Homology of Cell Complexes

Cell Complexes

Let us denote by $E^n$ the closed $n$-ball, whose boundary is the sphere $S^{n-1}$. A cell complex (or CW-complex) is a topological space $X$ equipped with a filtration by subspaces

$$X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \cdots \subseteq X^{(n)} \subseteq \cdots$$

where $X^{(0)}$ is discrete (i.e., has the discrete topology), such that $X = \cup_{n=0}^\infty X^{(n)}$ as a topological space, and, for all $n$, the space $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching a set of $n$-cells by means of a family of maps $\{\varphi_j: S^{n-1} \to X^{(n-1)}\}_{j \in J_n}$. Here $J_n$ is any set of indices and $X^{(n)}$ is therefore a quotient of the disjoint union $\coprod_{j \in J_n} E^n$ where we identify, for each point $x \in S^{n-1}$, the image $\varphi_j(x) \in X^{(n-1)}$ with the point $x$ in the $j$-th copy of $E^n$. It is customary to denote

$$X^{(n)} = X^{(n-1)} \cup_{\{\varphi_j\}} \{e^n_j\}$$

and view each $e^n_j$ as an “open $n$-cell”. The space $X^{(n)}$ is called the $n$-skeleton of $X$, and the filtration (1) together with the attaching maps $\{\varphi_j\}_{j \in J_n}$ for all $n$ is called a cell decomposition or CW-decomposition of $X$. Thus a topological space may admit many distinct cell decompositions (or none).

Cellular Chain Complexes

Let $X$ be a cell complex with $n$-skeleton $X^{(n)}$ and attaching maps $\{\varphi_j\}_{j \in J_n}$ for all $n$. We define

$$C_n(X) = H_n(X^{(n)}, X^{(n-1)})$$

for each $n$. Note that, since $X^{(n-1)}$ is a closed subspace of $X^{(n)}$ which is a strong deformation retract of an open neighbourhood, we have

$$C_n(X) \cong H_n(X^{(n)}/X^{(n-1)})$$

for all $n$. Moreover, $X^{(n)}/X^{(n-1)} \cong \vee_{j \in J_n} S^n$. Therefore $C_n(X)$ is a free abelian group with a free generator for each $j \in J_n$, or, in other words, for each $n$-cell of $X$. Thus we may write

$$C_n(X) \cong \oplus_{j \in J_n} \mathbb{Z} e^n_j.$$

We next show that $C_*(X)$ is in fact a chain complex whose homology groups are isomorphic to the homology groups of $X$. For this, we need to define a boundary operator $\partial_n: C_n(X) \to C_{n-1}(X)$ for each $n$. We pick $\partial_n = p_{n-1} \circ \Delta_n$, where

$$\Delta_n: H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)})$$

is the connecting homomorphism from the homology long exact sequence of the pair $(X^{(n)}, X^{(n-1)})$, and

$$p_{n-1}: H_{n-1}(X^{(n-1)}) \to H_{n-1}(X^{(n-1)}, X^{(n-2)})$$
comes from the homology long exact sequence of the pair \((X^{(n-1)}, X^{(n-2)})\). Hence,

\[
\partial_{n-1} \circ \partial_n = p_{n-2} \circ \Delta_{n-1} \circ p_{n-1} \circ \Delta_n = 0,
\]
since \(\Delta_{n-1} \circ p_{n-1} = 0\), as they are consecutive arrows in the same exact sequence.

Furthermore, observe that \(p_{n-1}\) is injective because \(H_{n-1}(X^{(n-2)}) = 0\), as one shows inductively. This fact yields

\[
H_n(C_n(X)) = \ker \partial_n / \operatorname{im} \partial_{n+1} = \ker \Delta_n / \operatorname{im} \partial_{n+1}
\]

\[
= \operatorname{im} p_n / \operatorname{im} \partial_{n+1} \cong H_n(X) / \operatorname{im} \Delta_{n+1} \cong H_n(X) / \ker i_n
\]

\[
\cong \operatorname{im} i_n \cong H_n(X^{(n+1)}) \cong H_n(X),
\]

where \(i_n : H_n(\partial_1 X) \to H_n(X^{(n+1)})\) is induced by the inclusion. Note that \(i_n\) is surjective since \(H_n(X^{(n+1)}, X^{(n)}) = 0\), and the isomorphism \(H_n(X^{(n+1)}) \cong H_n(X)\) is shown by applying the Mayer–Vietoris exact sequence to the \(k\)-cells with \(k > n+1\).

**Incidence Numbers**

Let us obtain a more useful description of the boundary operators \(\partial_n\) of the cellular chain complex. For a cell complex \(X\) with \(n\)-skeleton \(X^{(n)}\) and attaching maps \(\{\varphi_j\}_{j \in J_n}\), we may view \(\partial_n\) as a group homomorphism between free abelian groups:

\[
\partial_n : \bigoplus_{j \in J_n} \mathbb{Z}e_j^n \to \bigoplus_{i \in J_{n-1}} \mathbb{Z}e_i^{n-1}.
\]

Hence \(\partial_n\) is fully determined by an array of integers, which we call *incidence numbers* and denote as follows:

\[
\partial_n e_j^n = \sum_{i \in J_{n-1}} [e_i^{n-1} : e_j^n] e_i^{n-1}.
\]

Observe that \([e_i^{n-1} : e_j^n]\) is the degree of the map \(\phi_j : S^{n-1} \to S^{n-1}\) obtained by composing \(\varphi_j : S^{n-1} \to X^{(n-1)}\) with the collapse \(X^{(n-1)} \to X^{(n-1)}/(X^{(n-1)} \smallsetminus e_i^{n-1})\) and with the canonical homeomorphism of the latter with \(S^{n-1}\).

**Simplicial Homology**

Suppose that \(|K|\) is the geometric realization of an abstract simplicial complex \(K\). Then \(|K|\) admits a cell decomposition where the \(n\)-cells correspond to the \(n\)-faces of \(K\), and the cellular chain complex \(C_\ast(|K|)\) is isomorphic to the simplicial chain complex \(C_\ast(K)\). This shows that

\[
H_n(|K|) \cong H_n(K)
\]

for all \(n\), where the group on the left means singular homology while the group on the right means simplicial homology.

**Exercises**

26. Prove that, for every abstract simplicial complex \(K\), the boundary operator of the cellular chain complex \(C_\ast(|K|)\) coincides with the one of the simplicial chain complex \(C_\ast(K)\).

27. Find the homology groups of the space obtained by attaching an \((n+1)\)-cell to a sphere \(S^n\) by means of a map \(f : S^n \to S^n\) of degree \(k \geq 2\).

28. Prove that for every family \(\{A_n\}_{n \geq 1}\) of finitely generated abelian groups there exists a connected space \(X\) such that \(H_n(X) \cong A_n\) for all \(n\).