simplicial set in $\mathcal{U}^+$, which falls into $\mathcal{U}$ precisely when $\mathcal{C}$ is small. When we speak of a "set" without any further specification, the context should make clear whether a small set or a large set is intended. In Lemma 4.1 and in the rest of this section, nerves of categories, maps between nerves, and homotopies between these are implicitly defined in $\mathcal{U}^+$ whenever it is necessary.

We emphasize, however, that $N\mathcal{L}(X, Y)$ is *homotopically small* for all X and Y; that is, its set of connected components is small and its homotopy groups at every vertex are also small. Hence, as explained in [20, 2.2], the homotopy type of a homotopically small simplicial set is well defined within the universe $\mathcal{U}$ in which we agreed to work.

The following notions are dual to those introduced in [8, XI.9.1] and [8, XI.10.1]. For a small category $\mathcal{S}$, a functor $\Phi\colon \mathcal{S} \to \mathcal{C}$ is *right cofinal* if, for each object x in $\mathcal{C}$, the nerve of the undercategory $(x \downarrow \Phi)$ is contractible (i.e., weakly equivalent to a point). A category $\mathcal{C}$ is *right small* if there exists a right cofinal functor $\Phi\colon \mathcal{S} \to \mathcal{C}$ where $\mathcal{S}$ is small. If $\mathcal{C}$ is right small, then its nerve $N\mathcal{C}$ is homotopically small and weakly equivalent to $N\mathcal{S}$ for any right cofinal functor $\Phi\colon \mathcal{S} \to \mathcal{C}$. Indeed, $N\mathcal{S}$ is the homotopy colimit of the constant functor from $\mathcal{S}$ to simplicial sets sending all objects to a point, and, for a right small category $\mathcal{C}$, the homotopy type of this homotopy colimit does not depend on the choice of $\mathcal{S}$ or Φ; cf. [8, XI.10.3].

For a simplicial set K, we denote by ΔK the *category of simplices* of K, whose objects are the simplicial maps $\Delta[n] \to K$ with $n \geq 0$, and where a morphism from $\sigma\colon \Delta[n] \to K$ to $\tau\colon \Delta[m] \to K$ is a simplicial map $\varphi\colon \Delta[n] \to \Delta[m]$ with $\tau \circ \varphi = \sigma$. For each K, there is a natural weak equivalence $N\Delta K \to K$ from the nerve of this category onto K; see e.g. [23, 15.1.14 and 18.9.3].

If Y is a fibrant simplicial set, then for every X there is a functor

$$\Phi\colon \Delta\mathrm{map}(X,Y) \longrightarrow \mathcal{L}^{\mathrm{c}}(X,Y)$$

sending each object $\Delta[n] \to \mathrm{map}(X,Y)$ to the diagram

$$X \longrightarrow Y^{\Delta[n]} \longleftarrow Y,$$

where we write $Y^{\Delta[n]}$ instead of $\mathrm{map}(\Delta[n],Y)$ for convenience. The arrow $X \to Y^{\Delta[n]}$ corresponds to the given object by adjointness, and $Y \to Y^{\Delta[n]}$ is a trivial cofibration induced by the unique map $\Delta[n] \to \Delta[0]$.

Lemma 4.2 *If Y is fibrant, then $\Phi\colon \Delta\mathrm{map}(X,Y) \longrightarrow \mathcal{L}^{\mathrm{c}}(X,Y)$ is right cofinal.*

Proof. The argument is based on [20, § 6 and § 7]. For each object x in $\mathcal{L}^{\mathrm{c}}(X,Y)$, say $X \to C \leftarrow Y$, the undercategory $(x \downarrow \Phi)$ has objects the commutative diagrams

$$\begin{array}{ccccc} X & \rightarrow & C & \leftarrow & Y \\ {\scriptstyle =}\downarrow & & \downarrow & & {\scriptstyle =}\downarrow \\ X & \rightarrow & Y^{\Delta[n]} & \leftarrow & Y, \end{array}$$

and hence it is isomorphic to the category of simplices ΔF of the fibre F of

$$\mathrm{map}(C,Y) \longrightarrow \mathrm{map}(Y,Y) \tag{4.1}$$

over the identity, taken as basepoint of $\mathrm{map}(Y,Y)$. Since $Y \to C$ is a trivial cofibration, the map (4.1) is a trivial fibration and therefore F is contractible. This implies that the nerve of $(x \downarrow \Phi)$ is contractible, as claimed. $\square$

