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The Banzhaf Value in the Presence of Externalities

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Abstract: We propose two generalizations of the Banzhaf value for partition function form games. In both cases, our approach is based on probability distributions over the set of possible coalition structures that may arise for any given set of agents. First, we introduce a family of values, one for each collection of the latter probability distributions, defined as the Banzhaf value of an expected coalitional game. Then, we provide two characterization results for this new family of values within the framework of all partition function games. Both results rely on a property of neutrality with respect to amalgamation of players. Second, as this collusion transformation fails to be meaningful for simple games in partition function form, we propose another generalization of the Banzhaf value which also builds on probability distributions of the above type. This latter family is characterized by means of a neutrality property which uses an amalgamation transformation of players for which simple games are closed.

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1 Introduction

One of the most fruitful applications of game theory into political sciences is the study of power in decision making bodies. Typically, power is understood as the ability of an agent to affect the outcome of a voting procedure. A number of power indices have been proposed in the literature to quantify this ability. The most relevant of these measures are the Shapley-Shubik index (Shapley and Shubik, 1954) and the Banzhaf index (Banzhaf, 1964). Both of them build on a common underlying assumption: members of a legislature (be they individuals or parties) may form coalitions, and a coalition is *winning* if and only if the votes of its members account for at least as many as an exogenously given threshold (most often the majority). Accordingly, the bargaining possibilities of the members of a legislature can be described by a simple coalitional game, which assigns 1 to winning coalitions and 0 to the rest.

Nevertheless, quite often there are voting procedures in legislatures in which endogenous majority rules are used instead of fixed majority rules. For instance, when the proclamation of the president is a duty of the parliament and no candidate receives the support of at least half of the legislature, governments backed by only a minority of the legislators may be formed.[1] In these cases, the minority government takes hold because the parties that do not back it cannot agree on voting in favor of an alternative candidate. When endogenous voting rules apply, the bargaining possibilities of the parties cannot be fully captured by a coalitional game. The reason is the following: whether a coalition of parties is winning or not is not only determined by how many seats the coalition has, but also by the voting strategies of the remaining parties.

To encompass situations like the previous one, Thrall and Lucas (1963) devised the so-called *partition function form games*. In a partition function form game, the value of a coalition depends on how the players outside the coalition are organized, hence allowing for externalities to be incorporated into the framework of coalitional games. Some years later, Myerson (1977) proposed and characterized an extension of the Shapley value for partition function form games. It has only been in the later times, however, that these games have gained importance and some relevant contributions on the topic have been made. For instance, Hafalir (2007) studies the core of coalitional games with externalities and obtains necessary conditions for its non-emptiness. The problem of extending the Shapley value to games in partition function form has captured great attention and several alternatives to Myerson (1977) have been proposed. First, Macho-Stadler et al. (2007) characterize a value that can be obtained as the Shapley payoffs of an average game. Second, de Clippel and Serrano (2008) follow an axiomatic approach to single out a value that is not affected by externalities. Third, Dutta et al. (2010) provide a family of values using the notion of potential and characterize it by means of a reduced game property (see Hart and Mas-Colell (1989)). Other recent contributions include Albizuri et al. (2005) and Pham Do and Norde (2007). To the best of our knowledge, however, no extension of the Banzhaf

[1] For instance, this is the current situation of the regional governments of the Basque Country and Catalonia.

value has been yet proposed within the framework of all games in partition function form.

The Banzhaf value was first proposed for voting games (Banzhaf, 1964; Penrose, 1946) and later on extended to the class of all coalitional games by Owen (1975). As for the Shapley value, marginal contributions of players to coalitions are the basis of the Banzhaf value. However, whereas the Shapley value considers orderings of players, the Banzhaf value considers coalitions of players. From an axiomatic perspective, the main difference is that the Shapley value is efficient while the Banzhaf value satisfies interesting amalgamation neutrality properties (Haller, 1994; Nowak, 1997; Casajus, 2012).

In the first part of this paper, we propose a family of values for arbitrary games in partition function form that are obtained by applying the Banzhaf value to an expected coalitional game. Our approach is based on probability distributions over the set of possible coalition structures that may arise for any given set of players. Then, we provide two characterization results of these values by means of an appealing amalgamation neutrality property.

In the second part of the paper, we focus on partition function form games which are simple. As the collusion transformation used in the above characterization results fails to be meaningful for this subclass of games, we propose another family of values for arbitrary games of partition function form which are obtained by applying the Banzhaf value to a different expected coalitional game. These latter values are then characterized similarly as the former ones, with the modification that the previous amalgamation property is substituted by a different version that uses a collusion transformation for which simple games are closed. Both families of values coincide when there are no externalities.

We note that, in the framework of simple games with externalities, different extensions of the Banzhaf value have been studied. The paper with the most similar motivation to ours is Bolger (1990), where an extension of the Banzhaf value is proposed and characterized for multi-candidate voting games. Specifically, Bolger (1990) studies values that can be defined as weighted averages of marginal contributions, and he characterizes one particular value. When restricted to simple games, our family of indices is not included within the class of values studied by Bolger (1990), although, very interestingly, it satisfies the dummy-independence property of his characterization.

The rest of the paper is organized as follows. In Section 2, we present the first family of values for games in partition function form. In Section 3, we prove two characterization results. In Section 4, we turn our interest to simple games in partition function form, and we present and characterize the second family of values for games in partition function form.

2 A Banzhaf-based value

Let Ω be a (possibly infinite) set of potential players. We denote by $N \subseteq \Omega$ any finite set of players in Ω. We denote the cardinality of N by n. A *partition* of N, denoted by P, is a division of N into pairwise disjoint coalitions, i.e., $P \subseteq \{S : S \subseteq N\}$ such that $\cup_{S\in P} = N$ and for every $S, T \in P$ with $S \neq T$, $S \cap T = \emptyset$. By convenience, we assume that the empty set is an element of every partition. We denote by $\mathcal{P}(N)$ the set of all partitions of the finite set N. An *embedded coalition of* N is a pair (S, P) where $P \in \mathcal{P}(N)$ and $S \in P$. The set of embedded coalitions of N is $EC^N = \{(S, P) : P \in \mathcal{P}(N) \text{ and } S \in P\}$. Given $S \subseteq N$ and $i \in N$, S_{-i} (resp. S_{+i}) stands for the set $S \setminus \{i\}$ (resp. $S \cup \{i\}$). Similarly, given $P \in \mathcal{P}(N)$ and $S \in P$, $P_{-S} \in \mathcal{P}(N \setminus S)$ denotes the partition $P \setminus \{S\}$.

