The symmetric product $Sym(X;A)$
and the Dold-Thom theorem

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This poster is strictly expository, and the author takes no credit for the mathematics being presented.

Notation

Throughout this poster:

- X- (based) topological space
- A- abelian group
- X_+ - disjoint union of X with basept $+$
- X^+ - one point compactification of X
- I- finite set
- X^{I_+}- based maps from I_+ to X
- A^I- all maps from I to A

Introduction

The first goal of this poster is to give a definition for the notion of a "free A-module on a space, X", called the *symmetric product*. The symmetric product recovers the homology of the space via the Dold-Thom theorem, which states

$$\pi_*(Sym(X;A)) \cong \tilde{H}_*(X;A)$$

The second goal of this poster is to indicate the proof.

The Symmetric Product

As a set, define the *symmetric product of X with labels in A* to be pairs

$$Sym(X;A) := \{(S,l) : * \in S \subset X, |S| < \infty, l : (S \backslash *) \to A\}$$

where * denotes the basepoint of X. To topologize this set, we'll declare the following family of maps to be continuous:

$$X^{I_+} \times A^I \to Sym(X;A)$$
$$(c : I_+ \to X, l : I \to A) \mapsto (c(I_+), l' : s \mapsto \sum_{i \in c^{-1}(s)} l(i))$$

Taking the union of these maps over all finite sets, this surjects onto $Sym(X;A)$, and we say $U \subset Sym(X;A)$ is open iff its inverse images under these maps are open for all I-finite. In particular, this topology allows points to collide, whence their labels add.

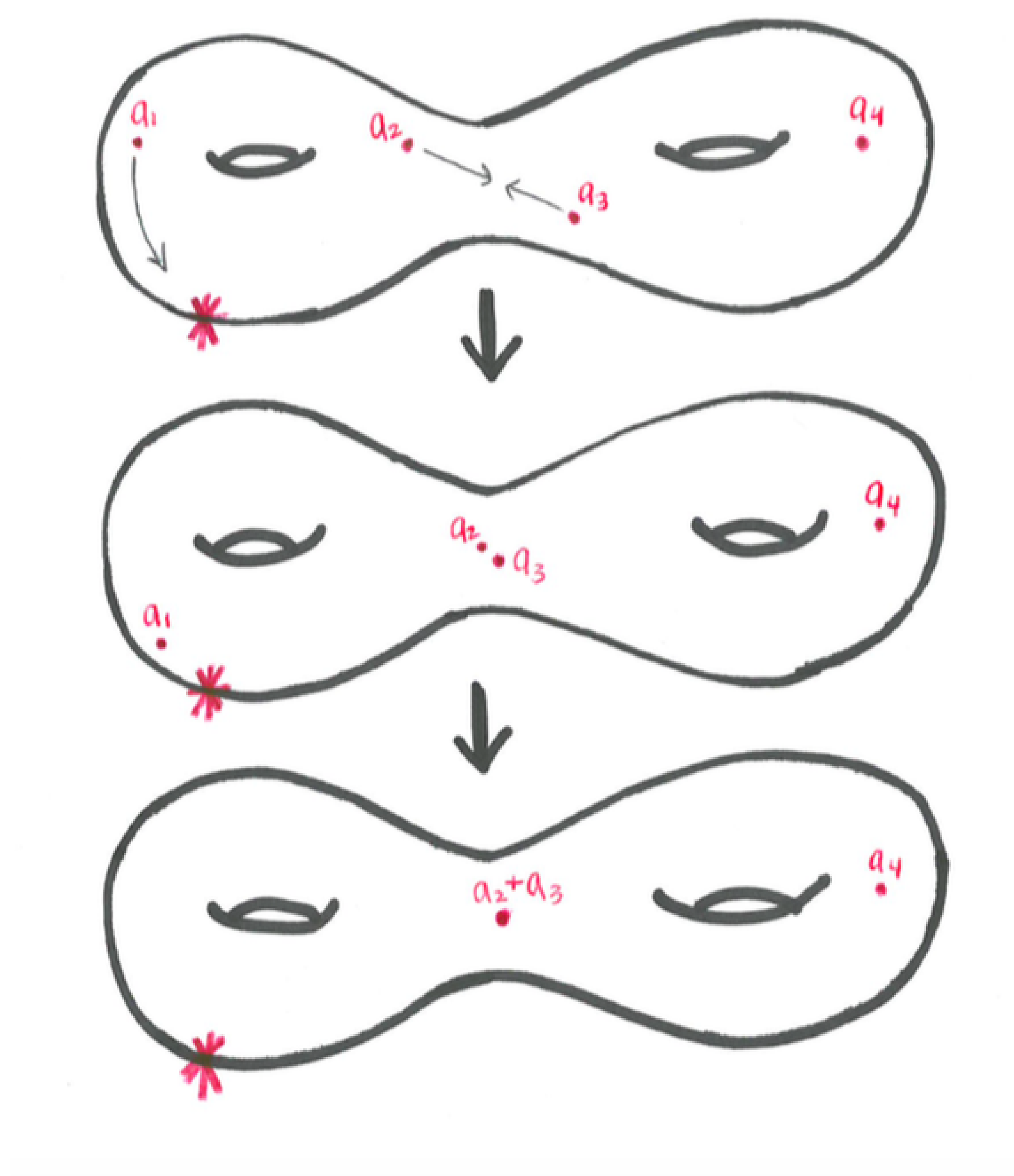

Figure 1: A continuous path in Sym$(\Sigma_2; A)$

Examples:

