Cellularization of structures in triangulated categories

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Cellularization functors were introduced by Farjoun in 1996 in the category of topological spaces.

- Given $A$ and $X$ two pointed topological spaces, $\text{Cell}_A X$ contains the information on $X$ that can be built up from $A$.
- $X$ is called $A$-cellular if $\text{Cell}_A X \simeq X$ and it is the smallest class that contains $A$ and it is closed under weak equivalences and homotopy colimits.
- $f : X \rightarrow Y$ is an $A$-cellular equivalence if

$$f_* : \text{map}_*(A, X) \rightarrow \text{map}_*(A, Y)$$

is a weak equivalence.
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Examples: $n$-connective covers, universal covers.

Cellularization for groups and modules has been studied by Farjoun-Göbel-Segev-Shelah and Rodríguez-Strüngmann.

Objectives

- Describe the formal properties of cellularization functors in triangulated categories.
- Study the algebraic structures preserved by these functors.
Introduction

Precedents

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Cellular and null objects

Let \((\mathcal{T}, \Sigma, [-,-])\) be a triangulated category with arbitrary coproducts and a set of generators.

**Definition**

Let \(A\) be any object of \(\mathcal{T}\).

i) A map \(f: X \rightarrow Y\) in \(\mathcal{T}\) is an \textit{A-cellular equivalence} if the induced map

\[
[\Sigma^k A, X] \xrightarrow{g_*} [\Sigma^k A, Y]
\]

is an isomorphism of abelian groups for all \(k \geq 0\).

ii) An object \(Z\) of \(\mathcal{T}\) is \textit{A-cellular} if the induced map

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is an isomorphism for every \(A\)-cellular equivalence \(f: X \rightarrow Y\) and for all \(k \geq 0\).
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Cellular and null objects

**Definition**

Let $A$ be any object of $\mathcal{T}$.

i) An object $X$ is **$A$-null** if $[\Sigma^k A, X] = 0$ for every $k \geq 0$.

ii) A map $g : X \to Y$ is an **$A$-null equivalence** if the induced map

$$[\Sigma^k Y, Z] \cong [\Sigma^k X, Z]$$

is an isomorphism of abelian groups for $k \geq 0$.

- An **$A$-cellularization functor** is a colocalization functor $(Cell_A, c)$ such that for every object $X$ of $\mathcal{T}$, the map $c_X : Cell_A X \to X$ is an $A$-cellular equivalence and $Cell_A X$ is $A$-cellular.

- An **$A$-nullification functor** is a localization functor $(P_A, l)$ such that for every object $X$ of $\mathcal{T}$, the map $l_X : X \to P_A X$ is an $A$-null equivalence and $P_A X$ is $A$-null.
Cellularization and nullification functors

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Cellular and null objects

- We say that $P_A$ or $Cell_A$ are \textit{exact} if they are triangulated functors.

Existence

- Assume that there is a stable model category $\mathcal{M}$ such that $\mathcal{T} = Ho(\mathcal{M})$. Cellularization and nullification functors always exist if $\mathcal{M}$ is a proper combinatorial model category.

- Examples to keep in mind: Spectra, $\mathcal{D}(R)$, $E$-local spectra, $\mathcal{D}(shv/X)$, ...
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Cellular and null objects

Closure properties

Let $X \longrightarrow Y \longrightarrow Z$ be an exact triangle in $\mathcal{T}$

i) If $Y$ and $Z$ are $A$-null then $X$ is $A$-null.

ii) If $X$ and $Z$ are $A$-null then $Y$ is $A$-null.

iii) If $X$ and $Y$ are $A$-cellular then $Z$ is $A$-cellular.

iv) If $X$ and $Z$ are $A$-cellular then $Y$ is not $A$-cellular in general.

v) The class of $A$-null objects and the class of $A$-cellular equivalences are closed under desuspensions.

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vii) If $P_A$ and $Cell_A$ are exact the above classes are closed under suspensions and desuspensions.

Colocalization functors satisfying the analog of condition ii) are called quasiexact.
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Exact triangles

There are natural maps

\[ \text{Cell}_A X \longrightarrow X \longrightarrow P_A X. \]

This is \textit{not} an exact triangle in general.

