Moduli spaces of unstable curves

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(based on work of Joshua Jackson and others)
Classification problems in geometry

Ingredients:
(1) **objects** (e.g. compact Riemann surfaces/nonsingular complex projective curves);
(2) **equivalence relation** $\sim$ (e.g. biholomorphism/isomorphism);
(3) **families** of objects parametrised by base spaces $S$ (typically $\pi : \mathcal{X} \to S$ with $\pi^{-1}(s)$ the object parametrised by $s \in S$).

Fix any discrete invariants (e.g. genus of compact Riemann surface/nonsingular projective curve). Then

**moduli space** $= \{\text{objects}\} / \sim$ with nice geometric structure.

E.g. $\mathcal{M}_g = \{\text{nonsingular curves of genus } g\}/\text{isomorphism}$
Moduli spaces often arise as (parameter space)/(group action).

E.g. (1) clock: \( S^1 \cong \mathbb{R}/\mathbb{Z} \)
where \( S^m = \{(y_0, y_1, \ldots, y_m) \in \mathbb{R}^m \mid y_0^2 + y_1^2 + \cdots + y_m^2 = 1\} \)
is a sphere of dimension \( m \) and \( \mathbb{Z} \) acts on \( \mathbb{R} \) by translation.

E.g. (2) complex projective space \( \mathbb{P}^n \)
\[ \mathbb{P}^n \cong S^{2n+1}/S^1 \]
\[ \cong (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* \]
where \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) (under multiplication) acts as scalar multiplication on \( \mathbb{C}^{n+1} \setminus \{0\} \).
A projective curve (over $\mathbb{C}$) of genus $g \geq 2$ is stable iff its singularities are at worst nodes and its automorphism group is finite.

$$\overline{M}_g = \{\text{stable curves of genus } g\} / \text{isomorphism}$$

is a projective variety containing $M_g$ as an open subset.

$$\overline{M}_g = K^s / SL(r + 1)$$

is the quotient by $SL(r + 1)$ in the sense of geometric invariant theory (GIT) of the closure

$$K = \{k - \text{canonically embedded nonsing curves of genus } g\}$$

in the Hilbert scheme of curves in $\mathbb{P}^r$ with Hilbert polynomial

$$P(m) = dm + 1 - g$$

for $r = (2k - 1)(g - 1) - 1$ and $d = 2k(g - 1)$, where $k \gg 1$ and $K^s$ is the stable subset of $K$. 

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A singular curve $C$ has a canonical ‘resolution of singularities’, its normalisation $p : \tilde{C} \to C$. Here $\tilde{C}$ is a nonsingular curve and $p$ is surjective and an isomorphism away from the singular locus of $C$.

**Rosenlicht–Serre description of singular curves** $C$ via the normalisation $p : \tilde{C} \to C$: Let $R$ be the equivalence relation on $Y = \tilde{C}$ defined by $p : \tilde{C} \to C$ with associated homeomorphism $Y/R \to C$ where $Y/R$ has the co-finite topology. For $z \in Y/R$ let

$$\mathcal{O}_z = \bigcap_{p(y) = z} \mathcal{O}_{y,Y} \quad \text{(semi-local ring)}$$

with radical $r_z$. Let $\mathcal{O}'_z$ be $\mathcal{O}_z$ if $z \notin (\text{Sing}(C))$ and $C + r_z \supseteq \mathcal{O}'_z \supseteq C + r_z^n$ for some $n \gg 1$. Then the ringed space $(Y/R, \mathcal{O}')$ is always a projective curve, and is isomorphic to the original curve $C$ for appropriate choices of $n \gg 1$ and of $\mathcal{O}'_z$ when $z \in (\text{Sing}(C))$. 
Unibranch case ($\tilde{p} : \tilde{C} \to C$ bijective):

Sherwood Ebey (1964) used the Rosenlicht–Serre construction to classify unibranched singularities using slices, given by constructible subsets, for a non-reductive linear algebraic group action on a variety (and showed in general $\exists$ nontrivial moduli). The same groups were used by Demailly in the 1990s to study jet differentials:

