Smooth Manifolds

Smooth Structures

Let M be a topological manifold of dimension $n \geq 1$. A *chart* in M is a pair (U, φ) where U is an open subset of M and φ is a homeomorphism from U into an open subset of $\mathbb{R}^n$. An *atlas* on M is a collection of charts $\{(U_i, \varphi_i)\}_{i \in I}$ such that $\cup_{i \in I} U_i = M$. An atlas $\{(U_i, \varphi_i)\}_{i \in I}$ is *smooth* if $\varphi_j \circ (\varphi_i)^{-1}$ is C^∞ on $\varphi_i(U_i \cap U_j)$ for all $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$. A *smooth structure* on M is an equivalence class of smooth atlases, where two smooth atlases are equivalent if their union is a smooth atlas. A *smooth manifold* is a topological manifold equipped with a smooth structure. The *canonical* smooth structure on $\mathbb{R}^n$ is the one given by the atlas consisting of a single chart with the identity map.

Smooth Maps

If $f\colon M \to N$ is a continuous map between smooth manifolds of respective dimensions m and n, we say that f is *smooth* if $\psi \circ f \circ \varphi^{-1}$ is C^∞ on $\varphi(U \cap f^{-1}(V))$ for each pair of charts (U, φ) in M and (V, ψ) in N such that $U \cap f^{-1}(V) \neq \emptyset$. If $f\colon M_1 \to M_2$ and $g\colon M_2 \to M_3$ are smooth maps, then $g \circ f$ is smooth. A smooth homeomorphism with smooth inverse is called a *diffeomorphism*.

Tangent Vectors

If M is a smooth manifold of dimension n, let $\mathcal{F}(M)$ denote the $\mathbb{R}$-algebra of smooth functions $M \to \mathbb{R}$, where $\mathbb{R}$ is equipped with the canonical smooth structure. A *derivation* at a point $p \in M$ (also called a *tangent vector* at p) is an $\mathbb{R}$-linear operator $X\colon \mathcal{F}(M) \to \mathbb{R}$ such that

$$X(fg) = X(f)g(p) + f(p)X(g)$$

for all smooth functions f and g. The *tangent space* of M at a point p is the $\mathbb{R}$-vector space of all derivations at p. It is denoted by T_pM. If (U, φ) is a chart in M with $p \in U$ and $\varphi = (x_1, \ldots, x_n)$, then $T_pM \cong T_pU$ and it is spanned as an $\mathbb{R}$-vector space by the derivations $(\partial/\partial x_1)_p, \ldots, (\partial/\partial x_n)_p$, where

$$\left(\frac{\partial}{\partial x_i}\right)_p (f) = \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}\,(p)$$

for each i and all $f \in \mathcal{F}(U)$. Thus T_pM has dimension n.

If $\gamma\colon (-\varepsilon, \varepsilon) \to M$ is a smooth curve with $\gamma(0) = p$, then the *tangent vector* to γ at p is the derivation $\gamma'(0)$ at p given by

$$\gamma'(0)(f) = \frac{d(f \circ \gamma)}{dt}(0)$$

for every $f \in \mathcal{F}(M)$. The collection of all tangent vectors at p to all smooth curves through a point $p \in M$ is equal to T_pM.

Tangent Bundle

The *tangent bundle* over a smooth manifold M of dimension n is the $2n$-dimensional manifold

$$TM = \{(p, v) \mid p \in M,\ v \in T_pM\}$$

together with the projection $TM \to M$ sending $(p, v) \mapsto p$. Then TM admits a smooth structure such that the projection is a smooth map. In fact TM is an n-dimensional smooth vector bundle over M. A manifold M is called *parallelizable* if its tangent vector bundle is trivial, i.e., $TM \cong M \times \mathbb{R}^n$ as vector bundles over M.

Vector Fields

A *smooth vector field* on a smooth manifold M is a smooth section $X\colon M \to TM$ of the tangent bundle over M. Thus a smooth vector field on M assigns to each point $p \in M$ a tangent vector $X_p \in T_pM$. If (U, φ) is a chart on M with $\varphi = (x_1, \ldots, x_n)$ then each smooth vector field X can be written uniquely as $X = X_1(\partial/\partial x_1) + \cdots + X_n(\partial/\partial x_n)$ where $X_1, \ldots, X_n$ are smooth functions.

Differential Forms

Let us denote by T^*M the dual of the tangent bundle. Thus T^*M is an n-dimensional vector bundle over M whose fibre at p is the dual vector space $(T_pM)^* = \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R})$. It is called the *cotangent bundle* over M.

A *smooth* 1-*form* or *differential* 1-*form* on M is a smooth section $\omega\colon M \to T^*M$ of the cotangent bundle over M. If X is a smooth vector field on M and ω is a smooth 1-form on M, then $\omega(X)$ is the smooth function given by $\omega(X)(p) = \omega_p(X_p)$ for each $p \in M$.

If $(dx_1)_p, \ldots, (dx_n)_p$ is the dual basis of $(\partial/\partial x_1)_p, \ldots, (\partial/\partial x_n)_p$ at any point p, then each smooth 1-form ω on M can be written uniquely as $\omega = \omega_1\, dx_1 + \cdots + \omega_n\, dx_n$ where $\omega_1, \ldots, \omega_n$ are smooth functions.