**Theorem**

Let $A$ and $X$ be two objects of $\mathcal{T}$.

i) There is an exact triangle $\text{Cell}_A X \longrightarrow X \longrightarrow P_A X$ if and only if the morphism of abelian groups $[\Sigma^{-1} A, \text{Cell}_A X] \longrightarrow [\Sigma^{-1} A, X]$ is injective (e.g. if $[\Sigma^{-1} A, \text{Cell}_A X] = 0$).

ii) There is an exact triangle $\text{Cell}_A X \longrightarrow X \longrightarrow P_{\Sigma A} X$ if and only if $[A, X] \longrightarrow [A, P_{\Sigma A} X]$ is the zero map (e.g. if $[A, X] = 0$).

iii) If $\text{Cell}_A$ or $P_A$ are exact, then $\text{Cell}_A X \longrightarrow X \longrightarrow P_A X$ is an exact triangle.
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Not every nullification and cellularization functor fitting into an exact triangle are exact.

If $\mathcal{T}$ is the stable homotopy category of spectra and $S$ is the sphere spectrum, then we have an exact triangle

$$\text{Cell}_S X \longrightarrow X \longrightarrow P_S X$$

for every $X$, but neither $\text{Cell}_S$ nor $P_S$ are exact.
Exact triangles

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Colocalizations associated to nullifications

Let $F_A X$ be the fiber of the map $X \rightarrow P_A X$

$$F_A X \rightarrow X \rightarrow P_A X$$

The universal property of $P_A$ and the fact that $P_A$ is quasiexact make $F_A$ a colocalization functor (augmented and idempotent).

Moreover

- $F_A$ is quasiexact
- $F_A$-colocal objects are closed under suspensions
- $[F_A X, P_A Y] = 0$ for every $X$ and $Y$ in $\mathcal{T}$.

Under Vopěnka’s principle $F_A = \text{Cell}_E$ for some $E$ [Chorny, 2008]. A construction of $E$ in pointed spaces is possible not relying on Vopěnka’s principle [Chacholski-Parent-Stanley, 2004].
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**Definition**

A *t-structure* on $\mathcal{T}$ is a pair of full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ such that, denoting $\mathcal{T}^{\leq n} = \Sigma^{-n}\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq n} = \Sigma^{-n}\mathcal{T}^{\geq 0}$, the following hold:

i) For every object $X$ of $\mathcal{T}^{\leq 0}$ and every object $Y$ of $\mathcal{T}^{\geq 1}$, $[X, Y] = 0$.

ii) $\mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 1}$ and $\mathcal{T}^{\geq 1} \subset \mathcal{T}^{\geq 0}$.

iii) For every object $X$ of $\mathcal{T}$, there is an exact triangle

$$U \rightarrow X \rightarrow V,$$

where $U$ is an object of $\mathcal{T}^{\leq 0}$ and $V$ is an object of $\mathcal{T}^{\geq 1}$.

The *core* of the *t*-structure is the full subcategory $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$. The core is always an abelian subcategory of $\mathcal{T}$. 

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A **$t$-structure** on $\mathcal{F}$ is a pair of full subcategories $(\mathcal{F}^{\leq 0}, \mathcal{F}^{\geq 0})$ such that, denoting $\mathcal{F}^{\leq n} = \Sigma^{-n}\mathcal{F}^{\leq 0}$ and $\mathcal{F}^{\geq n} = \Sigma^{-n}\mathcal{F}^{\geq 0}$, the following hold:

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The **core** of the $t$-structure is the full subcategory $\mathcal{F}^{\leq 0} \cap \mathcal{F}^{\geq 0}$. The core is always an abelian subcategory of $\mathcal{F}$. 
Theorem

For any object $A$ in $\mathcal{T}$ the full subcategory of $\Sigma A$-null objects and the full subcategory of $F_A$-colocal objects define a $t$-structure on $\mathcal{T}$.

- If $Cell_A$ and $P_A$ fit into an exact triangle, then the $t$-structure is given by the $A$-cellular objects and the $\Sigma A$-null objects.

- If $Cell_A$ and $P_A$ are exact, then the associated $t$-structure is trivial.
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- If \( \text{Cell}_A \) and \( P_A \) are exact, then the associated \( t \)-structure is trivial.
Example

Let $\mathcal{T}$ be a monogenic stable homotopy category with unit $S$, such that $[\Sigma^k S, S] = 0$ for every $k < 0$. Let $R$ denote the ring $[S, S]$. Then the functors $Cell_{\Sigma^k S}$ and $P_{\Sigma^k S}$ are the $k$-th connective cover functor and the $k$-th Postnikov section functor respectively:

$$[\Sigma^n S, Cell_{\Sigma^k S} X] = \begin{cases} 0 & \text{if } n < k \\ [\Sigma^n S, X] & \text{if } n \geq k \end{cases}$$

$$[\Sigma^n S, P_{\Sigma^k S} X] = \begin{cases} 0 & \text{if } n \geq k \\ [\Sigma^n S, X] & \text{if } n < k \end{cases}$$

