Simplicial Homology

Abstract Simplicial Complexes

An *abstract simplicial complex* is a collection K of finite subsets of a set V such that $\{v\} \in K$ for all $v \in V$, and $T \in K$ whenever $T \subset S$ with $S \in K$. The elements of V are called *vertices* of K and the elements of K are called *faces*. A face is *maximal* if it is not properly contained in any other face. Thus each abstract simplicial complex is uniquely determined by the list of its maximal faces. A face is *n-dimensional* or an *n-face* if it has cardinality $n+1$. An abstract simplicial complex is *ordered* if the set V is totally ordered. We usually choose $V = \{1, 2, 3, \dots\}$ and for shortness we denote an n-face by $v_1 \cdots v_{n+1}$ or $(v_1 \cdots v_{n+1})$ instead of $\{v_1, \dots, v_{n+1}\}$.

Geometric Realization

We denote by Δ^n the convex hull of the points $e_i = (0, \dots, 1, \dots, 0)$ in $\mathbb{R}^{n+1}$, where 1 appears in the i-th place for each $i = 1, \dots, n+1$. This topological space Δ^n is called *standard n-simplex*.

The *geometric realization* of an abstract simplicial complex K is the topological space $|K|$ obtained by picking a copy of Δ^n for each maximal face $\{v_1, \dots, v_{n+1}\}$ of K together with a bijection between $\{v_1, \dots, v_{n+1}\}$ and the vertices of Δ^n, and identifying each pair of faces of simplices that correspond to the same subset of V. The resulting topological space is called a *polyhedron* or a *geometric simplicial complex*. A *triangulation* of a topological space X is a homeomorphism $|K| \to X$ where K is an abstract simplicial complex.

Simplicial Chain Complexes

Every ordered abstract simplicial complex K determines a chain complex $C_*(K)$ by defining $C_n(K)$, for each n, as the free abelian group on the set of n-faces of K, and $\partial_n \colon C_n(K) \to C_{n-1}(K)$ as the group homomorphism given by

$$\partial_n(\{v_1, \dots, v_{n+1}\}) = \sum_{i=1}^{n+1} (-1)^{i-1} \{v_1, \dots, \hat{v}_i, \dots, v_{n+1}\}, \tag{1}$$

assuming that $v_1 < v_2 < \cdots < v_{n+1}$, where $\hat{v}_i$ means that v_i is missing. The *homology groups* of K are then defined as the homology groups of the chain complex $C_*(K)$:

$$H_n(K) = H_n(C_*(K)) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}. \tag{2}$$

We will call ∂_n the *n-th boundary operator* of $C_*(K)$. Elements in $\operatorname{Ker} \partial_n$ will be called *n-cycles* and elements in $\operatorname{Im} \partial_{n+1}$ will be called *n-boundaries*.

More generally, for each commutative ring R with 1, we define $C_n(K; R)$ as the free R-module on the set of n-faces of K, with boundary operators defined as in (1) for all n. Then the R-modules $H_n(K; R) = H_n(C_*(K; R))$ are called *homology of K with coefficients in R*.

Exercises

7. Prove that $\partial_n \circ \partial_{n+1} = 0$ for all n in the chain complex $C_*(K)$ of any ordered abstract simplicial complex K.

8. Compute the homology groups of the abstract simplicial complexes determined by the following lists of maximal faces:

   K: 12, 13, 14, 23, 24, 34.

   L: 123, 124, 134, 234.

   M: 1234.

   N: 123, 124, 134, 234, 145, 146, 156, 456.

   S: 123, 124, 134, 234, 15, 26, 37, 48.

9. Let X be the abstract simplicial complex determined by the following list of maximal faces:

   124, 125, 135, 136, 146, 234, 236, 256, 345, 456.

   a) Prove that the geometric realization of X is homeomorphic to the real projective plane.

   b) Compute the homology groups of X with coefficients in $\mathbb{Z}$, $\mathbb{Z}/2$ and $\mathbb{Q}$.

10. Prove that, if an abstract simplicial complex K has finitely many vertices, then $H_n(K)$ is a finitely generated abelian group for each n.

11. Let K be an abstract simplicial complex with finitely many vertices. For each $n \geq 0$, denote by α_n the number of n-faces of K.

    a) Prove that
    $$\sum_{n=0}^{\infty} (-1)^n \, \alpha_n = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} H_n(K).$$

    b) Prove that, if R is any field, then
    $$\sum_{n=0}^{\infty} (-1)^n \, \alpha_n = \sum_{n=0}^{\infty} (-1)^n \, \dim_R H_n(K; R).$$