Cohomological localization towers

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May 8, 2014

Abstract

We prove that, assuming the existence of cohomological localizations (which is known to follow from the existence of supercompact cardinals), the localization of a space or a spectrum X with respect to a cohomology theory E^* can be computed pointwise as a homotopy inverse limit.

Introduction

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Acknowledgements If any.

1 Completion with respect to group objects

Let $\mathcal{M}$ be a left proper pointed model category with functorial factorizations, and let $\mathcal{G}$ be any class of fibrant objects. A map $A \to B$ will be called *$\mathcal{G}$-monic*, as in [3, § 3.1], if the induced function $[\Sigma^n B, G] \to [\Sigma^n A, G]$ is surjective for each $G \in \mathcal{G}$ and $n \geq 0$. An object X is *$\mathcal{G}$-injective* if the function $[\Sigma^n B, X] \to [\Sigma^n A, X]$ is surjective for each $\mathcal{G}$-monic map $A \to B$ and all $n \geq 0$.

One says that $\mathcal{G}$ is a *class of injective models* if every object of $\mathcal{M}$ is the source of a $\mathcal{G}$-monic map to a $\mathcal{G}$-injective target. According to [3, § 4.5], if $\mathcal{G}$ is any set of group objects in $\mathrm{Ho}(\mathcal{M})$, then $\mathcal{G}$ is a class of injective models. The motivating example is the set

$$\mathcal{G} = \{\Sigma^n E_k \mid n \in \mathbb{Z},\ k \geq 0\}, \tag{1.1}$$

where E is a given Ω-spectrum and $\{E_k\}_{k\geq 0}$ are its representing spaces (i.e., pointed simplicial sets).

A class $\mathcal{G}$ of injective models is *functorial* in the sense of [3, § 4.2] if there exists a functor $\Gamma\colon \mathcal{M} \to \mathcal{M}$ and a natural transformation $\gamma\colon \mathrm{Id} \to \Gamma$ such that $\gamma_X\colon X \to \Gamma X$ is a $\mathcal{G}$-monic map and ΓX is $\mathcal{G}$-injective for each X in $\mathcal{M}$. From the assumption that the model category $\mathcal{M}$ has functorial factorizations it follows that every set $\mathcal{G}$ of group objects in $\mathrm{Ho}(\mathcal{M})$ is functorial; details are given in [3, § 4.5].

For an object X of $\mathcal{M}$ and a class of injective models $\mathcal{G}$, let $X \to X^\circ$ be a trivial cofibration into a fibrant object in the *$\mathcal{G}$-resolution model structure* on the category of cosimplicial objects over $\mathcal{M}$ given by [3, Theorem 3.3], where X is viewed as a constant cosimplicial object. This can be chosen functorially by [3, § 4.2] if $\mathcal{G}$ is functorial. Recall that, in the $\mathcal{G}$-resolution model structure, a weak equivalence is a map $f\colon X^\circ \to Y^\circ$ of cosimplicial objects inducing a weak equivalence of simplicial groups $[\Sigma^n Y^\circ, G] \simeq [\Sigma^n X^\circ, G]$ for all $G \in \mathcal{G}$ and $n \geq 0$, and the fibrations and cofibrations are described in [3, § 3.2].

Now define the *$\mathcal{G}$-completion* of an object X in $\mathcal{M}$ as

$$\widehat{L_{\mathcal{G}}} X = \mathrm{Tot}\, X^\circ,$$

as in [3, § 5.7]. If $\mathcal{G}$ is functorial, then $\widehat{L_{\mathcal{G}}}$ can be chosen as a functor $\mathcal{M} \to \mathcal{M}$ equipped with a natural transformation $\alpha\colon \mathrm{Id} \to \widehat{L_{\mathcal{G}}}$. Note, moreover, that $\widehat{L_{\mathcal{G}}} X$ is fibrant by [3, § 2.8].

We set, accordingly, $T_1 = \widehat{L_{\mathcal{G}}}$ and $\eta_1 = \alpha$. By [3, Corollary 8.2], the functor T_1 and the natural transformation η_1 are part of a monad on $\mathrm{Ho}(\mathcal{M})$. The main properties of this monad are collected in the following lemma.