- There is a homeomorphism,
$$Sym(I_+;A) \cong A^I$$
- Thinking of S^1 as an abelian group under rotations, there is a homotopy equivalence
$$Sym(S^1;\mathbb{Z}) = Sym(\mathbb{R}^+;\mathbb{Z}) \xrightarrow{\simeq} S^1$$
$$(S, S - 0 \xrightarrow{l} \mathbb{Z}) \mapsto (\sum_{s \in S} l(s)s)$$
- There is a continuous map
$$\mathbb{C}[z,z^{-1}]/\mathbb{C}^\times \to Sym(\mathbb{C}^+;\mathbb{Z})$$
polynomial $\mapsto$ zeroes and poles with multiplicity

Properties of $Sym(-;A)$

1. **Pointed:** $Sym(X;A)$ has a canonical basepoint, namely $(\emptyset_+, \emptyset \xrightarrow{!} A)$.
2. **Functoriality:** A pointed map $f : X \to Y$ induces a map $Sym(f;A) : Sym(X;A) \to Sym(Y;A)$. An open embedding $X \to Y$ induces a map $Y^+ \to X^+$ called the "collapse" map, which in turn (by the previous sentence) induces $Sym(Y^+;A) \to Sym(X^+;A)$. These respect compositions and identity.
3. **Monoidal:**
$$Sym(X \vee Y;A) \cong Sym(X;A) \times Sym(Y;A)$$
4. **Continuous:**
$$f \simeq g \Rightarrow Sym(f;A) \simeq Sym(g;A) : Sym(X;A) \to Sym(Y;A)$$
5. **Pullback:** If $U, V \subset X$ such that $X = U \cup V$ and X is locally compact, Hausdorff then we have the *pullback* diagram
$$\begin{array}{ccc} Sym(X^+;A) & \longrightarrow & Sym(U^+;A) \\ \downarrow & & \downarrow \\ Sym(V^+;A) & \longrightarrow & Sym((U \cap V)^+;A) \end{array}$$
6. **Homotopy pullback**: Given embeddings $X_L \xleftarrow{f} X_0 \xrightarrow{g} X_R$, can form the *homotopy pushout* space $X := X_L \cup_f X_0 \times [-1,1] \cup_g X_R$. Then there is a *homotopy pullback* diagram
$$\begin{array}{ccc} Sym(X^+;A) & \longrightarrow & Sym((X - X_R)^+;A) \\ \downarrow & & \downarrow \\ Sym((X - X_L)^+;A) & \longrightarrow & Sym((X_0 \times (-1,1))^+;A) \end{array}$$
This is the most technical property, and the crux of the proof is the fact that the vertical maps in the diagram are *quasifibrations*.

Consequences

Corollary 1 (Interaction w.suspension)**.** $Sym(X^+;A) \simeq \Omega Sym(\Sigma(X^+);A)$

Proof. Apply the homotopy pullback property to the diagram $X \xleftarrow{id} X \xrightarrow{id} X$ to obtain the homotopy pullback diagram
$$\begin{array}{ccc} Sym(X^+;A) & \longrightarrow & Sym((X \times [-1,1))^+;A) \simeq * \\ \downarrow & & \downarrow \\ * \simeq Sym((X \times (-1,1])^+;A) & \longrightarrow & Sym((X \times (-1,1))^+;A) \simeq Sym(\Sigma(X^+);A) \end{array}$$
Where the equivalences in the top right and lower left corners come from the fact that $(X \times [-1,1))^+ \simeq C(X^+) \simeq *$. □

Corollary 2. $A \simeq \Omega^n Sym((\mathbb{R}^n)^+;A)$

Proof. For the case $n = 0$, note $\Omega^0 Sym((\mathbb{R}^0)^+;A) \simeq Sym((\mathbb{R}^0)^+;A) \simeq A$. For the inductive step, assume we've shown $A \simeq \Omega^n Sym((\mathbb{R}^n)^+;A)$. Then note that from the above corollary,
$$Sym((\mathbb{R}^n)^+;A) \simeq \Omega Sym((\mathbb{R}^n \times (-1,1))^+;A) \simeq \Omega Sym((\mathbb{R}^{n+1})^+;A),$$
and applying Ω^n to each side gives $\Omega^n Sym((\mathbb{R}^n)^+;A) \simeq \Omega^{n+1} Sym((\mathbb{R}^{n+1})^+;A)$, which by inductive hypothesis $\simeq A$. □

Corollary 3 (Eilenberg-Maclane spaces)**.** $\pi_k Sym((\mathbb{R}^n)^+;A) \cong \begin{cases} A & k = n \\ 0 & else \end{cases}$

Proof. This follows from the previous corollary together with a little extra work. Namely, the previous corollary gives that it holds for $k \geq n$, for the other cases one must use a "general position" argument. □

The Dold-Thom Theorem

Theorem 4 (Dold-Thom)**.**
$$\pi_*(Sym(X;A)) \cong \tilde{H}_*(X;A)$$

Proof. By the Eilenberg-Steenrod theorem, singular homology $\tilde{H}_*(-;A)$ is the unique continuous, monoidal, excisive functor $Top_* \to Gr.Abe.Gp.$ such that it's value on $*_+$ is A. Meanwhile, the properties we enumerated give that $\pi_*(Sym(-;A))$ is another such functor, thus the result follows. □

Forthcoming Research

It is my hope to prove the Dold-Thom theorem directly using higher-categorical methods, for example to bypass the technical digression of checking that certain maps are quasifibrations.

Acknowledgements

Inspiration for this poster came from a lecture series by David Ayala given at IMPA. Videos of his lectures are available at http://video.impa.br/index.php?page=factorization-homology.

For the original proof circa 1958:

A Dold, R Thom, Quasifaserungen und unendliche symmetrische Produkte, Ann. of Math. 67 (1958), 239-281.