If Y is not fibrant, then $N\mathcal{L}^c(X,Y)$ still has the correct homotopy type of a function complex, while $\mathrm{map}(X,Y)$ does not, in general.

The main result in this section is the following.

Theorem 4.3 *Let E be any homotopy idempotent functor on simplicial sets, with coaugmentation η. Then, for all simplicial sets X and Z, the map*

$$\mathrm{map}(EX, EZ) \longrightarrow \mathrm{map}(X, EZ)$$

induced by $\eta_X\colon X \to EX$ is a weak equivalence.

Proof. Let us write Y instead of EZ for shortness, so Y denotes any E-local simplicial set. By the previous two lemmas, since Y is fibrant, the function complex $\mathrm{map}(X,Y)$ has the same homotopy type as the nerve of the category $\mathcal{L}(X,Y)$. In fact, there are natural weak equivalences

$$\mathrm{map}(X,Y) \longleftarrow N\Delta\mathrm{map}(X,Y) \longrightarrow N\mathcal{L}(X,Y).$$

Since the next argument involves simplicial homotopies, we need in addition a fibrant replacement of $N\mathcal{L}(X,Y)$. We may use, for instance, the Kan functor Ex^∞. To simplify the notation, let us write $M(X,Y)$ as an abbreviation of $\mathrm{Ex}^\infty N\mathcal{L}(X,Y)$.

What we achieve by using $M(X,Y)$ instead of the function complex $\mathrm{map}(X,Y)$ is that the functor E gives rise to a natural map

$$M(X,Y) \xrightarrow{\ \varepsilon\ } M(EX, EY)$$

for all X, Y, induced by the functor that sends $X \to C \leftarrow Y$ to $EX \to EC \leftarrow EY$.

Thus, in order to prove the theorem we may check that, if Y is E-local, then the map

$$(\eta_X)^*\colon M(EX,Y) \longrightarrow M(X,Y)$$

induced by $\eta_X\colon X \to EX$ is a weak equivalence. For shortness, we denote this map by f. Observe that it is induced by the functor $F\colon \mathcal{L}(EX,Y) \to \mathcal{L}(X,Y)$ given by

$$F\left(EX \xrightarrow{\varphi_1} C \xleftarrow{\varphi_2} Y\right) = X \xrightarrow{\varphi_1\circ\eta_X} C \xleftarrow{\varphi_2} Y.$$

Define another functor $G\colon \mathcal{L}(X,Y) \to \mathcal{L}(EX,Y)$ by

$$G\left(X \xrightarrow{\psi_1} D \xleftarrow{\psi_2} Y\right) = EX \xrightarrow{E\psi_1} ED \xleftarrow{E\psi_2 \circ \eta_Y} Y,$$

and let $g\colon M(X,Y) \to M(EX,Y)$ be the corresponding map, that is, $g = (\eta_Y)^* \circ \varepsilon$. We are going to prove that g is a homotopy inverse of f.

First, from the fact that η is a natural transformation it follows that, for every object $X \xrightarrow{\psi_1} D \xleftarrow{\psi_2} Y$ in $\mathcal{L}(X,Y)$, we have equalities $E\psi_1 \circ \eta_X = \eta_D \circ \psi_1$ and $E\psi_2 \circ \eta_Y = \eta_D \circ \psi_2$. These yield precisely a natural transformation $\mathrm{Id} \to F \circ G$, showing that $f \circ g$ is simplicially homotopic to the identity map.