$X$ complex manifold

$J_k \to X$ bundle of $k$-jets of holomorphic curves $f : (\mathbb{C}, 0) \to X$

Under composition modulo $t^{k+1}$ we have a group $\mathbb{G}_k$ acting on $J_k$ whose elements are $k$-jets of germs of biholomorphisms of $\mathbb{C}$

$$t \mapsto \phi(t) = a_1 t + a_2 t^2 + \ldots + a_k t^k, \quad a_j \in \mathbb{C}, \quad a_1 \neq 0.$$
This reparametrisation group $G_k$ is isomorphic to the group of matrices

$$\left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_k \\ 0 & a_2^2 & \ldots \\ & & \ddots \\ 0 & 0 & \ldots & a_k^k \end{pmatrix} : a_1 \in \mathbb{C}^*, a_2, \ldots a_k \in \mathbb{C} \right\}.$$

$G_k$ has a subgroup $\mathbb{C}^*$ (represented by $\phi(t) = a_1 t$) and a unipotent subgroup $U_k$ (represented by $\phi(t) = t + a_2 t^2 + \ldots + a_k t^k$) such that

$$G_k \cong U_k \rtimes \mathbb{C}^*;$$

$U_k$ is its unipotent radical.
Question: Can we modify the construction of

\[ \overline{M}_g = \mathcal{K}^s / SL(r + 1) \]

as the GIT quotient by \( G = SL(r+1) \) of \( \mathcal{K} \) to find discrete invariants of unstable curves such that moduli spaces of curves with these invariants fixed can also be constructed by GIT methods?

Answer: We need to use geometric invariant theory for non-reductive group actions, rather than classical GIT, even though \( G = SL(r + 1) \) is reductive.
Classical GIT tells us that \( \mathcal{K} \) has a stratification

\[
\mathcal{K} = \bigsqcup_{\beta \in \mathcal{B}} S_\beta
\]

indexed by a finite subset \( \mathcal{B} \) of a \( + \)ve Weyl chamber for \( SU(r+1) \), with (i) \( S_0 = \mathcal{K}^s \), and for each \( \beta \in \mathcal{B} \)
(ii) the closure of \( S_\beta \) is contained in \( \bigcup_{\gamma \geq \beta} S_\gamma \),
(iii) \( S_\beta \cong G \times_{P_\beta} Y^{ss}_\beta \) where \( P_\beta \) is a parabolic subgroup of \( G = SL(r+1) \) and \( Y^{ss}_\beta \) is an open subset of a projective subscheme \( \overline{Y}_\beta \) of \( \mathcal{K} \), determined by the action of the Levi subgroup of \( P_\beta \) with respect to a twisted linearisation.

To construct a quotient of (an open subset of) \( S_\beta \) by \( G \) we can study the linear action on \( \overline{Y}_\beta \) of the parabolic subgroup \( P_\beta \), twisted appropriately.
Recall: moduli spaces in algebraic geometry are often constructed as quotients of varieties by actions of linear algebraic groups.

Assume for simplicity that we are working over \( \mathbb{C} \). A linear algebraic group \( G \) is a semi-direct product of a unipotent group, which is its unipotent radical, by a reductive group.

**Example:** The automorphism group of the weighted projective plane \( \mathbb{P}(1, 1, 2) = (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^* \), where \( \mathbb{C}^* \) acts on \( \mathbb{C}^3 \) with weights 1,1 and 2, is

\[
\text{Aut}(\mathbb{P}(1,1,2)) \cong R \ltimes U
\]

with \( R \cong GL(2) \times_{\mathbb{C}^*} \mathbb{C}^* \cong GL(2) \) reductive

\[
U \cong (\mathbb{C}^+)^3 \text{ unipotent}
\]

where \( (x, y, z) \mapsto (x, y, z + \lambda x^2 + \mu xy + \nu y^2) \) for \( (\lambda, \mu, \nu) \in \mathbb{C}^3 \).
Mumford’s Geometric Invariant Theory (1960s)