A *game in partition function form* (or simply a *game*) is a pair (N, v), where $N \subseteq \Omega$ and v is a function that assigns to every embedded coalition its "worth", i.e., $v : EC^N \to \mathbb{R}$, with the convention that for every $P \in \mathcal{P}(N)$, $v(\emptyset, P) = 0$. The set of all games in partition function form with common player set N is denoted by $\mathcal{G}^N$. The set of games in partition function form with an arbitrary set of players is denoted by $\mathcal{G}$, i.e., $\mathcal{G} = \{\mathcal{G}^N : N \subseteq \Omega\}$. For each $N \subseteq \Omega$, $\mathcal{G}^N$ is a vector space. Let $(N, e_{(S,P)}) \in \mathcal{G}^N$ be the game defined for every $(T, Q) \in EC^N$ by

$$e_{(S,P)}(T, Q) = \begin{cases} 1 & \text{if } S \subseteq T \text{ and } \forall T' \in Q_{-T}, \exists S' \in P \text{ such that } T' \subseteq S', \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

From de Clippel and Serrano (2008), we know that $\{(N, e_{(S,P)}) : (S, P) \in EC^N \text{ and } S \neq \emptyset\}$ constitutes a basis of $\mathcal{G}^N$. Lastly, given $\mathcal{H} \subseteq \mathcal{G}$, a *value* on $\mathcal{H}$ is any mapping, f, that assigns to every game $(N, v) \in \mathcal{H}$ a vector $\mathsf{f}(N, v) \in \mathbb{R}^N$.

2.1 A family of values

In the following, we define a new family of values for games in partition function form. To do so, we require further concepts. For every $N \subseteq \Omega$, let $\lambda^N : EC^N \to \mathbb{R}_+$ be such that for every $S \subseteq N$,

$$\sum_{P\in\mathcal{P}(N):S\in P} \lambda^N(S, P) = 1.$$

I.e., λ^N yields, for any $S \subseteq N$, a probability distribution over the set $\big\{(R, P) \in EC^N : R = S\big\}$. Alternatively, we can interpret that λ^N contains the frequencies with which different coalition structures arise. This second interpretation has a special appeal when we consider the example of the legislature outlined in the Introduction. Indeed, coalitions formed in a parliament among parties vary depending on the bill being voted, and hence λ^N summarizes the average behavior of parties in terms of voting alliances.

We impose two technical conditions. First, we require that the above distributions depend only on the set of players not in S, i.e., for every $N, N' \subseteq \Omega$, $S \subseteq N$, $S' \subseteq N'$, $(S, P) \in EC^N$ and $(S', P') \in EC^{N'}$ such that $P_{-S} = P'_{-S'}$,

$$\lambda^N(S, P) = \lambda^{N'}(S', P'). \tag{2}$$

We denote by Λ any set of collections of probability distributions, one for each possible player set, satisfying Eq. (2), i.e.,

$$\Lambda = \left\{\lambda^N : N \subseteq \Omega, \text{ satisfying Eq. (2)}\right\}. \tag{3}$$

The idea behind (3) is to assume that there is an exogenously given belief on how any set of players organizes itself into a coalition structure.

Second, we say that Λ is *consistent* if for every $N \subseteq \Omega$, $j \in N$, and $(S, P) \in EC^{N_{-j}}$,

$$\lambda^{N_{-j}}(S, P) = \sum_{T \in P_{-S}} \lambda^N(S, P_{-T} \cup T_{+j}). \tag{4}$$

Thus, Λ is consistent if given some $S \subseteq N_{-j}$, the probability that players in $N_{-j} \setminus S$ are organized according to P_{-S} is independent of whether we consider j or not. Lastly, we use $\mathcal{L}$ to denote the set of all consistent Λ. We stress that there exist infinitely many such Λ.

Example 2.1. *In this example, we introduce two possible beliefs, Λ, that satisfy Eqs. (2) and (4). By notational convenience, for $N \subseteq \Omega$ and $S \subseteq N$, let $(S, *) \in EC^N$ and $(S, **) \in EC^N$ be such that $* = \{\emptyset, S, N \setminus S\} \in \mathcal{P}(N)$ and $** = \{\emptyset, S, \{i\}_{i \in N \setminus S}\} \in \mathcal{P}(N)$. Define Λ^* to be such that for every $N \subseteq \Omega$ and $(S, P) \in EC^N$,*

$$\lambda^N(S, P) = \begin{cases} 1 & \textit{if } (S, P) = (S, *), \\ 0 & \textit{otherwise.} \end{cases}$$

Similarly, let Λ^{**} be such that for every $N \subseteq \Omega$ and $(S, P) \in EC^N$,

$$\lambda^N(S, P) = \begin{cases} 1 & \textit{if } (S, P) = (S, **), \\ 0 & \textit{otherwise.} \end{cases}$$

It is straightforward to see that $\Lambda^*, \Lambda^{**} \in \mathcal{L}$. In a monotone context, Λ^* represents the pessimistic belief, whereas Λ^{**} is the optimistic one. In such a context, they constitute the two polar cases.

Following the average approach used in Macho-Stadler et al. (2007), given $\Lambda \in \mathcal{L}$, $N \subseteq \Omega$ and $(N, v) \in \mathcal{G}^N$, we can associate a number with any coalition $S \subseteq N$ by considering the expected worth of S in (N, v) according to $\lambda^N \in \Lambda$, i.e.[2]

$$v^\Lambda(S) = \sum_{P \in \mathcal{P}(N) : S \in P} \lambda^N(S, P) \cdot v(S, P). \tag{5}$$

In a sense, we have incorporated to the game the beliefs on the possible externalities that may arise. We are now in the position to introduce our family of values, one for every $\Lambda \in \mathcal{L}$.

[2]Note that we are actually associating a coalitional game with every game in partition function form.

Definition 2.1. *Given $\Lambda \in \mathcal{L}$, the* Λ-Banzhaf value*, Ba^Λ, is the value defined for every $(N,v) \in \mathcal{G}$ and $i \in N$ by*

$$\mathsf{Ba}_i^\Lambda(N,v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N_{-i}} \left[v^\Lambda(S_{+i}) - v^\Lambda(S) \right]. \tag{6}$$

The values in (6) are built by means of a two-step procedure. First, the beliefs obtained from Λ are used to define the average coalitional game (N, v^Λ). Second, we consider the Banzhaf value of the latter game. Observe that, even though our approach is very similar to Macho-Stadler et al. (2007), the assumptions on the weights differ from those they consider. The reason is that we work with variable player sets, as the next section will make it more evident. Moreover, we do not consider that the above weights are independent of the identity of the players.

