Factorization Homology Of Thom Spectra

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Thom spectra

An n-dimensional real vector bundle ζ on a (paracompact) space X is classified by a map

$$f : X \to BO(n)$$

The Thom space of ζ can be defined by $D(\zeta)/S(\zeta)$, the disk bundle of ζ mod its sphere bundle.
A map $f : X \to BO$ gives a sequence of maps

$$f_n : f^{-1}(BO(n)) \to BO(n)$$

and their Thom spaces assemble to give the **Thom spectrum** of f, denoted $Th(f)$.

Examples.

- Cobordism spectra MO, MU, MSO, $MSpin$,...
- $H\mathbb{Z}/2 \simeq Th(\Omega^2 S^3 \to BO)$ (Mahowald)
- $H\mathbb{Z}/p$, $H\mathbb{Z}_{(p)}$ are generalized Thom spectra

E_n-algebras

Lewis proved that the Thom spectrum $Th(f)$ of an n-fold loop map

$$f : \Omega^n X \to \Omega^n B^{n+1} O \simeq BO$$

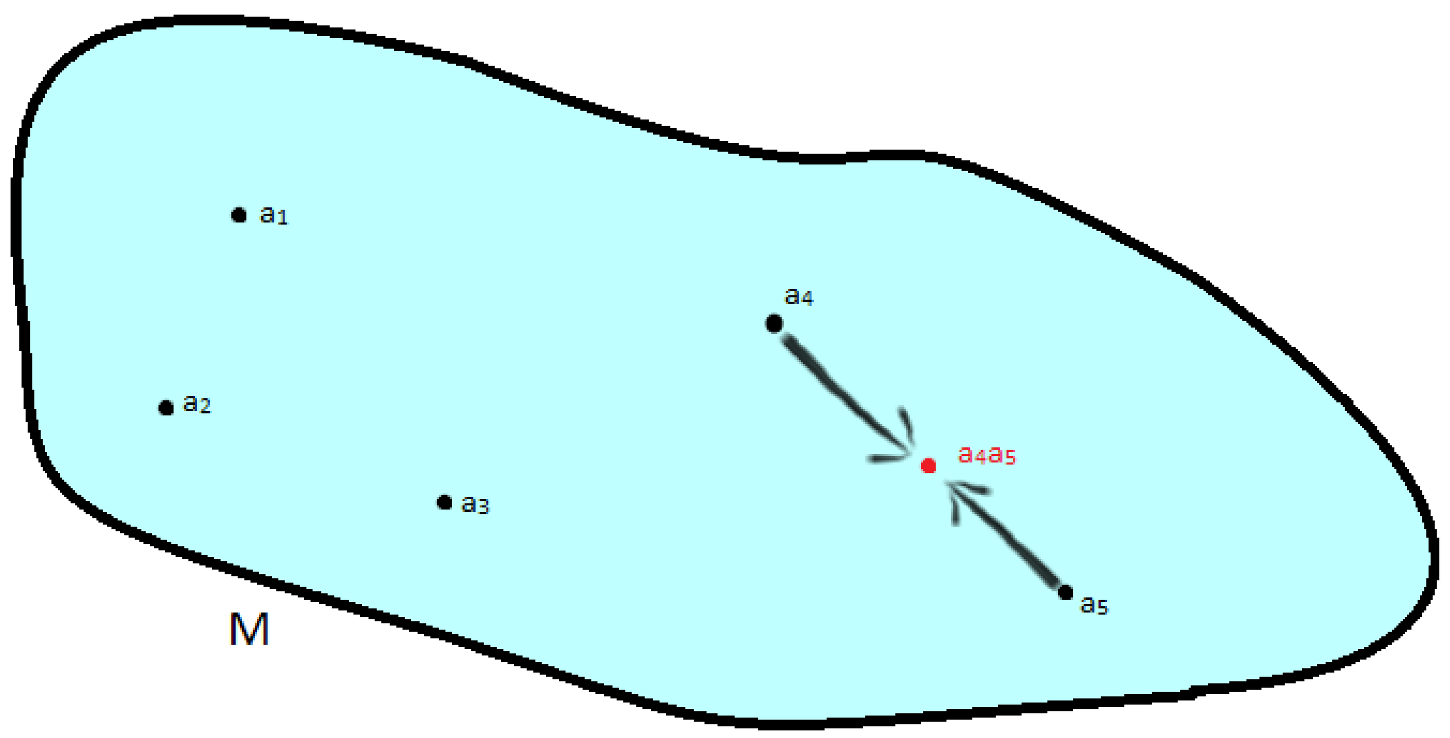
is an E_n-algebra in spectra.