We have an exact triangle

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Example

Let $\mathcal{T}$ be a monogenic stable homotopy category with unit $S$, such that $[\Sigma^k S, S] = 0$ for every $k < 0$. Let $R$ denote the ring $[S, S]$. Then the functors $\text{Cell}_{\Sigma^k S}$ and $P_{\Sigma^k S}$ are the $k$-th connective cover functor and the $k$-th Postnikov section functor respectively:

$$[\Sigma^n S, \text{Cell}_{\Sigma^k S} X] = \begin{cases} 0 & \text{if } n < k \\ [\Sigma^n S, X] & \text{if } n \geq k \end{cases}$$

$$[\Sigma^n S, P_{\Sigma^k S} X] = \begin{cases} 0 & \text{if } n \geq k \\ [\Sigma^n S, X] & \text{if } n < k \end{cases}$$

We have an exact triangle

$$\text{Cell}_{\Sigma^k S} X \to X \to P_{\Sigma^k S} X.$$
Example

The core of the associated $t$-structure is the full subcategory of $\mathcal{T}$ with objects $X$ such that such that $[\Sigma^n S, X] = 0$ if $n \neq k$ and it is equivalent to the category of $R$-modules. The objects in the core are called \textit{Eilenberg-Mac Lane objects}.

Note that $\text{Cell}_{\Sigma^k S}$ is not an exact functor. For example, if $[\Sigma^{k-1} S, X] \neq 0$, then $\text{Cell}_{\Sigma^k S} \Sigma X \neq \Sigma \text{Cell}_{\Sigma^k S} X$. 
Example

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Let $\mathcal{T}$ be a monoidal triangulated category with tensor product $\otimes$, unit $S$ and internal hom $F(\cdot, \cdot)$, such that

- $\mathcal{T}$ is monogenic.
- $\mathcal{T}$ is connective, i.e., $[\Sigma^k S, S] = 0$ for $k < 0$.

An object $X$ is called connective if $\text{Cell}_S X \cong X$ and if $X$ is connective, then

$$[X, \text{Cell}_S F(Y, Z)] \cong [X, F(Y, Z)] \cong [X \otimes Y, Z].$$

A ring $R$ in $\mathcal{T}$ is a monoid object and an $R$-module in $\mathcal{T}$ is a monoid over the monoid $R$. 
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An object $X$ is called connective if $Cell_S X \simeq X$ and if $X$ is connective, then

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A ring $R$ in $\mathcal{T}$ is a monoid object and an $R$-module in $\mathcal{T}$ is a monoid over the monoid $R$. 

Theorem

If $E$ is a connective ring object and $M$ is an $E$-module, then for any object $A$, the object $\text{Cell}_A M$ has an $E$-module structure such that the cellularization map $\text{Cell}_A M \to M$ is a map of $E$-modules. If $\text{Cell}_A$ is exact, we can avoid the connectivity condition.

The case for rings is more involved. If $R$ is a ring, then $\text{Cell}_A R$ will not be a ring in general, even if $\text{Cell}_A$ is exact!
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Cellularization of modules and rings

Let $C$ be the cofiber of $\text{Cell}_A E \rightarrow E$

$$\text{Cell}_A E \rightarrow E \rightarrow C$$

**Theorem**

Let $E$ be a ring object. Assume that either one of the following holds:

i) $\text{Cell}_A$ commutes with suspension, the morphism $\pi_1(E) \rightarrow \pi_1(C)$ is surjective and the morphism $\pi_0(C) \rightarrow \pi_{-1}(\text{Cell}_A E)$ is injective or

ii) $\text{Cell}_A E$ is connective, $\text{Cell}_A$ is of the form $F_B$ for some $B$, the morphism $\pi_1(E) \rightarrow \pi_1(P_B E)$ is surjective and $\pi_0(P_B E) = 0$.

Then $\text{Cell}_A E$ has a unique ring structure such that the cellularization map is a map of rings.
Let $C$ be the cofiber of $\text{Cell}_AE \to E$

\[ \text{Cell}_AE \to E \to C \]

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ii) $\text{Cell}_AE$ is connective, $\text{Cell}_A$ is of the form $F_B$ for some $B$, the morphism $\pi_1(E) \to \pi_1(P_BE)$ is surjective and $\pi_0(P_BE) = 0$.