3 Characterizations of the Λ-Banzhaf value

In what follows, we consider some properties that a value for games in partition function form may satisfy. In order to introduce the first two properties, we consider an adaptation of the dummy player property to our setting. Given $\Lambda \in \mathcal{L}$ and $N \subseteq \Omega$, we say that $i \in N$ is a Λ-*dummy player in* $(N,v) \in \mathcal{G}$ if for every $S \subseteq N_{-i}$,

$$v^\Lambda(S_{+i}) - v^\Lambda(S) = v^\Lambda(\{i\}), \tag{7}$$

or, equivalently,

$$\begin{aligned} &\sum_{P \in \mathcal{P}(N): S_{+i} \in P} \lambda^N(S_{+i}, P) \cdot v(S_{+i}, P) - \sum_{P \in \mathcal{P}(N): S \in P} \lambda^N(S, P) \cdot v(S, P) \\ = &\sum_{P \in \mathcal{P}(N): \{i\} \in P} \lambda^N(\{i\}, P) \cdot v(\{i\}, P). \end{aligned}$$

That is, i is a Λ-dummy player in $(N,v) \in \mathcal{G}$ if the value she expects to add to any coalition is, according to the probability distribution given by Λ, equal to the value she expects to obtain when she remains a singleton.

Λ-DPP Given $\Lambda \in \mathcal{L}$, a value, f, satisfies the Λ-*dummy player property* if for every $N \subseteq \Omega$ and $(N,v) \in \mathcal{G}^N$ such that $i \in N$ is a Λ-dummy player in (N,v),

$$\mathsf{f}_i(N,v) = v^\Lambda(\{i\}).$$

For convenience, we consider coalitional games (without externalities) as a subclass of games in partition function form. Formally, $(N,v) \in \mathcal{G}$ is a *coalitional game* if for every $S \subseteq N$ and $(S,P_1),(S,P_2) \in EC^N$ with $P_1 \neq P_2$,

$$v(S,P_1) = v(S,P_2).$$

In the case of coalitional games, we may simply denote by $v(S)$ the worth obtained by any coalition $S \subseteq N$, which is independent of how the remaining players organize themselves. The set of all coalitional games with common player set $N \subseteq \Omega$ is denoted by $\mathcal{CG}^N$. If we let $\mathcal{CG} = \cup_{N\subseteq\Omega}\mathcal{CG}^N$, we have $\mathcal{CG} \subseteq \mathcal{G}$. Note that if $(N,v) \in \mathcal{CG}$, i.e. there are no externalities, $v^\Lambda = v$. In this later case, if a player $i \in N$ is a Λ-dummy player in (N,v) for some $\Lambda \in \mathcal{L}$ then it is so for every $\Lambda' \in \mathcal{L}$. In such a situation, we will simply say that i is a dummy player in (N,v) with no reference to Λ. Next, we introduce a weaker version of Λ-DPP, obtained from demanding it only to games in $\mathcal{CG}$.

DPP(W) A value, f, satisfies the *dummy player property in the weak sense* if for every $N \subseteq \Omega$, $(N,v) \in \mathcal{CG}$ such that $i \in N$ is a dummy player in (N,v),

$$\mathsf{f}_i(N,v) = v(\{i\}).$$

In the following, we introduce a property that considers a certain merging of players. Given $\Lambda \in \mathcal{L}$, $N \subseteq \Omega$, $i,j \in N$, with $i \neq j$, and $(N,v) \in \mathcal{G}^N$, the $\{ij\}$*-reduced game*, which we denote by $(N_{-j}, v_{(\Lambda,ij)}) \in \mathcal{G}^{N_{-j}}$, is defined for every $(S,P) \in EC^{N_{-j}}$ by[3]

$$v_{(\Lambda,ij)}(S,P) = \begin{cases} v(S_{+j}, P_{-S} \cup S_{+j}) & \text{if } i \in S, \\ \dfrac{\sum_{T\in P_{-S}} \lambda^N(S,P_{-T}\cup T_{+j})\cdot v(S,P_{-T}\cup T_{+j})}{\sum_{T\in P_{-S}} \lambda^N(S,P_{-T}\cup T_{+j})} & \text{if } i \notin S, \end{cases} \tag{8}$$

if $\sum_{T\in P_{-S}} \lambda^N(S, P_{-T} \cup T_{+j}) > 0$ and

$$v_{(\Lambda,ij)}(S,P) = \begin{cases} v(S_{+j}, P_{-S} \cup S_{+j}) & \text{if } i \in S, \\ 0 & \text{if } i \notin S, \end{cases} \tag{9}$$

otherwise. That is, $(N_{-j}, v_{(\Lambda,ij)})$ is the resulting game in partition function form obtained from (N,v) when j delegates her role on i. More precisely, when i participates in a given coalition, then j goes along with her. However, when i does not participate in a coalition we compute the expected value that is obtained when j joins any other coalition updating the beliefs with Bayes' rule. When no confusion may arise we will denote $(N_{-j}, v_{(\Lambda,ij)})$ simply by (N_{-j}, v_{ij}).

Λ-DNP Given $\Lambda \in \mathcal{L}$, a value, f, satisfies the Λ-*delegation neutrality property* if for every $N \subseteq \Omega$, $(N,v) \in \mathcal{G}$, and $i,j \in N$, with $i \neq j$,

$$\mathsf{f}_i(N,v) + \mathsf{f}_j(N,v) = \mathsf{f}_i(N_{-j}, v_{(\Lambda,ij)}).$$

[3] We recall that $\emptyset$ is a member of any partition.

A version of the above property is first considered in the framework of coalitional games with no externalities by Haller (1994). Recently, Casajus (2012) used the latter to characterize the Banzhaf value.[4] In the particular case of two-player games, i.e., when $N = \{i, j\}$ with $i \neq j$, externalities cannot arise as the only partitions are $\{\emptyset, \{i\}, \{j\}\}$ and $\{\emptyset, \{i, j\}\}$. Hence, the set of games in partition function form coincides in this case with the set of coalitional games. The property below only applies to two-player games.

2-PSP A value, f, satisfies the *2-player standard payoff property* if for every $i, j \in \Omega$ with $i \neq j$ and $(\{i, j\}, v) \in \mathcal{G}^{\{i,j\}}$,

$$\mathsf{f}_i(\{i, j\}, v) = \frac{1}{2}\left[v(\{i, j\}) + v(\{i\}) - v(\{j\})\right].$$

A value satisfying 2-PSP gives for two-player games the payoffs prescribed by many common solution concepts, like the Shapley value, the Banzhaf value or the nucleolus.

