Infinity Yoneda

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1.1 Simplicial nerve

The *coherent nerve* or *simplicial nerve* of a simplicial category $\mathcal{S}$ is the simplicial set defined as

$$(N\mathcal{S})_n = \mathrm{sCat}(\mathbb{C}^n, \mathcal{S}), \tag{1.1}$$

where $\mathbb{C}^n$ is the simplicial category whose set of objects is $\{0, \ldots, n\}$ and whose space of maps from i to j is $\mathrm{Map}(i, j) = NP_{i,j}$ where $P_{i,j}$ is the partially ordered set of subsets of $\{i, \ldots, j\}$ containing i and j for $i \leq j$, and $P_{i,j} = \emptyset$ if $i > j$. Composition maps

$$\mathrm{Map}(i_0, i_1) \times \cdots \times \mathrm{Map}(i_{m-1}, i_m) \longrightarrow \mathrm{Map}(i_0, i_m)$$

are given by the functions $P_{i_0,i_1} \times \cdots \times P_{i_{m-1},i_m} \to P_{i_0,i_m}$ sending $(I_1, \ldots, I_m)$ to $I_1 \cup \cdots \cup I_m$.

In fact we may define a colimit-preserving functor $\mathbb{C}\colon \mathbf{sSet} \to \mathrm{sCat}$ by setting $\mathbb{C}\Delta[n] = \mathbb{C}^n$ and passing to its left Kan extension, that is,

$$\mathbb{C}X = \operatorname*{colim}_{(\Delta \downarrow X)} \mathbb{C}(-) \circ U,$$

where $U\colon (\Delta \downarrow X) \to \Delta$ is the forgetful functor. The simplicial category $\mathbb{C}X$ may be viewed as a free simplicial category over the simplicial set X.

Then the equality (1.1) takes the form

$$\mathbf{sSet}(\Delta[n], N\mathcal{S}) \cong \mathrm{sCat}(\mathbb{C}\Delta[n], \mathcal{S}),$$

telling us that $\mathbb{C}$ is left adjoint of the simplicial nerve functor $N\colon \mathrm{sCat} \to \mathbf{sSet}$. This adjunction is a Quillen equivalence if $\mathbf{sSet}$ is equipped with the Joyal model structure (where the fibrant objects are the quasicategories) and sCat is equipped with the Bergner model structure (where the weak equivalences are the Dwyer–Kan equivalences). The simplicial nerve $N\mathcal{S}$ of a simplicial category $\mathcal{S}$ is a quasicategory if $\mathcal{S}$ has the property that $\mathrm{Map}_{\mathcal{S}}(X, Y)$ is a Kan complex for all X and Y in $\mathcal{S}$.

The *homotopy category* of a simplicial set X is the homotopy category of the associated simplicial category $\mathbb{C}X$. A map $f\colon X \to Y$ of simplicial sets is a *categorical equivalence* if the simplicial functor $\mathbb{C}f\colon \mathbb{C}X \to \mathbb{C}Y$ is a Dwyer–Kan equivalence. This notion extends the notion of equivalences of quasicategories. Since the adjoint pair $(\mathbb{C}, N)$ is a Quillen equivalence, every simplicial set X is categorically equivalent to a quasicategory, namely $N\mathbb{C}X$.

If X is a quasicategory, then the space of morphisms $\mathrm{map}(x, y)$ between two objects of X is not isomorphic to the simplicial set $\mathrm{Map}_{\mathbb{C}X}(x, y)$. However, it is weakly equivalent to it. Note that $\mathrm{map}(x, y)$ is always a Kan complex, while $\mathrm{Map}_{\mathbb{C}X}(x, y)$ need not be so. On the other hand, $\mathrm{Map}_{\mathbb{C}X}(-, -)$ is equipped with a composition law while $\mathrm{map}(-, -)$ is not.

1.2 The Yoneda functor

Every simplicial set X has an *opposite* X^{op} with $(X^{\mathrm{op}})_n = X_n$ and $(d^{\mathrm{op}})^n_i = d^n_{n-i}$ and $(s^{\mathrm{op}})^n_i = s^n_{n-i}$. For example, if σ is a 1-simplex in X from $x = d^1_1\sigma$ to $y = d^1_0\sigma$ then σ is also a 1-simplex in X^{op} from $(d^{\mathrm{op}})^1_1\sigma = y$ to $(d^{\mathrm{op}})^1_0\sigma = x$. Note that, for a small category $\mathcal{C}$, there is a canonical isomorphism $(N\mathcal{C})^{\mathrm{op}} \cong N(\mathcal{C}^{\mathrm{op}})$. Note also that, if X is a quasicategory, then X^{op} is a quasicategory with the same objects and morphisms reversed.

Let Kan denote the full subcategory of simplicial sets whose objects are the Kan complexes. It is a simplicial category where the space of maps from X to Y is the simplicial set $\mathrm{Map}(X, Y)$, which is also a Kan complex. The *quasicategory of spaces* is the simplicial nerve $N(\mathrm{Kan})$, which will be denoted by $\mathbf{S}$.

For a quasicategory X, the *quasicategory of presheaves* on X is defined as

$$P(X) = \mathrm{Fun}(X^{\mathrm{op}}, \mathbf{S}).$$

The *Yoneda functor*

$$¥\colon X \longrightarrow P(X)$$

is adjunct to the functor

$$X^{\mathrm{op}} \times X \longrightarrow \mathbf{S},$$

which, in its turn, is adjunct to the composite

$$\mathbb{C}(X^{\mathrm{op}} \times X) \longrightarrow (\mathbb{C}X)^{\mathrm{op}} \times \mathbb{C}X \longrightarrow \mathrm{Kan},$$

where the first arrow is induced by the projections from $X^{\mathrm{op}} \times X$ to its factors together with the natural isomorphism $(\mathbb{C}X)^{\mathrm{op}} \cong \mathbb{C}(X^{\mathrm{op}})$, and the second arrow sends each pair of objects x and y of X to the Kan complex $\mathrm{Sing}|\,\mathrm{Map}_{\mathbb{C}X}(x, y)|$.

The functor ¥ is fully faithful and has the property that

$$\mathrm{map}_{P(X)}(¥(p), F) \simeq F(p)$$

for all X and every functor $F\colon X^{\mathrm{op}} \to \mathbf{S}$. This is the infinity-version of the Yoneda lemma.

The images under ¥ of the objects of X generate $P(X)$ under colimits, in the sense that no proper full subcategory of $P(X)$ closed under colimits contains the essential image of ¥. This is analogous to the fact that every presheaf is a colimit of representables.

1.3 Compact objects

For a quasicategory X, every object $p \in X$ yields a functor

$$¥_p\colon X \longrightarrow \mathbf{S}$$

defined as the composite of ¥ with the *evaluation* functor $\mathrm{ev}_p\colon P(X) \to \mathbf{S}$ sending $F \mapsto F(p)$. For a regular cardinal λ, an object $p \in X$ is called λ-*compact* if $¥_p$ preserves λ-filtered colimits, that is, for every functor $f\colon K \to X$ where K is small and λ-filtered admitting a colimit $\tilde{f}\colon K^{\triangleright} \to X$ the composite $¥_p \circ \tilde{f}$ is also a colimit.