Exactly the same argument shows that the composite

$$M(EX,Y) \xrightarrow{\varepsilon} M(EEX,EY) \xrightarrow{(\eta_{EX})^*} M(EX,EY) \xrightarrow{(\eta_Y)^*} M(EX,Y)$$

is homotopic to the identity map. Moreover, observe that the composites

$$M(EX,Y) \xrightarrow{\varepsilon} M(EEX,EY) \xrightarrow{(E\eta_X)^*} M(EX,EY)$$

and

$$M(EX,Y) \xrightarrow{(\eta_X)^*} M(X,Y) \xrightarrow{\varepsilon} M(EX,EY)$$

are induced by the same functor. Finally, recall that the maps $E\eta_X$ and η_{EX} are homotopic, since E is assumed to be homotopy idempotent. Thus, let $\tau\colon EX \to EEX^{\Delta[1]}$ be adjoint to a simplicial homotopy from $E\eta_X$ to η_{EX}, and consider the functor

$$T\colon \mathcal{L}(EEX,EY) \longrightarrow \mathcal{L}(EX,EY)$$

sending each diagram $EEX \xrightarrow{\varphi_1} C \xleftarrow{\varphi_2} EY$ to $EX \longrightarrow C^{\Delta[1]} \xleftarrow{c \circ \varphi_2} EY$, where the first arrow is $\varphi_1^{\Delta[1]} \circ \tau$ and c is adjoint to the constant homotopy at the identity. This functor T comes equipped with natural transformations $T \to (E\eta_X)^*$ and $T \to (\eta_{EX})^*$ by means of τ. Hence $(E\eta_X)^* \simeq (\eta_{EX})^*$ after taking nerves and a fibrant approximation. Therefore,

$$g \circ f = (\eta_Y)^* \circ \varepsilon \circ (\eta_X)^* \simeq (\eta_Y)^* \circ (E\eta_X)^* \circ \varepsilon \simeq (\eta_Y)^* \circ (\eta_{EX})^* \circ \varepsilon \simeq \mathrm{id},$$

as needed. We are thankful to Oriol Raventós for his help in fixing the last details of this proof, and to Bill Dwyer for useful comments. $\square$

Corollary 4.4 *Let E be a homotopy idempotent functor on simplicial sets.*

(a) *If $f\colon A \to B$ is an E-equivalence, then all E-local simplicial sets are f-local, and all f-equivalences are E-equivalences.*

(b) *The class of E-equivalences coincides with the simplicial orthogonal complement of the class of E-local simplicial sets.*

Proof. In part (a), it suffices to prove that every E-local simplicial set is f-local. Now, for every E-local simplicial set Z, the map

$$\mathrm{map}(B, Z) \longrightarrow \mathrm{map}(A, Z)$$

induced by f is a weak equivalence, since Ef is a weak equivalence by assumption and, by Theorem 4.3, the vertical arrows are weak equivalences in the commutative diagram

$$\begin{array}{ccc} \mathrm{map}(EB, Z) & \longrightarrow & \mathrm{map}(EA, Z) \\ \downarrow & & \downarrow \\ \mathrm{map}(B, Z) & \longrightarrow & \mathrm{map}(A, Z). \end{array}$$

To prove (b), denote by $\mathcal{S}$ the simplicial orthogonal complement of the class of E-local simplicial sets. Then every map in $\mathcal{S}$ is an E-equivalence. Conversely, if $g\colon X \to Y$ is an E-equivalence, then, by (a), all E-local simplicial sets are g-local, and this means that g is in $\mathcal{S}$. $\square$

Corollary 4.5 *If E is any homotopy idempotent functor on simplicial sets and some E-equivalence $f\colon A \to B$ is not bijective on connected components, then EX is contractible for every nonempty simplicial set X.*

Proof. By part (a) of Corollary 4.4, all E-local simplicial sets are f-local. According to [31, Theorem 4.1], if the function $\pi_0(f)$ is not bijective, then nonempty f-local simplicial sets are contractible. $\square$

5 Existence results

We will use Vopěnka's principle in this section. One of its various equivalent formulations [1, Ch. 6] says that *no locally presentable category contains a rigid proper class of objects*. (A class of objects is called *rigid* if it admits no other morphisms than identities.) Thus, according to Vopěnka's principle, given a proper class of objects A_i in any locally presentable category, there is a nonidentity morphism $A_i \to A_j$ for some indices i and j. Under this axiom, as stated in [1, 6.24], every full subcategory closed under limits in a locally presentable category is a small-orthogonality class (hence reflective). Since the category of groups is locally presentable, we have the following.