In the following we prove a series of results which lead to two characterization results of the family of Λ-Banzhaf values by means of Λ-DNP.

Proposition 3.1. *Let $\Lambda \in \mathcal{L}$. Then, the Λ-Banzhaf value satisfies Λ-DPP, Λ-DNP and 2-PSP.*

Proof. Let $N \subseteq \Omega$, $(N, v) \in \mathcal{G}^N$ and $i \in N$. First, if i is a Λ-dummy player in (N, v),

$$\mathsf{Ba}_i^\Lambda(N, v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N_{-i}} \left[v^\Lambda(S_{+i}) - v^\Lambda(S)\right] = \frac{1}{2^{n-1}} \sum_{S \subseteq N_{-i}} v^\Lambda(\{i\}) = v^\Lambda(\{i\}),$$

where the first equality holds by definition of Ba^Λ and the second equality from the fact that i is a Λ-dummy player in (N, v). Second, let $j \in N_{-i}$. Then,[5]

$$\begin{aligned} \mathsf{Ba}_i^\Lambda(N, v) + \mathsf{Ba}_j^\Lambda(N, v) =& \mathsf{Ba}_i(N, v^\Lambda) + \mathsf{Ba}_j(N, v^\Lambda) = \mathsf{Ba}_i(N_{-j}, (v^\Lambda)_{ij}) \\ =& \mathsf{Ba}_i(N_{-j}, (v_{ij})^\Lambda) = \mathsf{Ba}_i^\Lambda(N_{-j}, v_{ij}), \end{aligned} \tag{10}$$

where the first and last equalities hold from the definition of Ba^Λ, the second equality holds because Ba satisfies the 2-efficiency property for coalitional games as considered in Casajus (2012), and the third equality holds if

$$(v^\Lambda)_{ij} = (v_{ij})^\Lambda. \tag{11}$$

Hence, it only remains to prove Eq. (11). Let $i, j \in N$, with $i \neq j$, and $S \subseteq N_{-j}$. We distinguish two cases.

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[4]See Alonso-Meijide et al. (2012) for a comparison in the framework of coalitional games with no externalities between this property and the 2-efficiency property considered by Nowak (1997).

[5]We denote by Ba the Banzhaf value of coalitional games. We note that it can be defined for every $(N, v) \in \mathcal{CG}$ by $\mathsf{Ba}(N, v) = \mathsf{Ba}^\Lambda(N, v)$, where $\Lambda \in \mathcal{L}$ is arbitrary. The $\{ij\}$-reduced game for coalitional games is then independent of Λ and coincides with the one used in Casajus (2012).

Case 1: $i \in S$.

On the one hand,

$$(v^\Lambda)_{ij}(S) = v^\Lambda(S_{+j}) = \sum_{P' \in \mathcal{P}(N): S_{+j} \in P'} \lambda^N(S_{+j}, P') \cdot v(S_{+j}, P').$$

On the other hand,

$$\begin{aligned}(v_{ij})^\Lambda(S) &= \sum_{P \in \mathcal{P}(N_{-j}): S \in P} \lambda^{N_{-j}}(S, P) \cdot v_{ij}(S, P) \\ &= \sum_{P \in \mathcal{P}(N_{-j}): S \in P} \lambda^{N_{-j}}(S, P) \cdot v(S_{+j}, P_{-S} \cup S_{+j}).\end{aligned}$$

We note that by setting $P' = P_{-S} \cup S_{+j}$, we define a one-to-one correspondence between the set of partitions $P \in \mathcal{P}(N_{-j})$ such that $S \in P$ and the set of partitions $P' \in \mathcal{P}(N)$ such that $S_{+j} \in P'$. Hence, from the two expressions above and Eq. (2) it follows that $(v^\Lambda)_{ij}(S) = (v_{ij})^\Lambda(S)$.

Case 2: $i \notin S$.

On the one hand,

$$\begin{aligned}(v^\Lambda)_{ij}(S) = v^\Lambda(S) &= \sum_{P \in \mathcal{P}(N): S \in P} \lambda^N(S, P) \cdot v(S, P) \\ &= \sum_{P \in \mathcal{P}(N_{-j}): S \in P} \left(\sum_{T \in P_{-S}} \lambda^N(S, P_{-T} \cup T_{+j}) \cdot v(S, P_{-T} \cup T_{+j}) \right).\end{aligned}$$

On the other hand,

$$\begin{aligned}&(v_{ij})^\Lambda(S) = \sum_{P \in \mathcal{P}(N_{-j}): S \in P} \lambda^{N_{-j}}(S, P) \cdot v_{ij}(S, P) \\ &= \sum_{P \in \mathcal{P}(N_{-j}): S \in P} \lambda^{N_{-j}}(S, P) \cdot \left(\frac{\sum_{T \in P_{-S}} \lambda^N(S, P_{-T} \cup T_{+j}) \cdot v(S, P_{-T} \cup T_{+j})}{\sum\limits_{T \in P_{-S}} \lambda^N(S, P_{-T} \cup T_{+j})} \right) \\ &= \sum_{P \in \mathcal{P}(N_{-j}): S \in P} \left(\frac{\lambda^{N_{-j}}(S, P)}{\sum\limits_{T \in P_{-S}} \lambda^N(S, P_{-T} \cup T_{+j})} \right) \left(\sum_{T \in P_{-S}} \lambda^N(S, P_{-T} \cup T_{+j}) \cdot v(S, P_{-T} \cup T_{+j}) \right),\end{aligned}$$

where the first (maybe empty) summation after the second equality is taken only for those $P \in \mathcal{P}(N_{-j})$ such that $\sum_{T \in P_{-S}} \lambda^N(S, P_{-T} \cup T_{+j}) > 0$. When the summation is empty we set it to zero. Under Eq. (4), we obtain $(v^\Lambda)_{ij}(S) = (v_{ij})^\Lambda(S)$.

Third, and last, it is trivial to check that the Λ-Banzhaf value satisfies 2-PSP. □

Next, we provide for every $\Lambda \in \mathcal{L}$ two characterizations of the Λ-Banzhaf value in the framework of all games in partition function form.

Proposition 3.2. *Let $\Lambda \in \mathcal{L}$. There is at most one value that satisfies* DPP(W) *and* Λ-DNP.