Lemma 1.1. (i) *If an object X in $\mathcal{M}$ is $\mathcal{G}$-injective, then $\eta_1\colon X \to T_1 X$ is a weak equivalence.*

(ii) *Given a map $f\colon X \to Y$ in $\mathcal{M}$, the map $T_1 f\colon T_1 X \to T_1 Y$ is a weak equivalence if and only if the function*

$$[f, G]\colon [\Sigma^n Y, G] \longrightarrow [\Sigma^n X, G]$$

is a bijection for all $G \in \mathcal{G}$ and $n \geq 0$.

Proof. Part (i) is stated in [3, Corollary 6.6] and part (ii) follows from [3, Lemma 8.4]. □

The maps $X \to Y$ such that the induced function $[\Sigma^n Y, G] \to [\Sigma^n X, G]$ is bijective for all $G \in \mathcal{G}$ and $n \geq 0$ are called *$\mathcal{G}$-equivalences.*

2 A long tower

We next define a *long tower* as in [7], starting with $T_1 = L_{\mathcal{G}}^{\widehat{}}$ as defined in the previous section, where $\mathcal{G}$ is a functorial class of injective models in a left proper pointed model category $\mathcal{M}$. Thus we define inductively, for each ordinal α and each object X in $\mathcal{M}$,

$$T_{\alpha+1}X = \operatorname{holim} \mathbf{T}_\alpha X$$

where $\mathbf{T}_\alpha X$ is the restricted cosimplicial object with $(\mathbf{T}_\alpha X)^k = T_\alpha^{k+1}X$ for $k \geq 0$; cf. [7, § 3]. For a limit ordinal λ, we let

$$T_\lambda X = \operatorname*{holim}_{\alpha<\lambda} T_\alpha.$$

Each T_α is a functor on $\mathcal{M}$ preserving weak equivalences, equipped with a natural transformation $\eta_\alpha \colon \mathrm{Id} \to T_\alpha$, and it is part of a monad $(T_\alpha, \eta_\alpha, \mu_\alpha)$ on $\mathrm{Ho}(\mathcal{M})$. These functors form an inverse system

$$\cdots \longrightarrow T_{\alpha+1} \xrightarrow{\varphi_\alpha} T_\alpha \longrightarrow \cdots \longrightarrow T_2 \xrightarrow{\varphi_1} T_1 \tag{2.1}$$

indexed by all ordinals, which we call the *$\mathcal{G}$-completion tower*, in which the equality $\varphi_\alpha \circ \eta_{\alpha+1} = \eta_\alpha$ holds for all α. Moreover, by [7, Lemma 3.4], each map $(\varphi_\alpha)_X \colon T_{\alpha+1}X \to T_\alpha X$ is a fibration.

We call *T_α-equivalences* those maps f such that $T_\alpha f$ is a weak equivalence.

Lemma 2.1. *The class of T_α-equivalences is the class of $\mathcal{G}$-equivalences for all ordinals $\alpha \geq 1$.*

Proof. One implication follows from the fact that each T_α preserves weak equivalences and from the homotopy invariance of holim. The converse is shown using the same argument as in [3, Lemma 8.4]. THIS NEEDS TO BE WRITTEN DOWN PROPERLY. □

Lemma 2.2. *If X is cofibrant and $\eta_\alpha \colon X \to T_\alpha X$ has a left inverse for some ordinal α, then $\eta_{\alpha+1} \colon X \to T_{\alpha+1}X$ is a weak equivalence.*

Proof. This follows from the Collapse Lemma proved in [7, § 3.6]. CHECK THAT IT HOLDS FOR EVERY MODEL CATEGORY. □

3 Main result

Let $\mathcal{M}$ be a left proper pointed model category and let $\mathcal{G}$ be a set of group objects in $\mathrm{Ho}(\mathcal{M})$. Let $\{T_\alpha : \alpha \in \mathrm{Ord}\}$ be the corresponding $\mathcal{G}$-completion tower (2.1).