Theorem 5.1 *Suppose that Vopěnka's principle holds. Let E be any idempotent functor on the category of groups. Then there exists a homomorphism φ such that $E \cong L_\varphi$.*

Proof. Our assumptions ensure that the class $\mathcal{D}$ of E-local groups is a small-orthogonality class, that is, there is a homomorphism φ such that $\mathcal{D} = \{\varphi\}^\perp$. This means that the classes of φ-local groups and E-local groups coincide and our claim follows. $\square$

Now we turn to the analogous situation in homotopy theory. We are going to prove that, assuming that Vopěnka's principle is true, every homotopy idempotent functor E on simplicial sets is equivalent to L_f for some map f.

Lemma 5.2 *Let $\mathcal{D}$ be any class of fibrant simplicial sets. Then the class of $\mathcal{D}$-equivalences is closed under filtered colimits.*

Proof. Let $f_i\colon X_i \to Y_i$ be a diagram of maps in the class $\mathcal{S}$ of $\mathcal{D}$-equivalences, where the index i ranges over a filtered set I. We wish to prove that the induced map

$$\operatorname{colim} f_i\colon \operatorname{colim} X_i \longrightarrow \operatorname{colim} Y_i$$

is in $\mathcal{S}$. Since homotopy groups commute with filtered colimits of simplicial sets, the horizontal arrows in the commutative diagram

$$\begin{array}{ccc} \operatorname{hocolim} X_i & \longrightarrow & \operatorname{colim} X_i \\ \downarrow & & \downarrow \\ \operatorname{hocolim} Y_i & \longrightarrow & \operatorname{colim} Y_i \end{array}$$

are weak equivalences; cf. [8, XII.3.5]. Therefore, it suffices to show that the left vertical arrow $\operatorname{hocolim} f_i$ is in $\mathcal{S}$. But this follows from the natural equivalences

$$\operatorname{map}(\operatorname{hocolim} Y_i, Z) \simeq \operatorname{holim} \operatorname{map}(Y_i, Z) \simeq \operatorname{holim} \operatorname{map}(X_i, Z) \simeq \operatorname{map}(\operatorname{hocolim} X_i, Z),$$

for every simplicial set Z in $\mathcal{D}$. $\square$

The following theorem is the basic result that explains the role of large-cardinal axioms in homotopy theory.

Theorem 5.3 *Suppose that Vopěnka's principle is true. Let $\mathcal{D}$ be any class of fibrant simplicial sets. Then there is a map f such that the class of f-equivalences is equal to the class of $\mathcal{D}$-equivalences.*

Proof. According to [1, 6.6] and [1, 6.18], assuming Vopěnka's principle, every full subcategory which has λ-directed colimits for some regular cardinal λ in a locally presentable category $\mathcal{C}$ is accessible. We will use this fact in the category $\mathcal{C}$ whose objects are maps between simplicial sets and whose morphisms are commutative squares. Thus, let $\mathcal{S}$ be the simplicial orthogonal complement of a given class $\mathcal{D}$ of fibrant simplicial sets, and denote with the same letter $\mathcal{S}$ the full subcategory of $\mathcal{C}$ with these objects. Lemma 5.2 says that $\mathcal{S}$ is closed under filtered colimits. That is, $\mathcal{S}$ has filtered colimits, and the inclusion of $\mathcal{S}$ into $\mathcal{C}$ preserves filtered colimits. Therefore, $\mathcal{S}$ is accessible. (This illustrates well the distinction between locally presentable and accessible categories; $\mathcal{S}$ need not be closed under arbitrary colimits, so it is not cocomplete.) Thus, for a certain regular cardinal λ_0, the class $\mathcal{S}$ contains a set $\mathcal{X}$ of λ_0-presentable objects such that every object of $\mathcal{S}$ is a λ_0-directed colimit of objects from $\mathcal{X}$.

