Presheaves and the Yoneda Embedding

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1.1 Presheaves

A (set-valued) *presheaf* on a category $\mathcal{C}$ is a functor

$$F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

The motivating example is the category $\mathcal{O}_X$ of open sets in a topological space X, with morphisms the inclusions. Thus a presheaf on $\mathcal{O}_X$ is a functor assigning a set FU to each open set U in X and a *restriction* function $r_{V,U}\colon FV \to FU$ whenever $U \subseteq V$. Functoriality ensures that $r_{U,U}$ is the identity for all U and $r_{W,V} \circ r_{V,U} = r_{W,U}$ if $U \subseteq V \subseteq W$.

We denote by $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ the category of presheaves on a category $\mathcal{C}$, where morphisms are natural transformations. Categories of presheaves are complete and cocomplete; in fact, limits and colimits are created in the category of sets.

A *simplicial set* is a presheaf on the category Δ whose objects are the ordered sets $[n] = \{0, \ldots, n\}$ and whose morphisms are order-preserving functions. Further details about simplicial sets will be given in a next lecture.

1.2 The Yoneda embedding

Given any category $\mathcal{C}$, we consider the functor

$$¥\colon \mathcal{C} \longrightarrow \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$$

sending each object X to $\mathcal{C}(-,X)$ and each morphism $f\colon X \to Y$ to the natural transformation $\mathcal{C}(-,f)\colon \mathcal{C}(-,X) \to \mathcal{C}(-,Y)$. The functor $\mathcal{C}(-X)$ is said to be *represented* by X, or a *representable* functor.

The *Yoneda Lemma* states that there is a natural bijection

$$\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(\mathcal{C}(-,X), F) \cong FX$$

for all $X \in \mathcal{C}$ and every presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. As a special case of this general fact, there are natural bijections

$$\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(\mathcal{C}(-,X), \mathcal{C}(-,Y)) \cong \mathcal{C}(X,Y)$$

for all X and Y. This tells us that the functor ¥ is fully faithful. Hence we may view it as the inclusion of a full subcategory, and we call it the *Yoneda embedding*. Consequently, every category $\mathcal{C}$ embeds into a category that is complete and cocomplete. Moreover, as we next explain, every object of $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is a colimit of objects in the image of ¥, that is, representable functors.

1.3 The Density Theorem

Suppose given functors $F\colon \mathcal{A} \to \mathcal{C}$ and $G\colon \mathcal{B} \to \mathcal{C}$. The *comma category* $(F \downarrow G)$ has objects the morphisms $f\colon FA \to GB$ with $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $f \in \mathcal{C}(FA, GB)$. Morphims in $(F \downarrow G)$ are natural transformations.

As special cases, for an object $X \in \mathcal{C}$ we defined the *slice* category $(\mathcal{C} \downarrow X)$ as the comma category $(\mathrm{Id}_{\mathcal{C}} \downarrow X)$, where X is viewed as a functor $* \to \mathcal{C}$ with value X. Similarly, the *coslice* category $(X \downarrow \mathcal{C})$ is the comma category $(X \downarrow \mathrm{Id}_{\mathcal{C}})$.

For a simplicial set X, the *category of simplices* of X is the slice category $(\Delta \downarrow X)$, that is, the comma category $(\yen \downarrow X)$ where $\yen$ is the Yoneda embedding of Δ into the category $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ of simplicial sets.

Every simplicial set X has the property that

$$X \cong \operatorname*{colim}_{(\Delta \downarrow X)} \Delta[n],$$

and this is a special case of the following general result:

Theorem 1.1 (Density Theorem)**.** *Let $\mathcal{C}$ be any category and X a presheaf on $\mathcal{C}$. Then X is isomorphic to the colimit of the diagram*

$$(\yen \downarrow X) \xrightarrow{U} \mathcal{C} \xrightarrow{\yen} \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$$

where U sends each pair (A, a) with $A \in \mathcal{C}$ and $a \in XA$ to A, and $\yen(A) = \mathcal{C}(-, A)$.

This colimit is often denoted as a *coend*

$$\int^{\mathcal{C}^{\mathrm{op}}} XA \times \mathcal{C}(-, A).$$

In other words, the Density Theorem says that the left Kan extension of the Yoneda embedding along itself is the identity functor of $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

As a consequence of the Density Theorem, every presheaf on $\mathcal{C}$ is a canonical colimit of representable presheaves.

1.4 Geometric realization

The *geometric realization* of a simplicial set X is the topological space defined as

$$|X| = \int^{\Delta^{\mathrm{op}}} X_n \times \Delta^n,$$

where $\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \le t_i \le 1,\ \Sigma t_i = 1\}$. In other words, geometric realization is the left Kan extension of the Yoneda embedding from Δ to simplicial sets, that sends $[n]$ to $\Delta[n]$ for all n. We may paraphrase this fact by saying that $|X|$ is constructed by means of copies of Δ^n for all n in the same way as X is constructed by means of copies of $\Delta[n]$ for all n.