Let $\mathcal{L}$ denote the closure of $\mathcal{G}$ under homotopy limits and define an increasing sequence of classes $\mathcal{L}_\alpha$, where $\mathcal{L}_0$ is the class of $\mathcal{G}$-injective objects in $\mathcal{M}$ and $\mathcal{L}_{\alpha+1}$ is the class of objects that are homotopy limits of diagrams with values in $\mathcal{L}_\alpha$. For a limit ordinal λ, the class $\mathcal{L}_\lambda$ is the union of $\mathcal{L}_\alpha$ for $\alpha < \lambda$. Then the union of the sequence $\{\mathcal{L}_\alpha : \alpha \in \mathrm{Ord}\}$ is closed under homotopy limits, and hence it is equal to $\mathcal{L}$. Compare this sequence $\mathcal{L}_\alpha$ with the hierarchy I_α defined in [7, § 1.1].

By construction, each object of $\mathcal{L}$ has a *complexity* with respect to $\mathcal{G}$. Namely, $c(X) = 0$ if X is $\mathcal{G}$-injective, and $c(X) = \alpha$ if α is the smallest ordinal for which $X \in \mathcal{L}_\alpha$.

Theorem 3.1. *Let X be any object in $\mathcal{M}$, which we assume fibrant and cofibrant. If X is in $\mathcal{L}$ and $c(X) \leq \alpha$ for an ordinal α, then the map $\eta_{\alpha+1} \colon X \to T_{\alpha+1}X$ is a weak equivalence.*

Proof. By Lemma 1.1, $\eta_1 \colon X \to T_1 X$ is a weak equivalence if $X \in \mathcal{L}_0$. Now argue by transfinite induction as follows. Suppose that $\eta_\alpha \colon Z \to T_\alpha Z$ is a weak equivalence whenever Z is in $\mathcal{L}$ and $c(Z) < \alpha$. Then, given X in $\mathcal{L}$ with $c(X) = \alpha$, we may write $X \simeq \operatorname{holim} Z_i$ where each Z_i is in $\mathcal{L}$ and $c(Z_i) < \alpha$.

Since, by induction hypothesis, $\eta_\alpha \colon Z_i \to T_\alpha Z_i$ is a weak equivalence for all i, the induced map $X \to \operatorname{holim}(T_\alpha Z_i)$ is a weak equivalence. If $\gamma_i \colon X \to Z_i$ is the ith structure map of X, then the maps $T_\alpha \gamma_i \colon T_\alpha X \to T_\alpha Z_i$ yield together a map $T_\alpha X \to \operatorname{holim}(T_\alpha Z_i) \simeq X$ which is a homotopy left inverse of $\eta_\alpha \colon X \to T_\alpha X$.

Since X is fibrant and cofibrant, $\eta_\alpha \colon X \to T_\alpha X$ also admits a strict left inverse. This implies, by Lemma 2.2, that $\eta_{\alpha+1} \colon X \to T_{\alpha+1} X$ is a weak equivalence, as claimed. □

Our main goal in the article is to prove the following result. Recall that a full subcategory $\mathcal{L}$ of a category $\mathcal{C}$ is *reflective* if the inclusion $\mathcal{L} \hookrightarrow \mathcal{C}$ has a left adjoint $L \colon \mathcal{C} \to \mathcal{L}$. In this case, we denote by $l \colon \mathrm{Id} \to L$ the unit of the adjunction and call $l_X \colon X \to LX$ a *reflection* of X onto $\mathcal{L}$.

Theorem 3.2. *Let $\mathcal{M}$ be a left proper pointed model category and let $\mathcal{G}$ be a set of group objects in* $\mathrm{Ho}(\mathcal{M})$ *closed under suspension. Let $\{T_\alpha : \alpha \in \mathrm{Ord}\}$ be the $\mathcal{G}$-completion tower. If the closure $\mathcal{L}$ of $\mathcal{G}$ under homotopy limits is reflective in* $\mathrm{Ho}(\mathcal{M})$*, then for each cofibrant object X in $\mathcal{M}$ there is an ordinal κ such that the map $\eta_{\kappa+1} \colon X \to T_{\kappa+1} X$ is a reflection of X onto $\mathcal{L}$.*

Proof. Since $\mathcal{L}$ is reflective in $\mathrm{Ho}(\mathcal{M})$, for every object X in $\mathcal{M}$ there is a map $l_X \colon X \to LX$ where LX is in $\mathcal{L}$ and l_X is a T_1-equivalence by part (ii) of Lemma 1.1. Then l_X is a T_α-equivalence for all ordinals α, by Lemma 2.1. Since LX is in $\mathcal{L}$, there is an ordinal κ such that $c(LX) = \kappa$.