Let f be the disjoint union of all the maps in $\mathcal{X}$. Since every simplicial set in $\mathcal{D}$ is f-local, every f-equivalence is in $\mathcal{S}$. Conversely, every map g in $\mathcal{S}$ is a λ_0-directed colimit of maps in $\mathcal{X}$. Since all maps in $\mathcal{X}$ are rendered invertible by L_f, they are f-equivalences. Therefore, g is also an f-equivalence, by Lemma 5.2. This finishes the proof. $\square$

Corollary 5.4 *Suppose that Vopěnka's principle is true. If h^* is any generalized cohomology theory, then there is a map f such that the class of f-equivalences coincides with the class of h^*-equivalences.*

Proof. Let $\mathcal{D}$ be a set of fibrant simplicial sets $\{K_n\}$, $n \in \mathbf{Z}$, representing the cohomology theory h^*. Then a map $g\colon X \to Y$ is a $\mathcal{D}$-equivalence if and only if

$$\mathrm{map}(g, K_n)\colon \mathrm{map}(Y, K_n) \longrightarrow \mathrm{map}(X, K_n)$$

is a weak equivalence for all n. Now, for every n and every X, the function complex $\mathrm{map}(X, K_n)$ is an infinite loop space, so all its connected components have the same homotopy type. Its group of connected components is $[X, K_n] \cong h^n(X)$ and the homotopy groups of the basepoint component are

$$\pi_i(\mathrm{map}(X, K_n), *) \cong \pi_0(\Omega^i \mathrm{map}(X, K_n)) \cong \pi_0(\mathrm{map}(X, \Omega^i K_n)) \cong [X, \Omega^i K_n] \cong h^{n-i}(X).$$

Hence, the class of $\mathcal{D}$-equivalences is precisely the class of h^*-equivalences, and our claim follows from Theorem 5.3. □

In other words, Vopěnka's principle implies the existence of arbitrary cohomological localizations. So far, no proof of this fact has been given using only ZFC.

We also infer the following key result.

Theorem 5.5 *Suppose that Vopěnka's principle is true. If E is any homotopy idempotent functor on simplicial sets, then there is a map f such that $L_f \simeq E$.*

Proof. Let $\mathcal{D}$ be the class of E-local simplicial sets. It follows from Corollary 4.4 that the class of $\mathcal{D}$-equivalences coincides with the class of E-equivalences. Then Theorem 5.3 says that there is a map f such that the f-equivalences are precisely the E-equivalences. This implies that the functors E and L_f are homotopy equivalent. □

Note, however, that we have not proved the consistency relative to ZFC of the assertion that every homotopy idempotent functor is equivalent to L_f for some f, since the validity of this assertion depends on large-cardinal hypotheses. What we have shown is that it is unreasonable to try to find a counterexample in ZFC, since such a counterexample would imply that Vopěnka's principle is inconsistent. The extent to which the strongest large-cardinal principles not known to be inconsistent (such as Vopěnka's principle) should be considered "true" is discussed e.g. at the beginning and end of § 24 in [25]. We quote the following paragraph: "Of course, a new inconsistency result would be an exciting development, but as time goes by, the further analysis and application of these hypotheses suggest that they may be approached with increasing confidence if not acceptance."

In the next section we show that, under different set-theoretical assumptions, the conclusion of Theorem 5.5 is different.