E_n-algebras. An E_n-algebra (e.g. in spaces, spectra, or chain complexes) can be thought of as having n compatible multiplications, or space of multiplications $\simeq S^{n-1}$. For $n = 1$, this is just an algebra/ring/monoid.
In **spaces**: think of $\Omega^n X$. n compatible multiplications = n loop directions.

Examples above of Thom spectra are all **ring spectra**, even commutative, and all arise from n-fold loop maps for some n, as in Lewis's theorem.

Intuition for factorization homology

Input: (framed) manifold M^n, E_n-algebra A (in spaces, spectra, chain complexes...)
Output: $\int_M A$ (a space, spectrum, chain complex... respectively)



$\int_M A$: space of configurations in M with labels in A; when points collide, their labels multiply. A is not necessarily commutative, so take into account direction of collision. E_n-algebra has S^{n-1} multiplications, framed n-manifold has S^{n-1} directions of collision.
Dold-Thom theorem: If A is a discrete abelian group, then $\pi_*(\int_M A) \cong H_*(M; A)$.

Results

Let $f : \Omega^n X \to BO$ be an n-fold loop map (X $(n-1)$-connected), and M^n a properly embedded framed submanifold of $\mathbb{R}^{n+k}$, $i : M \times \mathbb{R}^k \hookrightarrow \mathbb{R}^{n+k}$.

Then $\int_M Th(f)$ is equivalent to the Thom spectrum of a map we denote $\int_M f$:

$$Map_*(M^+, X) \to Map_*(M^+, B^{n+1}O) \simeq Map_*(\Sigma^k M^+, B^{n+k+1}O) \to Map_*(S^{n+k}, B^{n+k+1}O) \simeq BO$$

The first arrow is induced by $B^n f : X \to B^{n+1}O$.
The second arrow is induced by the Pontryagin-Thom collapse map associated to i, $\tau_i : S^{n+k} \to \Sigma^k M^+$.

Twisted non-abelian Poincaré duality. This can be seen as a twisted version of non-abelian Poincaré duality (see to the right), which describes the factorization homology of a Thom spectrum as a twisted suspension spectrum over a mapping space.

Generalizations.

- Generalizes to stably framed manifolds (M such that $M \times \mathbb{R}^k$ is framed) given a more highly commutative algebra.
- Also works for generalized Thom spectra, i.e. Thom spectra of maps $\Omega^n X \to BGL_1(R)$, R a commutative ring spectrum.

Non-abelian Poincaré duality

Theorem (Segal, Salvatore, Lurie) If M^n is a framed manifold and X is $(n-1)$-connected, there is a scanning map giving an equivalence

$$\int_M \Omega^n X \simeq Map_*(M^+, X)$$

M^+ denotes the 1-point compactification of M.
Poincaré duality? When $X = K(A, n) = B^n A$, A a discrete abelian group, specializes to Poincaré duality. (Homotopy groups of LHS = homology, homotopy groups of RHS = compactly supported cohomology).
Spectra? This applies to E_n-algebras in spaces, but also gives, for suspension spectra:

$$\int_M \Sigma^\infty_+ \Omega^n X \simeq \Sigma^\infty_+ Map_*(M^+, X)$$

Suspension spectrum = Thom spectrum of a trivial map, $\Sigma^\infty_+ \Omega^n X \simeq Th(* : \Omega^n X \to BO)$, so our theorem can be thought of as a twisted version of the above equivalence, twisted by a vector bundle over $\Omega^n X$.

Corollaries

- Gives calculations of factorization homology of cobordism spectra (MO, MU,...) over stably framed manifolds. Also $\int_M M\Sigma$, $\int_M MGL(\mathbb{Z})$.
- $\int_{\Sigma_g} H\mathbb{Z}/p$, $\int_{\Sigma_g} H\mathbb{Z}$ for Σ_g a genus g surface (and for other orientable surfaces).
  e.g. $\int_{\Sigma_g} H\mathbb{Z}/p \simeq H\mathbb{Z}/p \wedge (S^3 \times (\Omega S^3)^{2g})_+$

References

[1] David Ayala and John Francis. Factorization homology of topological manifolds. *Journal of Topology*, 2015.

[2] L Gaunce Lewis Jr, J Peter May, and Mark Steinberger. Equivariant stable homotopy theory. 1986.

[3] Paolo Salvatore. Configuration spaces with summable labels. In *Cohomological methods in homotopy theory*, pages 375–395. Springer, 2001.

[4] Graeme Segal. Configuration-spaces and iterated loop-spaces. *Inventiones Mathematicae*, 21(3):213–221, 1973.