$G$ complex reductive group
$X$ complex projective variety acted on by $G$

We require a linearisation of the action (i.e. an ample line bundle $L$ on $X$ and a lift of the action to $L$; think of $X \subseteq \mathbb{P}^n$ and the action given by a representation $\rho : G \to GL(n + 1)$).

\[
X \sim A(X) = \mathbb{C}[x_0, \ldots, x_n]/\mathcal{I}_X = \bigoplus_{k=0}^{\infty} H^0(X, L^\otimes k)
\]

$X//G \Leftarrow A(X)^G$ algebra of invariants

$G$ reductive implies that $A(X)^G$ is a finitely generated graded complex algebra so that $X//G = \text{Proj}(A(X)^G)$ is a projective variety.
The rational map $X \dashrightarrow X//G$ fits into a diagram

\[
\begin{array}{ccc}
X & \dashrightarrow & X//G \\
\cup & & \| \\
\text{semistable} \quad X^{ss} & \overset{\text{onto}}{\longrightarrow} & X//G \\
\cup & & \cup \\
\text{stable} \quad X^s & \rightarrow & X^s/G
\end{array}
\]

where the morphism $X^{ss} \rightarrow X//G$ is $G$-invariant and surjective.

Topologically $\boxed{X//G = X^{ss}/\sim}$ where $x \sim y \Leftrightarrow \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset$. 

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A partial desingularisation of $X//G$

$G$ complex reductive group
$X$ complex projective variety acted on linearly by $G$

There is a partial desingularisation $\tilde{X}//G = \tilde{X}^{ss}/G$ of $X//G$ which is a geometric quotient by $G$ of an open subset $\tilde{X}^{ss} = \tilde{X}^s$ of a $G$-equivariant blow-up $\tilde{X}$ of $X$.

$\tilde{X}^{ss}$ is obtained from $X^{ss}$ by successively blowing up along the subvarieties of semistable points stabilised by reductive subgroups of $G$ of maximal dimension and then removing the unstable points in the resulting blow-up.

$X$ nonsingular $\Rightarrow \tilde{X}//G$ is an orbifold.
**Good case:** $X^{ss} = X^s \neq \emptyset$

Then

$$X^s/G = X//G = \text{Proj}(A(X)^G)$$

is a projective variety, geometric quotient of $X^s$.

**More generally:** $X^s \neq \emptyset$

Then

$$\tilde{X}/G \text{ proj variety}$$

$$\tilde{X}^s/G \text{ geom quotient}$$

$$\bigcup \text{ open} \quad X^s/G \text{ geom quotient}$$
So what happens if $G$ is not reductive?

**Problem:** We can’t define a projective variety

$$X//G = \text{Proj}(A(X)^G)$$

because $A(X)^G$ is not necessarily finitely generated.

**Question:** Can we define a sensible ‘quotient’ variety $X//G$ when $G$ is not reductive? If so, can we understand it geometrically?

**Answer:** We can define open subsets $X^s$ (‘stable points’) and $X^{ss}$ (‘semistable points’) with a geometric quotient $X^s \to X^s/G$ and an ‘enveloping quotient’ $X^{ss} \to X//G$. BUT $X//G$ is not necessarily projective and $X^{ss} \to X//G$ is not necessarily onto.
**Defn:** Call a unipotent linear alg group $U$ **graded unipotent** if there is a homomorphism $\lambda : \mathbb{C}^* \to Aut(U)$ with the weights of the $\mathbb{C}^*$ action on $\text{Lie}(U)$ all **strictly positive**. Then let

$$\hat{U} = U \rtimes \mathbb{C}^* = \{(u, t) : u \in U, t \in \mathbb{C}^*\}$$

with multiplication $(u, t) \cdot (u', t') = (u(\lambda(t)(u')), tt')$.