Differential of a Smooth Map

If $f\colon M \to N$ is a smooth map between smooth manifolds, then $df\colon TM \to TN$ is the fibrewise $\mathbb{R}$-linear map defined at each point p by $(df)_p\gamma'(0) = (f \circ \gamma)'(0)$ where γ is any smooth curve with $\gamma(0) = p$. This map $(df)_p$ is called the *differential* of f at p. If (U, φ) and (V, ψ) are charts in M and N with $p \in U$ and $f(p) \in V$, and we denote $\varphi = (x_1, \ldots, x_n)$, then the matrix of $(df)_p$ is the *Jacobian* matrix $(\partial(\psi \circ f \circ \varphi^{-1})_i/\partial x_j)$ at $\varphi(p)$. As a special case, the differential of a smooth function $f\colon M \to \mathbb{R}$ is the 1-form given by $df = (\partial(f \circ \varphi^{-1})/\partial x_1)dx_1 + \cdots + (\partial(f \circ \varphi^{-1})/\partial x_n)dx_n$ in each chart (U, φ) of M.

A smooth map $f\colon M \to N$ is an *immersion* if $(df)_p$ is injective for all $p \in M$. It is a *submersion* if $(df)_p$ is surjective for all $p \in M$. It is an *embedding* if it is an injective immersion inducing a homeomorphism $M \cong f(M)$. A *smooth submanifold* of N is the image of an embedding $f\colon M \to N$.

The *Regular Value Theorem* states that, if a map $f\colon M \to N$ is given and $q \in N$ is a point such that $f^{-1}(q) \neq \emptyset$ and $(df)_p$ is surjective for each $p \in f^{-1}(q)$, then $f^{-1}(q)$ is a smooth submanifold of M of dimension $\dim M - \dim N$. A point $q \in N$ such that $f^{-1}(q) \neq \emptyset$ and $(df)_p$ is surjective for each $p \in f^{-1}(q)$ is called a *regular value* of f. A point $p \in M$ for which $(df)_p$ is not surjective is called a *singular point* of f.

Exercises

61. Let $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid (x_1)^2 + \cdots + (x_{n+1})^2 = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$. Consider the open subsets $U_1 = S^n \smallsetminus \{(0, \dots, 0, 1)\}$, $U_2 = S^n \smallsetminus \{(0, \dots, 0, -1)\}$, and the stereographic projections $\phi_i \colon U_i \to \mathbb{R}^n$ given, respectively, by

$$\phi_1(x_1, \dots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \dots, x_n),$$

$$\phi_2(x_1, \dots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \dots, x_n).$$

Prove that the charts (U_1, ϕ_1) and (U_1, ϕ_2) form a smooth atlas on S^n.

62. Let $\mathbb{R}P^n$ denote the real projective space of dimension n, defined as the quotient of $\mathbb{R}^{n+1} \smallsetminus \{(0, \dots, 0)\}$ by the relation $(x_0, \dots, x_n) \sim \lambda(x_0, \dots, x_n)$ for $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Denote by $[x_0 : \cdots : x_n]$ the equivalence class of $(x_0, \dots, x_n)$. For $i \in \{0, \dots, n\}$, consider the open subsets $U_i = \{[x_0 : \cdots : x_n] \mid x_i \neq 0\}$ and the maps $\phi_i \colon U_i \to \mathbb{R}^n$ defined as

$$\phi_i([x_0 : \cdots : x_n]) = (x_0/x_i, \dots, x_{i-1}/x_i,\ x_{i+1}/x_i, \dots, x_n/x_i).$$

Prove that $\{(U_i, \phi_i)\}_{0 \le i \le n}$ is a smooth atlas on $\mathbb{R}P^n$.

63. Prove that, if M and N are smooth manifolds, then $M \times N$ admits a smooth structure such that the projections on each factor are smooth maps. Use this fact to endow $S^1 \times \cdots \times S^1$ with a smooth structure for every number of factors.

64. Prove that the map $F \colon S^2 \to \mathbb{R}^4$ defined by $F(x, y, z) = (x^2 - y^2,\ xy,\ xz,\ yz)$ yields a smooth embedding of the real projective plane $\mathbb{R}P^2$ into $\mathbb{R}^4$.

65. Prove that $M = \{(x, y, z, t) \in \mathbb{R}^4 \mid x^4 + y^4 + z^4 + t^4 = 1\}$ is a smooth submanifold of $\mathbb{R}^4$ and find a basis of the tangent space $T_pM \subset T_p(\mathbb{R}^4)$ for $p = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

66. Prove that $V = \{(x, y, z, t) \in \mathbb{R}^4 \mid x^3 + y^3 + z^3 + t^3 = 0,\ x^2 + y^2 + z^2 + t^2 = 1\}$ is a smooth submanifold of $\mathbb{R}^4$ and find a basis of the tangent space of V at each of the points $p = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\right)$ and $q = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$.

67. Prove that the vector fields in $\mathbb{R}^4$ defined as

$$X = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} - t\,\frac{\partial}{\partial z} + z\,\frac{\partial}{\partial t}$$

$$Y = -z\,\frac{\partial}{\partial x} + t\,\frac{\partial}{\partial y} + x\,\frac{\partial}{\partial z} - y\,\frac{\partial}{\partial t}$$

$$Z = -t\,\frac{\partial}{\partial x} - z\,\frac{\partial}{\partial y} + y\,\frac{\partial}{\partial z} + x\,\frac{\partial}{\partial t}$$

are tangent to the unit sphere S^3 and X_p, Y_p, Z_p are linearly independent for each $p \in S^3$. Hence S^3 is parallelizable.