Classical Operators

Curl, Divergence, Gradient

For a smooth vector field $F = (F_1, F_2, F_3)$ on $\mathbb{R}^3$, the *rotational* or *curl* of F is the vector field defined as

$$\operatorname{curl} F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right),$$

and the *divergence* of F is the function

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

The *gradient* of a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$ is the vector field

$$\operatorname{grad} f = \nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

Exercise

83. Prove that

$$\boxed{\operatorname{curl} \circ \operatorname{grad} = 0} \qquad \text{and} \qquad \boxed{\operatorname{div} \circ \operatorname{curl} = 0}$$

in $\mathbb{R}^3$, and show that these two equations correspond to the fact that the de Rham differential d satisfies $d \circ d = 0$, through the correspondence between 1-forms and vector fields given by $f\,dx + g\,dy + h\,dz \leftrightarrow (f, g, h)$, together with the correspondence between k-forms and $(3-k)$-forms given by *Hodge duality*:

$$f\,dx + g\,dy + h\,dz \longleftrightarrow f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy$$

$$f \longleftrightarrow f\,dx \wedge dy \wedge dz$$

Divergence Theorem

For a compact subset K whose boundary ∂K is a smooth surface in $\mathbb{R}^3$, Stokes' Theorem specializes to the *Divergence Theorem* for each smooth vector field F:

$$\boxed{\int_K \operatorname{div} F\, dV = \int_{\partial K} F \cdot N\, dA}$$

Here the 3-form $dV = dx \wedge dy \wedge dz$ on $\mathbb{R}^3$ is called *volume element* and $\int_K dV$ is the Euclidean volume of K. In the right-hand integral, N denotes the outward unit normal vector field on ∂K and dA is called *surface area element*, which is determined by the expression

$$F \cdot N\, dA = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy.$$

Kelvin–Stokes Theorem

Suppose now that M is a compact oriented smooth surface with boundary in $\mathbb{R}^3$ and let N be the outward unit normal vector field on M. Then, for every smooth vector field F on an open neighbourhood of M, Stokes' Theorem yields

$$\boxed{\int_M \operatorname{curl} F \cdot N \, dA = \int_{\partial M} F \cdot T \, dL}$$

where $F \cdot T \, dL = F_1 \, dx + F_2 \, dy + F_3 \, dz$. Here T denotes the unit tangent vector field along ∂M and dL is called *arc length element.*

Green Theorem

As a special case of the Kelvin–Stokes Theorem, one obtains *Green's Theorem*: If S is a compact subset with smooth boundary in $\mathbb{R}^2$, then, for every smooth vector field $F = (F_1, F_2)$ on S,

$$\int_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx \wedge dy = \int_{\partial S} F_1 \, dx + F_2 \, dy.$$

Exercises

84. Compute $\int_\gamma y^3 dx - x^3 dy$ where γ is the positively oriented circle of radius 2 centered at the origin in $\mathbb{R}^2$.

85. Let $T^2 = S^1 \times S^1 \subset \mathbb{R}^4$ denote the 2-torus, defined by $w^2 + x^2 = y^2 + z^2 = 1$. Compute
$$\int_{T^2} xyz \, dw \wedge dy.$$

86. A cycloid arc is given by $x = a(t - \sin t)$ and $y = a(1 - \cos t)$ where $t \in [0, 2\pi]$. Find the area between the arc and the x-axis. (*Hint:* Use Green's Theorem with the vector field $F = (-y, x)$.)

87. Let S be the surface obtained by rotating the curve in the xz-plane given by $x = \cos u$, $z = \sin 2u$, $-\pi/2 \le u \le \pi/2$ around the z axis. Compute the volume of the region in $\mathbb{R}^3$ bounded by S. (*Hint:* Apply the Divergence Theorem to a suitable vector field.)