Relations between Sarkisov links of surfaces over a perfect field

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Abstract

The Cremona group $\mathrm{Cr}_n(k)$ or $\mathrm{Bir}(P^n)$ is the group of birational transformations of the projective $n$-space over a field $k$. In dimension $n = 2$ it is known [CLC11, Lon16] that the Cremona group over any field is not simple. It was recently shown [BLZ19] that the same holds over (subfields of) the field of complex numbers for higher dimensions $n \geq 3$. The goal of this article is to adapt the strategy of [BLZ19] to dimension two over perfect fields and find new simple subgroups of $\mathrm{Cr}_2(k)$.

1. Introduction

Let $k$ be a perfect field. (Readers unfamiliar with perfect fields are encouraged to think of $k = \mathbb{Q}$.) We are interested in birational maps between minimal surfaces and their factorization into “simple” maps. Whereas points can be blown up over algebraically closed fields, over perfect fields one has to blow up an entire orbit of $\text{Gal}(k)$.

Example. Any birational map $P^2 \to P^2$ can be factorized into links as depicted in the following figure (following the classification of Sarkisov links [Isk96]):

- $X \to P^2, K_X^2 = 2$
- $X \to P^2, K_X^2 = 8$
- $X \to P^2, K_X^2 = 5$

The $X \to P^2$ with $K_X^2 = 8$ are Hirzebruch surfaces, and the $Q \to X$ are quadrics in $P^3$. The arrows denote the direction of the link labels of the form “$X \to Y$” denote a blow-up of an orbit of size of $X$ and “$\text{Hir}_n,k$” denotes a link that is given by blowing up an orbit of size of $X$, followed by the contraction to an orbit of size $Y$.

Note that not all surfaces that arise in the figure of the example are minimal surfaces. For example, the first Hirzebruch surface $\mathbb{F}_1$ appears, but it is not minimal since it is the blow-up of $P^2$ in one point.

Definition. A morphism $\phi : X \to B$, where $X$ is a smooth surface and $B$ is smooth, is called a Mori fiber space if the following conditions are satisfied:

1. $(\dim B) < (\dim X)$,
2. $\phi_*\mathcal{O}_X(\mathcal{O}_X/B) = \mathcal{O}_B$ (where $\mathcal{O}_X/B = \mathcal{O}_X/\mathcal{O}_X(B)$ is the relative Picard group),
3. $K_X - B > 0$ for all $D \subset \phi(X)$.

There are only two possibilities:

1. $(\dim B) = 1$, so $B$ is a curve and $X \to B$ is a conic bundle (that is a general fiber is isomorphic to $P^1$ and any simple fiber is the union of two (-1)-curves intersecting at one point), or
2. $B = \ast$ with $\phi_*\mathcal{O}_X(\mathcal{O}_X/B)$ and so $X$ is a del Pezzo surface.

Note that algebraically closed fields, the only Mori fiber spaces are $\mathbb{P}^2$ and ruled surfaces (which are Hirzebruch surfaces if they are rational).

Definition. A Sarkisov link is a birational map $\phi : X_1 \to X_2$ between two Mori fiber spaces $\pi_1 : X_1 \to B_1, \pi_2 : X_2 \to B_2$ that is of one of the following four types:

- **Type I**: $X_1 \to X_2$ where $\varphi^{-1}(X_2) \to X_1$ is the blow-up of an orbit.
- **Type II**: $\phi = [\pi_2 \circ \pi_1^{-1}]$, where $\pi_1 : X_2 \to X_1$ is the blow-up of an orbit.
- **Type III**: $X_1 \to X_2$ where $\varphi : X_1 \to X_2$ is the blow-up of an orbit.
- **Type IV**: $\phi = [\pi_2 \circ \pi_1^{-1}]$, where $B_1$ and $B_2$ are curves of genus 0, and $\varphi$ is an isomorphism that does not preserve the fibration.

2. Results

In [LZ19] they focused on Birational involutions, which correspond to links of type II based on an orbit of size 8 in $P^2$. Our method lies in studying links of type II between conic bundles that have a “large” base orbit. For our purposes, “large” means that the cardinality of the orbit is $\geq 16$. (We would like it to mean $\geq 8$, but some technicalities deny us this pleasure.) In fact, links of type I, III, and IV all have base orbits of size $\leq 8$. This is the reason why the links of type II between conic bundles are colored in red in the figure of the starting example, since these are the only birational maps that can have base orbits of cardinality $\geq 9$.

We will say that a birational map $\phi$ has cardinality $d$ if the maximal cardinality of orbits in $\text{Bir}(\phi)$ and $\text{Bir}(\phi^{-1})$ is $d$.

**Theorem: Generating Relations**

Let $X$ be a Mori fiber space. Relations of the groupoid $\text{Bir}(\text{Mori}(X))$ are generated by the following relations:

(a) $\phi_1 \circ \phi_2 = id$, where the cardinality of all $\phi_i$ is $\leq 15$, and
(b) $\phi_1 \circ \phi_2 = id$, where $\phi_i : X_i \to X_i$ are links of type II between conic bundles of cardinality $\geq 16$ such that

- $\phi_i^{-1}$ is a local isomorphism on the fiber on $X_i$ containing $\text{Bir}(\phi_i)$,
- $\phi_i$ is centered at $\phi_2(\text{Bir}(\phi_i^{-1}))$,
- $\phi_i$ is centered at $\phi_2(\text{Bir}(\phi_i^{-1}))$.

**Theorem: Group homomorphism**

Let $X$ be a Mori fiber space. There exists a group homomorphism $\text{Bir}(\text{Mori}(X)) : \mathbb{Z}/2$. In dimension two over non-closed fields, it is known [CLC11, Lon16] that the Cremona group over the field of complex numbers for higher dimensions $n \geq 3$. The goal of this article is to adapt the strategy of [BLZ19] to dimension two over perfect fields and find new simple subgroups of $\text{Cr}_2(k)$.

**Example.** Consider the birational map $\varphi : P^1 \to P^1 \to P^1 \to P^1$ that is given by

$[ax : bx : y : z] \mapsto [ay : bx : y : z]$,

and on the points $p_i = [0, 1, 1, 1]$. One can check that $\varphi$ is the composition of a link of type II $P^1 \to P^1 \to P^1$ centered at the orbit $[p_1, p_2, p_3, p_4]$ followed by a link $P^1 \to P^1$, type II of cardinality $1$ for $n = 3, 4, \ldots$.

Corollary

For each perfect field $k$ such that $|k : k| > 2$, there is a surjective group homomorphism $\text{Bir}(\text{Mori}(P^n)) : \mathbb{Z}/2$ whose kernel contains $\text{Aut}(\text{Bir}(P^n)) = PGL_2(k)$.

This result was already obtained in [LZ19] if $k$ has an extension of degree $8$, but the normal subgroup they constructed is different. However, it contrasts with [CLC11, Lon16]. The normal subgroups of the Cremona group that they found do not contain any automorphism except the identity.

References