Proof. Before showing the desired result, we note that $\{e^N_{(S,*)} : S \subseteq N \text{ and } S \neq \emptyset\}$ forms a basis of the vector subspace $\mathcal{CG}^N$.[6] Additionally, it is useful to describe how the elements of this basis are affected by the merging of players. Indeed, for every $N \subseteq \Omega$, $i, j \in N$ and $S \subseteq N$, it holds that

$$(e^N_{(S,*)})_{ij} = \begin{cases} e^{N_{-j}}_{(S,*)} & \text{if } i \notin S \text{ and } j \notin S, \\ e^{N_{-j}}_{(S_{-j},*)} & \text{otherwise.} \end{cases} \tag{12}$$

First, we prove uniqueness for two-player games.[7] W.l.o.g., let $N = \{1,2\} \subseteq \Omega$ and take $(N, v) \in \mathcal{G}^{\{1,2\}}$. Then, there are $\lambda_1, \lambda_2, \lambda_{12} \in \mathbb{R}$ such that

$$v = \lambda_1 \cdot e_{(\{1\},*)} + \lambda_2 \cdot e_{(\{2\},*)} + \lambda_{12} \cdot e_{(N,*)}.$$

Let $M = \{1,2,3,4\}$ and $(M, z) \in \mathcal{G}^M$ be defined by

$$z = (\lambda_1 - \lambda_2) \cdot e_{(\{4\},*)} + \lambda_2 \cdot \sum_{l \in \{1,2,3\}} (e_{(\{l\},*)} - e_{(N \setminus \{l,4\},*)}) + (\lambda_{12} + 2\lambda_2) \cdot e_{(\{1,2,3\},*)}. \tag{13}$$

Using (12), it easily follows that

$$z_{13} = z_{23} = (\lambda_1 - \lambda_2) \cdot e_{(\{4\},*)} + \lambda_2 \cdot \big(e_{(\{1\},*)} + e_{(\{2\},*)}\big) + \lambda_{12} \cdot e_{(\{1,2\},*)}.$$

Let now $(M_{-3}, w) \in \mathcal{G}^{M_{-3}}$, where $w = z_{13}$. By Λ-DNP,

$$\mathsf{f}_1(M_{-3}, w) = \mathsf{f}_1(M_{-3}, z_{13}) = \mathsf{f}_1(M, z) + \mathsf{f}_3(M, z) \tag{14}$$

and

$$\mathsf{f}_2(M_{-3}, w) = \mathsf{f}_1(M_{-3}, z_{23}) = \mathsf{f}_2(M, z) + \mathsf{f}_3(M, z). \tag{15}$$

From (12) it follows that 1 is a dummy player in the weak sense in $((M_{-3})_{-2}, w_{12})$, implying that

$$\begin{aligned} 2\lambda_2 + \lambda_{12} = \mathsf{f}_1((M_{-3})_{-2}, w_{12}) &= \mathsf{f}_1(M_{-3}, w) + \mathsf{f}_2(M_{-3}, w) \\ &= \mathsf{f}_1(M, z) + \mathsf{f}_2(M, z) + 2\mathsf{f}_3(M, z), \end{aligned} \tag{16}$$

where the first equality holds by DPP(W), the second equality by Λ-DNP and the last equality is due to Eqs. (14) and (15). As 1, 2, and 3 play symmetric roles in z, we can repeat the procedure above exchanging the roles of the three players in the definition of z to obtain

$$\mathsf{f}_1(M, z) = \mathsf{f}_2(M, z) = \mathsf{f}_3(M, z) = \frac{\lambda_2}{2} + \frac{\lambda_{12}}{4}.$$

[6] It is equivalent to the classical basis of coalitional games defined on the set N consisting of unanimity games.

[7] This part is based on the proof of Theorem 7 in Casajus (2012).

Note that, by plugging the above equation into Eqs. (14) and (15), we obtain $\mathsf{f}_1(M_{-3}, w) = \lambda_2 + \frac{\lambda_{12}}{2}$. As $(M_{-3}, w) \in CG^{M_{-3}}$ and 4 is a dummy player in the weak sense in (M_{-3}, w), by DPP(W) we have

$$\mathsf{f}_4(M_{-3}, w) = \lambda_1 - \lambda_2.$$

Lastly, observe that $(M_{-3})_{-4} = \{1, 2\}$ and, by Eq. (12),

$$w_{14} = \lambda_1 \cdot e_{(\{1\},*)} + \lambda_2 \cdot e_{(\{2\},*)} + \lambda_{12} \cdot e_{(\{1,2\},*)} = v.$$

As a consequence,

$$\mathsf{f}_1(N, v) = \mathsf{f}_1(M_{-3}, w) + \mathsf{f}_4(M_{-3}, w) = \lambda_1 + \frac{\lambda_{12}}{2},$$

where the second equality holds by Λ-DNP. Observe that, from (12), we have

$$v_{12} = (\lambda_1 + \lambda_2 + \lambda_{12}) \cdot e_{(1,*)}.$$

Since $(\{1\}, v_{12}) \in \mathcal{G}^{\{1\}}$ and 1 is a dummy player in the weak sense in $(\{1\}, v_{12})$,

$$\lambda_1 + \lambda_2 + \lambda_{12} = \mathsf{f}_1(\{1\}, v_{12}) = \mathsf{f}_1(N, v) + \mathsf{f}_2(N, v),$$

where the last equality holds by Λ-DNP. Therefore, $\mathsf{f}_2(v) = \lambda_2 + \frac{\lambda_{12}}{2}$. All in all, we have seen that $\mathsf{f}(N, v) = \mathsf{Ba}(N, v)$.

Second, suppose that uniqueness holds for every $(N', v) \in \mathcal{G}^{N'}$ where $|N'| < n$ (with $n > 2$) and let $(N, v) \in \mathcal{G}^N$.[8] By Λ-DNP, it holds, for every $i, j \in N$,

$$\mathsf{f}_i(N, v) + \mathsf{f}_j(N, v) = \mathsf{f}_i(N_{-j}, v_{ij}). \tag{17}$$

The right-hand side of Eq. (17) is unique by the inductive hypothesis. Since Eq. (17) holds for every pair $i, j \in N$, with $i \neq j$, we have a system of $\binom{n}{2}$ equations and n unknown variables. It is easy to check that the system has at most one solution. Moreover, by Proposition 3.1 the system is compatible.

□

A careful analysis of the proof reveals that the result in Proposition 3.2 remains true even if we restrict DPP(W) to apply only to games $(N, v) \in \mathcal{G}^N$ where $n \leq 4$. The following result is a straightforward consequence of Propositions 3.1 and 3.2.