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Cellularization of modules and rings

Let $C$ be the cofiber of $Cell_A E \to E$

\[ Cell_A E \to E \to C \]

**Theorem**

Let $E$ be a ring object. Assume that either one of the following holds:

1. $Cell_A$ commutes with suspension, the morphism $\pi_1(E) \to \pi_1(C)$ is surjective and the morphism $\pi_0(C) \to \pi_{-1}(Cell_A E)$ is injective or

2. $Cell_A E$ is connective, $Cell_A$ is of the form $F_B$ for some $B$, the morphism $\pi_1(E) \to \pi_1(P_B E)$ is surjective and $\pi_0(P_B E) = 0$.

Then $Cell_A E$ has a unique ring structure such that the cellularization map is a map of rings.
Let \( A = S \), then \( \text{Cell}_A E \) is the connective cover of \( E \). There is an exact triangle

\[
\text{Cell}_S E \rightarrow E \rightarrow P_S E
\]

where \( P_S \) is the Postnikov section functor, i.e., it kills all the homotopy groups in dimensions bigger or equal to zero. So \( \pi_1 P_S E = \pi_0 P_S E = 0 \) and by part ii) of the previous theorem we have that if \( E \) is a ring object, then so is its connective cover \( \text{Cell}_S E \).
Some computations

How to compute $\text{Cell}_A \Sigma^k H G$ for any abelian group $G$.

**Theorem**

Let $G$ be any abelian group, $n \in \mathbb{Z}$ and $A$ be any object in $\mathcal{T}$. Then

$$\text{Cell}_A \Sigma^n H G \simeq \Sigma^{n-1} H B \vee \Sigma^n H C$$

for some abelian groups $B$ and $C$. Moreover

i) $\text{Hom}(B, B) \oplus \text{Ext}(B, C) \cong \text{Ext}(B, G)$.

ii) $\text{Hom}(C, C) \cong \text{Hom}(C, G)$.

iii) $\text{Hom}(B, C) \cong \text{Hom}(B, G)$.

If $G$ is divisible, then $\text{Cell}_A \Sigma^n H G$ is either zero or $\Sigma^n H C$ for some abelian group $C$. 
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How to compute $\text{Cell}_A \Sigma^k HG$ for any abelian group $G$.

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Some computations

Example

For every object $A$ in $\mathcal{T}$ and any integer $m$, we have that

$$Cell_A \Sigma^m H\mathbb{Z}/p^n \simeq \Sigma^m H\mathbb{Z}/p^j,$$

where $1 \leq j \leq n$.

If $A = \Sigma^m H\mathbb{Z}/p^k$, then

$$Cell_A \Sigma^m H\mathbb{Z}/p^n \simeq \begin{cases} \Sigma^m H\mathbb{Z}/p^k & \text{if } n \geq k \\ \Sigma^m H\mathbb{Z}/p^n & \text{if } n < k. \end{cases}$$

This shows that $Cell_{H\mathbb{Z}/p}$ is not quasiexact, since $H\mathbb{Z}/p$ is $A$-cellular but $H\mathbb{Z}/p^2$ is not.
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If $A = \Sigma^m \mathbb{H}\mathbb{Z}/p^k$, then

$$Cell_A \Sigma^m \mathbb{H}\mathbb{Z}/p^n \simeq \prod \begin{cases} \Sigma^m \mathbb{H}\mathbb{Z}/p^k & \text{if } n \geq k \\ \Sigma^m \mathbb{H}\mathbb{Z}/p^n & \text{if } n < k. \end{cases}$$

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For every object $A$ in $\mathcal{C}$ and any integer $m$, we have that

$$Cell_A \Sigma^m H\mathbb{Z}/p^n \simeq \Sigma^m H\mathbb{Z}/p^j,$$

where $1 \leq j \leq n$.

If $A = \Sigma^m H\mathbb{Z}/p^k$, then

$$Cell_A \Sigma^m H\mathbb{Z}/p^n \simeq \begin{cases} 
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Some computations

Let $\mathcal{I}$ be the homotopy category of spectra. Let $E$ be any spectrum and let $L_E$ be homological localization with respect to $E$. Bousfield proved that there is another spectrum $A$, such that $L_E X \simeq P_A X$ for every $X$. Since $L_E$ commutes with suspension there is an exact triangle

$$Cell_A X \rightarrow X \rightarrow P_A X,$$

where $Cell_A$ is the $E$-acyclization functor (in Bousfield language).

Example

The cellularization $Cell_A H\mathbb{Z}$ is either zero or one of the following three possibilities

$$H\mathbb{Z}, \quad \Sigma^{-1} H(\oplus_{p \in P} \mathbb{Z}/p^\infty), \quad \Sigma^{-1} H((\prod_{p \in P} \hat{\mathbb{Z}}_p)/\mathbb{Z}).$$

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