Hence $T_{\kappa+1}l_X\colon T_{\kappa+1}X \simeq T_{\kappa+1}LX$ because l_X is a $T_{\kappa+1}$-equivalence, and $\eta_{\kappa+1}\colon LX \simeq T_{\kappa+1}LX$ by Theorem 3.1. □

4 Cohomological localizations

We now specialize to the case $\mathcal{G} = \{\Sigma^n E_k \mid n \in \mathbb{Z},\, k \geq 0\}$ where $E = \{E_k\}_{k\geq 0}$ is an Ω-spectrum and $\mathcal{M}$ is the category of pointed simplicial sets. Let E^* be the reduced cohomology theory represented by E, that is, $E^k(X) \cong [X, E_k]$ for all k.

Recall that a map $f\colon X \to Y$ is an *E-equivalence* if $E^k f\colon E^k X \to E^k Y$ is an isomorphism for all k, and a space Z is *E-local* if every E^*-equivalence $f\colon X \to Y$ induces a bijection $[Y, Z] \cong [X, Z]$.

Proposition 4.1. *Let $E = \{E_k\}_{k\geq 0}$ be an Ω-spectrum and let $\mathcal{L}$ be the closure of the set $\mathcal{G} = \{\Sigma^n E_k \mid n \in \mathbb{Z},\, k \geq 0\}$ under homotopy limits. If $\mathcal{L}$ is reflective in the homotopy category, then $\mathcal{L}$ is equal to the class of E^*-local spaces.*

Proof. By definition, E_k is E^*-local for all k. Since the class of E^*-local spaces is closed under homotopy limits, it follows that $\mathcal{L}$ is contained in the class of E^*-local spaces.

Conversely, observe that the class of E^*-local spaces is the orthogonal complement, by definition, of the class of E^*-equivalences, and the latter is the orthogonal complement of $\mathcal{G}$; that is, $({}^{\perp}\mathcal{G})^{\perp}$ is equal to the class of E^*-local spaces. Since $\mathcal{G}$ is contained in $\mathcal{L}$, we infer that $({}^{\perp}\mathcal{G})^{\perp}$ is contained in $({}^{\perp}\mathcal{L})^{\perp}$. Now, since we are assuming that $\mathcal{L}$ is reflective, we have $({}^{\perp}\mathcal{L})^{\perp} = \mathcal{L}$ and the argument is complete. □

What we have shown is that, assuming that E^*-localization exists (which is true if supercompact cardinals exist, as shown in [1]), then, for each pointed simplicial set X, the homotopy type of its E^*-localization $L_E X$ can be obtained as the homotopy inverse limit of the tower $\{T_\alpha X : \alpha \in \mathrm{Ord}\}$, which eventually stabilizes at some ordinal depending on the complexity of $L_E X$, in the sense of the previous section.

5 Homological localizations of spectra

This is, as of today, speculative. For a *ring* spectrum E, take $T_1 X$ to be the Tot of the Bendersky–Thompson cosimplicial spectrum, which is probably a homotopy inverse limit of E-modules. Hence, our construction starts from the monad $X \mapsto X \wedge E$, which we could call T_0. Note that, indeed, the T_0-equivalences are the E_*-equivalences, and we hope that the T_α-equivalences will be the E_*-equivalences for all ordinals α.

It is unclear if the class $\mathcal{L}$ of E_*-local spectra is the closure under homotopy inverse limits of the class of E-modules. Since E is a ring, each E-module is E_*-local, and therefore every homotopy inverse limit of E-modules is E_*-local.

If this works, then the results in this article generalize those in [7] to every homology theory represented by a ring spectrum; see [8] for a related discussion.

CAN WE EXPAND THIS SECTION SO THAT IT ALSO HOLDS FOR SPACES?

6 Final remarks

It follows from our results that localization with respect to a *cohomology* theory E^* is the *idempotent approximation*, in the sense of [5], of the E-completion functor on the homotopy category of spaces (OR PROBABLY ALSO SPECTRA).

Some of the philosophy of the present article can already be found in Pfenniger's thesis [9].

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