Theorem 3.1. *Let $\Lambda \in \mathcal{L}$. The Λ-Banzhaf value is the unique value on $\mathcal{G}$ that satisfies Λ-DNP and DPP(W).*[9]

The next result shows that the Λ-Banzhaf value can be characterized without the need of a dummy player property.

[8] This part is based on Lehrer (1988).

[9] The independence of the axioms in general can be proved analogously as in Casajus (2012).

Theorem 3.2. *Let $\Lambda \in \mathcal{L}$. The Λ-Banzhaf value is the unique value on $\mathcal{G}$ that satisfies* Λ-DNP *and* 2-PSP.[10]

Proof. As in the proof of Proposition 3.2, uniqueness can be shown by induction. The case $n = 2$ follows immediately from 2-PSP. The case $n = 1$ then follows from the case $n = 2$ by Λ-DNP. The inductive reasoning for $n \geq 2$ is the same as in the proof of Proposition 3.2, as it only depends on Λ-DNP. □

4 Simple Games (and Amalgamation Neutrality)

In this section, we focus our attention on a particular subclass of games in partition function form, the so-called *simple games in partition function form.* This subclass comprises those games where the worth of any embedded coalition takes only one of two values. To be consistent with the previous literature on simple (coalitional) games, we further demand a certain notion of monotonicity. Formally, for each $N \subseteq \Omega$ and $(N, v) \in \mathcal{G}$, we consider the following properties:

(i) For every $(S, P) \in EC^N$, $v(S, P) \in \{0, 1\}$,

(ii) $v(N, \{\emptyset, N\}) = 1$,

(iii) For every $S, T \subseteq N$ and $P \in \mathcal{P}(N)$ with $P = \{\emptyset, S, S_1, \ldots, S_m\}$,

$$v(S, P) \leq v(S \cup T, \{\emptyset, S \cup T, S_1 \setminus T, \ldots, S_m \setminus T\}).$$

First, *(i)* requires that an embedded coalition is either winning (and the value is 1) or losing (and the value is 0). Second, *(ii)* requires that the grand coalition is always winning. Third, *(iii)* requires that a winning embedded coalition S can never become losing when more agents join it. A game $(N, v) \in \mathcal{G}$ is a *simple game in partition function form* (just *simple game*) if it satisfies *(i)*, *(ii)* and *(iii)*. Let $\mathcal{SG}$ be the subset of $\mathcal{G}$ that consists of simple games.

The class of games just introduced was studied some time ago under the name of *multi-candidate voting games*. Two generalizations of the Banzhaf value for this class of games are proposed in Bolger (1983, 1986). Later on, Bolger (1990) characterized one of his proposals within the class of values that can be built using a particular weight system. It is worth to mention that the family of values proposed in the current work does not lie within the class of values considered by Bolger (1990), although the Λ-Banzhaf value satisfies the dummy player property of the characterization. Our model should not be confused with the model of games with r-alternatives (see for instance Freixas and Zwicker (2003)). The latter is a model that allows for different levels of approval; that is, besides knowing which coalitions are formed, it is also known what each of the coalitions is voting for. For instance, voting situations where abstention is present can be modelled using a simple game with 3-alternatives.

[10] The independence of the axioms is trivial.

The Λ-Banzhaf value introduced in Section 2 has been proved in Section 3 to be the only value within the framework of all games in partition function form that is invariant with respect to a certain amalgamation of players, namely Λ-DNP, and that satisfies an additional property, either DPP(W) or 2-PSP. When we restrict our attention to simple games, however, the aforementioned amalgamation property makes use of collusion transformation for which simple games are not closed in general. I.e., given $(N, v) \in \mathcal{SG}$ a game with externalities, and $i, j \in N$, we have that $(N_{-j}, v_{(\Lambda,ij)})$ is not in general a simple game, as it can be immediately seen from an inspection of Eq. (8).

In the remaining part of this section, we tackle the imbalance mentioned for simple games between the above value, on the one hand, and the (desired) amalgamation neutrality property, on the other hand, by means of three steps. First, we show that there do exist reasonable amalgamation neutrality properties in the spirit of Λ-DNP that are meaningful in the framework of simple games. Second, for each Λ that satisfies certain mild properties, we define a new value for arbitrary partition function form games, which we call the *modified* Λ*-Banzhaf value*. This value is in general different than the Λ-Banzhaf value, although both values coincide when there are no externalities. Third, we show that the modified Λ-Banzhaf value is the unique value – either within the framework of simple games or the framework of all games – that satisfies a certain amalgamation neutrality property and, additionally, either DPP(W) or 2-PSP.

4.1 A different amalgamation neutrality property

In the following, we modify the property Λ-DNP in a way that it turns out to be meaningful within the framework of simple games. To do so, it is convenient to assume that Λ is such that in many circumstances there is always a unique coalition structure that is the most likely configuration. Formally, this translates into two technical requirements. First, given $N \subseteq \Omega$ and $S \subseteq N$, we assume that there exists a unique $P^{N,S} \in \mathcal{P}(N)$ with $S \in P^{N,S}$ such that

$$\left\{P^{N,S}\right\} = \operatorname*{arg\,max}_{P \in \mathcal{P}(N): S \in P} \lambda^N(S, P). \tag{18}$$

Second, given $N \subseteq \Omega$, $j \in N$ and $(S, P) \in EC^{N_{-j}}$, we assume that there exists a unique $T^{N,j,(S,P)} \in P_{-S}$ such that

$$\left\{T^{N,j,(S,P)}\right\} = \operatorname*{arg\,max}_{T \in P_{-S}} \lambda^N(S, P_{-T} \cup T_{+j}). \tag{19}$$

Additionally, we say that Λ is *consistent for simple games* if for every $N \subseteq \Omega$, $j \in N$ and $S \subseteq N_{-j}$,

$$\operatorname*{arg\,max}_{P \in \mathcal{P}(N): S \in P} \lambda^N(S, P) = \operatorname*{arg\,max}_{P \in \mathcal{X}^{N,j,S}} \lambda^N(S, P), \tag{20}$$

where

$$\mathcal{X}^{N,j,S} := \left\{ P'_{-T} \cup T_{+j} : T \in P'_{-S}, P' \in \operatorname*{arg\,max}_{P'' \in \mathcal{P}(N_{-j}): S \in P''} \lambda^{N_{-j}}(S, P'') \right\}. \tag{21}$$

Thus, Λ is consistent for simple games if given some $S \subseteq N_{-j}$, the most likely configuration of players in $N_{-j} \setminus S$ according to Λ is independent of whether we consider j or not. We let $\mathcal{L}^{\mathcal{SG}}$ denote the set of Λ that satisfy Eqs. (2), (18), (19), and (21).[11] We stress that $\mathcal{L}^{\mathcal{SG}} \neq \emptyset$ as it contains the two Λ defined in Example 2.1.