Suppose that $\hat{U}$ acts linearly (with respect to an ample line bundle $L$) on a projective variety $X$. We can twist the action of $\hat{U}$ by any (rational) character. If we are willing to twist by an appropriate character then GIT for the $\hat{U}$ action is nearly as well behaved as in the classical case for reductive groups.
Thm: (Berczi, Doran, Hawes, K) Let $U$ be graded unipotent acting linearly on a projective variety $X$, and suppose that the action extends to $\hat{U} = U \times \mathbb{C}^*$. Suppose also that

\[ (*) \quad x \in X_{\min}^{\mathbb{C}^*} \Rightarrow \dim \text{Stab}_U(x) = \min_{y \in X} \dim \text{Stab}_U(y) \]

where $X_{\min}^{\mathbb{C}^*}$ is the union of connected components of $X^{\mathbb{C}^*}$ where $\mathbb{C}^*$ acts on the fibres of $L$ with minimum weight. Twist the action of $\hat{U}$ by a (rational) character so that 0 lies in the lowest bounded chamber for the $\mathbb{C}^*$ action on $X$. Then

(i) the ring $A(X)^{\hat{U}}$ of $\hat{U}$-invariants is finitely generated, so that $X//\hat{U} = \text{Proj}(A(X)^{\hat{U}})$ is projective;

(ii) $X//\hat{U}$ is a geometric quotient of $X_{ss}^{\hat{U}} = X^{s,\hat{U}}$ by $\hat{U}$.

Moreover, even without condition $(*)$ there is a projective completion of $X_{ss}^{s,\hat{U}}//\hat{U}$ which is a geometric quotient by $\hat{U}$ of an open subset $\tilde{X}^{ss}$ of a $\hat{U}$-equivariant blow-up $\tilde{X}$ of $X$. 

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Recall that when $G$ is reductive and acts linearly on a projective variety $X$, it has a stratification

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_{\beta}$$

indexed by a finite subset $\mathcal{B}$ of a +ve Weyl chamber, with

(i) $S_0 = X^{ss}$, and for each $\beta \in \mathcal{B}$
(ii) the closure of $S_{\beta}$ is contained in $\bigcup_{\gamma \geq \beta} S_{\gamma}$,
(iii) $S_{\beta} \cong G \times P_{\beta} Y_{\beta}^{ss}$ where $P_{\beta}$ is a parabolic subgroup of $G$ and $Y_{\beta}^{ss}$ is an open subset of a projective subvariety $\overline{Y}_{\beta}$ of $X$.

$P_{\beta} = U_{\beta} \rtimes L_{\beta}$, where its unipotent radical $U_{\beta}$ is graded by a central 1-parameter subgroup $\mathbb{C}^* \to L_{\beta}$ of its Levi subgroup $L_{\beta}$. To construct a quotient of (an open subset of) $S_{\beta}$ by $G$ we can study the linear action on $\overline{Y}_{\beta}$ of the parabolic subgroup $P_{\beta}$, twisted as in the previous theorem, and quotient first by $\widehat{U}_{\beta}$ and then by the residual action of the reductive group $P_{\beta}/\widehat{U}_{\beta} = L_{\beta}/\mathbb{C}^*$.
Moduli spaces of sheaves of fixed Harder–Narasimhan type over a nonsingular projective variety $W$

There is a well-known construction due to Simpson of the moduli space of semistable pure sheaves on $W$ of fixed Hilbert polynomial as the GIT quotient of a linear action of a special linear group $G$ on a scheme $Q$ (closely related to a quot-scheme) which is $G$-equivariantly embedded in a projective space. This construction can be chosen so that elements of $Q$ which parametrise sheaves of a fixed Harder–Narasimhan type form a stratum in the stratification of $Q$ associated to the linear action of $G$, at least modulo taking connected components of strata (Hoskins). One can then try to use non-reductive GIT for the associated linear action of a parabolic subgroup of $G$, appropriately twisted, to construct moduli spaces of sheaves of fixed Harder–Narasimhan type over $W$. 

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Moduli spaces of unstable curves of fixed ‘type’

Similarly we can define the ‘Rosenlicht–Serre type’ of a projective curve, measuring to some extent how unstable it is.

Non-reductive GIT for linear actions of suitable non-reductive linear algebraic groups with internally graded unipotent radicals can then be used to construct moduli spaces of unstable curves of fixed Rosenlicht–Serre type.

(Joshua Jackson, Oxford thesis, 2018)