6 A counterexample

It was shown in Proposition 2.1 of [11] that, in the category of groups, for every (possibly proper) class $\mathcal{S}$ of epimorphisms, the orthogonal complement $\mathcal{S}^{\perp}$ is reflective (cf. also Exercise 1.n in [1, p. 63] and Theorem 1 in [13]). Specifically, for a group G, let TG be the intersection of all kernels of epimorphisms from G onto groups in $\mathcal{S}^{\perp}$. Then $EG = G/TG$ is the desired reflection. In the special case when $\mathcal{S}$ is a class of homomorphisms of the form $A_\alpha \to 0$, where A_α ranges over a set or a class $\mathcal{A}$ of groups, the corresponding reflection will be called $\mathcal{A}$-*reduction* and denoted $P_{\mathcal{A}}$. Thus, a group G is $\mathcal{A}$-*reduced* if and only if the set $\mathrm{Hom}(A_\alpha, G)$ is trivial for every A_α in $\mathcal{A}$. Contrary to what happens with more general localization functors, if two reduction functors annihilate the same groups, then they coincide; cf. [11, Theorem 2.3].

For each cardinal κ, we denote by $\mathbf{Z}^\kappa$ the cartesian product of κ copies of the additive group of integers; that is, $\mathbf{Z}^\kappa$ is the abelian group of all functions $f\colon \kappa \to \mathbf{Z}$. For a function $f \in \mathbf{Z}^\kappa$, the support $\mathrm{supp}(f)$ is the set of indices $i \in \kappa$ for which $f(i) \neq 0$. We write $\mathbf{Z}^{<\kappa}$ to designate the set of all functions $f \in \mathbf{Z}^\kappa$ such that the cardinality of $\mathrm{supp}(f)$ is smaller than κ. We will focus attention on the quotient $\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$. (In the special case $\kappa = \aleph_0$, this is of course $\prod_{i=1}^{\infty}\mathbf{Z}/\oplus_{i=1}^{\infty}\mathbf{Z}$.)

Lemma 6.1 *Let κ be any regular cardinal and let G be a group with* $\mathrm{card}(G) < \kappa$. *Then every homomorphism $\beta\colon G \to \mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$ can be lifted to a homomorphism $\alpha\colon G \to \mathbf{Z}^\kappa$.*

Proof. This is a particular instance of Lemma 2.6 in [17], which we adapt to our purposes. For each element $g \in G$, pick a representative $\phi(g) \in \mathbf{Z}^\kappa$ of the image $\beta(g)$. Thus, for each pair of elements g and h of G, the element $\phi(g) + \phi(h) - \phi(gh)$ lies in $\mathbf{Z}^{<\kappa}$. Let S be the union of the supports of the elements $\phi(g) + \phi(h) - \phi(gh)$ for all pairs of elements g and h of G. The assumption that $\mathrm{card}(G) < \kappa$ ensures that $\mathrm{card}(S) < \kappa$ as well, since κ is regular. Thus, if we now define $\alpha(g)$ by setting to zero all the components in S of the element $\phi(g)$, then $\alpha(g)$ and $\phi(g)$ define the same element in $\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$, and $\alpha\colon G \to \mathbf{Z}^\kappa$ is in fact a homomorphism. □

Let $\mathcal{A}$ be the class of groups $\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$ for all cardinals κ, so that a group G is $\mathcal{A}$-reduced if and only if $\mathrm{Hom}(\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}, G) = 0$ for all κ.

Recall that an uncountable cardinal λ is *measurable* if it admits a nontrivial, two-valued, λ-additive measure, that is, if a function μ can be defined on any set X of cardinality λ assigning to each subset of X a value 0 or 1, in such a way that $\mu(X) = 1$,

$\mu(x) = 0$ for all $x \in X$, and $\mu\left(\cup_i A_i\right) = \Sigma_i\ \mu(A_i)$ if the subsets A_i are pairwise disjoint and the set of indices i has cardinality smaller than λ. The existence of measurable cardinals cannot be proved in ZFC, since every measurable cardinal is (strongly) inaccessible; see [1, A.10] or [24, 5.27].