We are now in the position to introduce the *modified* $\{ij\}$*-reduced game*, which we denote by $(N_{-j}, \tilde{v}_{(\Lambda,ij)}) \in \mathcal{G}^{N_{-j}}$ and is defined, for every $(S,P) \in EC^{N_{-j}}$, by[12]

$$\tilde{v}_{(\Lambda,ij)}(S,P) = \begin{cases} v(S_{+j}, P_{-S} \cup S_{+j}) & \text{if } i \in S, \\ v\left(S, P_{-T^*} \cup T^*_{+j}\right) & \text{if } i \notin S, \text{ where } T^* = T^{N,j,(S,P)}. \end{cases} \tag{22}$$

It is a matter of straightforward calculations to check that $(N_{-j}, \tilde{v}_{(\Lambda,ij)})$ is a simple game if (N,v) is a simple game. When no confusion may arise, we write $(N_{-j}, \tilde{v}_{ij})$ instead of $(N_{-j}, \tilde{v}_{(\Lambda,ij)})$.

Finally, we consider the following amalgamation neutrality property, which can be seen as one natural adaptation of Λ-DNP into the framework of simple games.

$\tilde{\Lambda}$-DNP(S) Let $\Lambda \in \mathcal{L}^{\mathcal{SG}}$. A value on $\mathcal{H} \subseteq \mathcal{G}$, f, satisfies the Λ-*delegation neutrality property for simple games* if for every $N \subseteq \Omega$, $(N,v) \in \mathcal{G}^N \cap \mathcal{H}$, and $i,j \in N$, with $i \neq j$,

$$\mathsf{f}_i(N,v) + \mathsf{f}_j(N,v) = \mathsf{f}_i(N_{-j}, \tilde{v}_{(\Lambda,ij)}).$$

4.2 Another family of values

Next, we introduce a new family of values for arbitrary games, although we focus our attention primarily on simple games. Indeed, note that given a simple game $(N,v) \in \mathcal{SG}$ and $\Lambda \in \mathcal{L}^{\mathcal{SG}}$, the coalitional game (N, v^Λ) defined in Eq. (5) is not in general a simple game. Given $(N,v) \in \mathcal{G}^N$ and $\Lambda \in \mathcal{L}^{\mathcal{SG}}$, we can alternatively define the game $(N, \tilde{v}^\Lambda)$ such that, for each $S \subseteq N$,[13]

$$\tilde{v}^\Lambda(S) = v\left(S, P^{N,S}\right). \tag{23}$$

Trivially, $(N, \tilde{v}^\Lambda)$ is a simple coalitional game. Using Eq. (23), we can now define a new family of values, one for every $\Lambda \in \mathcal{L}^{\mathcal{SG}}$.

Definition 4.1. *Given* $\Lambda \in \mathcal{L}^{\mathcal{SG}}$*, the* modified Λ-Banzhaf value*,* $\tilde{\mathsf{Ba}}^\Lambda$*, is the value defined for every* $(N,v) \in \mathcal{G}$ *and* $i \in N$ *by*

$$\tilde{\mathsf{Ba}}_i^\Lambda(N,v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N_{-i}} \left[\tilde{v}^\Lambda(S_{+i}) - \tilde{v}^\Lambda(S)\right]. \tag{24}$$

[11] Without Eqs. (18) and (19), we would need to assume that $\mathcal{P}(M)$, with $M \subseteq \Omega$, can be completely and exogenously ordered. This complete order would then serve as a tie-breaking rule in the event that two partitions are equally likely.

[12] Note that different Λ define the same amalgamation game as long as the relative likelihood of coalition structures remains unaltered.

[13] Note that different Λ define the same coalitional game as long as the relative likelihood of coalition structures remains unaltered.

It is immediate to check that when $(N, v) \in \mathcal{G}$ is a coalitional game, i.e., it considers no externalities, we have $\tilde{\mathsf{Ba}}^{\Lambda}(N, v) = \mathsf{Ba}^{\Lambda}(N, v)$. For arbitrary simple games, however, both values are in general different.

4.3 More characterization results

We are now in the position to prove two results that characterize the modified Λ-Banzhaf value within the framework of simple games.

Theorem 4.1. *Let $\Lambda \in \mathcal{L}^{\mathcal{SG}}$. The modified Λ-Banzhaf value is the only value on $\mathcal{SG}$ that satisfies $\tilde{\Lambda}$-*DNP(S) *and* DPP(W).

Proof. First, to prove existence, we can simply repeat the proof of Proposition 3.1, which now hinges on

$$\widetilde{(\tilde{v}^{\Lambda})}_{ij} = \widetilde{(\tilde{v}_{ij})}^{\Lambda}. \tag{25}$$

In order to prove Eq. (25), let $i, j \in N$ and $S \subseteq N_{-j}$. We distinguish two cases.

Case 1: $i \in S$.

In this case, let $P^* := P^{N_{-j}, S} \in \mathcal{P}(N_{-j})$ be such that $S \in P^*$ and

$$\left\{P^*_{-S}\right\} = \underset{P_{-S} \in \mathcal{P}(N_{-j} \setminus S)}{\arg\max} \lambda^{N_{-j}}(S, P_{-S} \cup S) = \underset{P_{-S} \in \mathcal{P}(N \setminus S_{+j})}{\arg\max} \lambda^{N}(S_{+j}, P_{-S} \cup S_{+j}),$$

where the second equality holds by Eq. (2). On the one hand,

$$\widetilde{(\tilde{v}^{\Lambda})}_{ij}(S) = \tilde{v}^{\Lambda}(S_{+j}) = v(S_{+j}, P^*_{-S} \cup S_{+j}).$$

On the other hand,

$$\widetilde{(\tilde{v}_{ij})}^{\Lambda}(S) = \tilde{v}_{ij}(S, P^*) = v(S_{+j}, P^*_{-S} \cup S_{+j}).$$

Therefore, Eq. (25) holds.

Case 2: $i \notin S$.

As in the previous case, let $P^* := P^{N_{-j}, S} \in \mathcal{P}(N_{-j})$ be such that $S \in P^*$ and

$$\{P^*\} = \underset{P \in \mathcal{P}(N_{-j}) : S \in P}{\arg\max} \lambda^{N_{-j}}(S, P).$$

Additionally, let $P^{**} := P^{N,S} \in \mathcal{P}(N)$ be such that $S \in P^{**}$ and

$$\{P^{**}\} = \underset{P \in \mathcal{P}(N) : S \in P}{\arg\max} \lambda^{N}(S, P).$$

On the one hand,

$$\widetilde{(\tilde{v}^{\Lambda})}_{ij}(S) = \tilde{v}^{\Lambda}(S) = v\left(S, P^{**}\right) = v\left(S, P^{**}_{-T^{**}} \cup T^{**}\right),$$

where $T^{**} \in P^{**}_{-S}$ is such that $j \in T^{**}$. On the other hand,

$$\widetilde{(\tilde{v}_{ij})}^{\Lambda}(S) = \tilde{v}_{ij}(S, P^*) = v\left(S, P^*_{-T^*} \cup T^*_{+j}\right),$$

where $T^* = T^{N,j,(S,P^*)}$, i.e.,

$$\{T^*\} = \underset{T \in P^*_{-S}}{\arg\max} \quad \lambda^N\left(S, P^*_{-T} \cup T_{+j}\right).$$

Lastly, from Eqs. (20) and (21) it follows that $P^{**}_{-T^{**}} = P^*_{-T^*}$, hence implying that $P^{**}_{-T^{**}} \cup T^{**} = P^*_{-T^*} \cup T^*_{+j}$. As a consequence, Eq. (25) holds.

Second, in order to prove uniqueness for $n = 2$, it suffices to repeat the argument of Corollary 9 in Casajus (2012), noting that his proof for the case $n = 2$ only considers coalitional games, i.e. games without externalities. The induction argument is analogous to the proof of Theorem 3.1. □

Following the main lines of the proof of the above theorem, we can also obtain the next result.

Theorem 4.2. *Let $\Lambda \in \mathcal{L}^{\mathcal{SG}}$. The modified Λ-Banzhaf value is the only value on $\mathcal{SG}$ that satisfies $\tilde{\Lambda}$-*DNP(S) *and* 2-PSP.[14]

Since the distributions given by $\Lambda \in \mathcal{L}^{\mathcal{SG}}$ are independent of the game, the modified Banzhaf value is additive. Moreover, due to the fact that all games defined in (1) constitute a basis of the vector space of all partition function games with common player set N and, additionally, they are simple games, the characterization results in Theorems 4.2 and 4.2 extend to the framework of all games in partition function form.

Corollary 4.1. *Let $\Lambda \in \mathcal{L}^{\mathcal{SG}}$. The modified Λ-Banzhaf value is the only value on $\mathcal{G}$ that satisfies $\tilde{\Lambda}$-*DNP(S) *and* DPP(W).

Corollary 4.2. *Let $\Lambda \in \mathcal{L}^{\mathcal{SG}}$. The modified Λ-Banzhaf value is the only value on $\mathcal{G}$ that satisfies $\tilde{\Lambda}$-*DNP(S) *and* 2-PSP.

In combination with Theorems 3.1 and 3.2, the two above results permit us comparing the two families of values introduced in this paper. On the one hand, we see that demanding $\tilde{\Lambda}$-DNP(S) leads to a value, namely $\tilde{\mathsf{Ba}}^{\Lambda}$, defined in a way that the information contained in Λ is aggregated too abruptly, as opposed to Ba^{Λ}, where the information contained in Λ is aggregated following an average approach.[15] On the other hand, $\tilde{\mathsf{Ba}}^{\Lambda}$, but not Ba^{Λ}, is characterized by the same two results in the framework of all games and in the framework of simple games. As a

[14] The independence of the axioms is trivial.

[15] Of course, there exist other neutrality properties that use different, meaningful collusion transformations for which simple games are closed. The investigation of other such properties lies outside the scope of this paper.

consequence, depending on the specific setting where we want to consider our values, either one or the other value will be more suitable.

Lastly, we want to point out that Theorem 4.2 is interesting from the point of view of the study of the distribution of power in legislatures. Indeed, we say that $i \in N$ *has veto power within* (N, v) if $v(\{S, P\}) = 1$ implies $i \in S$. Then, consider the following property that a value on $\mathcal{SG}$ might satisfy:

2-VPB A value on $\mathcal{SG}$, f, satisfies the *veto property for bilateral systems* if for every $N \subseteq \Omega$ with $n = 2$ and $(N, v) \in \mathcal{SG}$ where only one player, $i \in N$, has veto power,

$$\mathsf{f}_i(N, v) = v(N, \{\emptyset, N\}).$$

The appeal of the above property within two-party systems is quite obvious – see e.g. May (1952). Corollary 4.3 below reveals that if we require the latter property to hold for two-party systems and we want the distribution of power to be (on expectation according to Λ) unaffected by the party agreements that may subscribed within a multi-party system, only one possibility remains: the modified Λ-Banzhaf value.

Corollary 4.3. *Let* $\Lambda \in \mathcal{L}^{\mathcal{SG}}$. *The modified* Λ*-Banzhaf value is the only value on* $\mathcal{SG}$ *that satisfies* $\tilde{\Lambda}$-DNP(S) *and* 2-VPB.[16]

Proof. By repeating the induction argument in the proof of Theorem 3.2, we can focus our attention to simple games $(N, v) \in \mathcal{SG}$ with $n = 2$. Note that there are only four such games. Indeed, let w.l.o.g. $N = \{1, 2\}$. Then, $(N, e_{(\emptyset,*)}), (N, e_{(\{1\},*)}), (N, e_{(\{2\},*)})$ and $(N, e_{(N,*)})$ are the only simple games that exist. Moreover, since $n = 2$, they are all coalitional games. For coalitional games, $\tilde{\Lambda}$-DNP(S) implies symmetry for any Λ.[17] In particular, for every N with $n = 2$ and $(N, v) \in \mathcal{SG}$ where either all players have veto power or no player does, $\mathsf{f}_i(N, v) = \mathsf{f}_j(N, v)$ for all $i, j \in N$. That is, a value that satisfies $\tilde{\Lambda}$-DNP(S) and 2-MPB within $\mathcal{SG}$ is uniquely determined for $(N, e_{(\emptyset,*)}), (N, e_{(\{1\},*)}), (N, e_{(\{2\},*)})$ and $(N, e_{(N,*)})$. □

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[16] The independence of the axioms is trivial.

[17] See Theorem 1 in Casajus (2012).

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