We note that in [21] —as in many other references— measurable cardinals are defined by imposing only that the measure μ be countably additive. However, the existence of cardinals λ with a nontrivial two-valued λ-additive measure is equivalent to the existence of cardinals with a nontrivial two-valued countably additive measure; see Theorem 6.1.11 in [3] or Lemma 27.1 in [24].

Proposition 6.2 *Let $\mathcal{A}$ be the class of groups $\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$ for all cardinals κ. Then the statement that the additive group of integers $\mathbf{Z}$ is $\mathcal{A}$-reduced is equivalent to the statement that all cardinals are nonmeasurable.*

Proof. Assuming that measurable cardinals do not exist, we have $\mathrm{Hom}(\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}, \mathbf{Z}) = 0$ for all κ; see Theorem 94.4 in Fuchs' book [21]. On the other hand, if we admit the existence of a measurable cardinal λ, then $\mathrm{Hom}(\mathbf{Z}^\lambda/\mathbf{Z}^{<\lambda}, \mathbf{Z}) \neq 0$, since we can define a nonzero homomorphism $\varphi\colon \mathbf{Z}^\lambda \to \mathbf{Z}$ with $\varphi(\mathbf{Z}^{<\lambda}) = 0$, by assigning to each function $f\colon \lambda \to \mathbf{Z}$ the unique integer z such that $f^{-1}(z)$ has measure 1; cf. [17, p. 83]. $\square$

Theorem 6.3 *Suppose that all cardinals are nonmeasurable. If $\mathcal{A}$ is the class of groups $\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$ for all cardinals κ, then there is no single group homomorphism φ such that φ-localization is isomorphic to $\mathcal{A}$-reduction on the category of groups.*

Proof. First we show that there exists no group G such that G-reduction coincides with $\mathcal{A}$-reduction. Suppose that there is such a group G. Then, since $P_{\mathcal{A}}(\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}) = 0$ for all κ, we must have $\mathrm{Hom}(G, \mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}) \neq 0$ for all κ as well. Let κ be a regular cardinal that is bigger than the cardinality of G. Let $\beta\colon G \to \mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$ be a nonzero homomorphism. By Lemma 6.1, β can be lifted to a homomorphism $\alpha\colon G \to \mathbf{Z}^\kappa$, which is of course nonzero. Hence, composition with a suitable projection yields a nonzero homomorphism $G \to \mathbf{Z}$ and this implies that $\mathbf{Z}$ is not G-reduced. From our assumption it follows that $\mathbf{Z}$ is not $\mathcal{A}$-reduced and this contradicts Proposition 6.2.

Now suppose that there is a homomorphism φ such that L_φ coincides with $P_{\mathcal{A}}$. Let G be a universal φ-acyclic group, as in [29, § 3]. Thus, P_G and $P_{\mathcal{A}}$ annihilate the same groups and hence they coincide. That is, G-reduction is the same as $\mathcal{A}$-reduction, which is, as we know, impossible. $\square$

Localizations of groups extend to localizations of groupoids as described in [9]. In particular, for each set or proper class of groups $\mathcal{A}$, the functor $P_{\mathcal{A}}$ extends over the category of groupoids. For each groupoid G, the $\mathcal{A}$-reduction morphism $\eta\colon G \to P_{\mathcal{A}}G$ induces a bijection of connected components and an isomorphism $P_{\mathcal{A}}\pi_1(G,v) \cong \pi_1(P_{\mathcal{A}}G,\, \eta(v))$ at each object v of G.

Theorem 6.4 *Suppose that all cardinals are nonmeasurable. Then there is a homotopy idempotent functor E on simplicial sets that is not equivalent to f-localization for any map f.*

Proof. For each simplicial set X, define

$$EX = NP_{\mathcal{A}}\pi X,$$

where πX is the fundamental groupoid of X, the letter N denotes the nerve, and the class $\mathcal{A}$ consists of all groups $\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$. The map $\eta_X\colon X \to EX$ is the composite of the natural map $X \to N\pi X$ with the map induced by the $\mathcal{A}$-reduction morphism $\pi X \to P_{\mathcal{A}}\pi X$. This functor E is homotopy idempotent, since $\pi EX \cong P_{\mathcal{A}}\pi X$ and the functor $P_{\mathcal{A}}$ is homotopy idempotent on the model category of groupoids [9].

Now suppose that there is a map $f\colon R \to S$ of simplicial sets such that L_f is equivalent to E. We may assume with no loss of generality that R and S are connected (using Corollary 4.5), and furthermore that they have a single vertex. Let $\varphi\colon \pi R \to \pi S$ be the morphism induced by f on fundamental groupoids (in fact, these are groups, since they have only one object). A group G is φ-local if and only if φ induces a bijection of sets

$$\mathrm{Hom}(\pi S, G) \cong \mathrm{Hom}(\pi R, G),$$

that is, if and only if f induces a weak equivalence of function complexes

$$\mathrm{map}(S, NG) \simeq \mathrm{map}(R, NG).$$

Hence, G is φ-local if and only if NG is f-local, that is, if and only if NG is E-local, which is the same as imposing that G be $\mathcal{A}$-reduced. Therefore, the class of φ-local groups coincides with the class of $\mathcal{A}$-reduced groups. This is impossible, according to Theorem 6.3. $\square$

The confrontation of Theorem 5.5 with Theorem 6.4 was indeed one of the main goals of this article.

7 Universal acyclic spaces

As shown in [29, § 3], for each localization on the category of groups with respect to a homomorphism φ there exists a (not necessarily unique) universal acyclic group, that is, a group G such that the G-reduction P_G and the φ-localization L_φ annihilate the same groups. As we next explain, it cannot be proved in ZFC that arbitrary localizations on the category of groups admit universal acyclic groups. Let E be an idempotent functor on groups. On one hand, under Vopěnka's principle, we know that $E \cong L_\varphi$ for some φ and hence a universal E-acyclic group exists. On the other hand, assuming that no measurable cardinal exists, Theorem 6.3 yields a reduction functor which is not isomorphic to P_G for any group G. Such a functor does not admit a universal acyclic group.

The same result holds for simplicial sets, as shown below. Thus, Bousfield's result in [7], asserting that for every map f there is a universal f-acyclic space, cannot be extended to arbitrary homotopy idempotent functors. Recall that, if E is a homotopy idempotent functor on simplicial sets, a simplicial set X is called *E-acyclic* if EX is contractible. A *universal* E-acyclic space is a simplicial set U such that the nullification P_U kills the same simplicial sets as E does.

Theorem 7.1 *The existence of a universal E-acyclic space for every homotopy idempotent functor E on simplicial sets is ensured by Vopěnka's principle. However, if we assume that measurable cardinals do not exist, then there are homotopy idempotent functors on simplicial sets for which no universal acyclic space exists.*

Proof. Let E be given. First, if Vopěnka's principle holds, then $E \simeq L_f$ for some map f, by Theorem 5.5. Hence, it follows from [7, Theorem 4.4] that a universal E-acyclic space exists. Secondly, assume that there are no measurable cardinals. As in Section 6, consider the functor $EX = NP_{\mathcal{A}}\pi X$, where the class $\mathcal{A}$ consists of all groups $\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$. We claim that there is no universal E-acyclic space. Indeed, suppose that universal E-acyclic spaces exist, and choose one with a single vertex. Call it U and let G be its fundamental group. Let κ be a regular cardinal that is bigger than the cardinality of G. Then the simplicial set $N(\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa})$ is E-acyclic and hence it is P_U-acyclic. This implies that there are essential maps $U \to N(\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa})$. Therefore, $\mathrm{Hom}(G, \mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}) \neq 0$. From Lemma 6.1 it follows that $\mathrm{Hom}(G, \mathbf{Z}) \neq 0$. Therefore, there are essential maps $U \to S^1$ and, as shown in [12], S^1 is then P_U-acyclic. It follows that S^1 is E-acyclic, which implies that $P_{\mathcal{A}}\mathbf{Z} = 0$, and this contradicts Proposition 6.2. $\square$

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