Quantitative Equidistribution for the Solutions of Systems of Sparse Polynomial Equations

Carlos D'Andrea

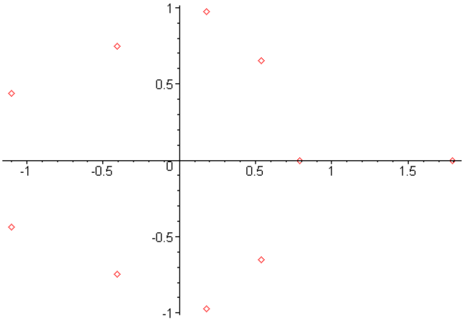
SIAM Conference on Applied Algebraic Geometry – Raleigh NC
October 2011

Roots of polynomials vs "controlled" coefficients

Let f be a polynomial of degree $d \gg 0$ with coefficients in $\{-1, 0, 1\}$. I will plot all complex solutions of $f = 0$ ...

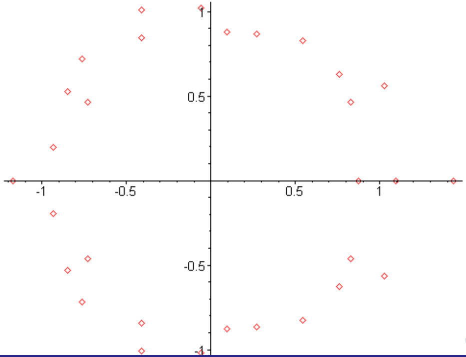
For instance, let $d = 10$ and

$$f = -x^{10} + x^9 + x^8 + x^6 + x^5 - x^4 + x^3 - x^2 + x - 1$$

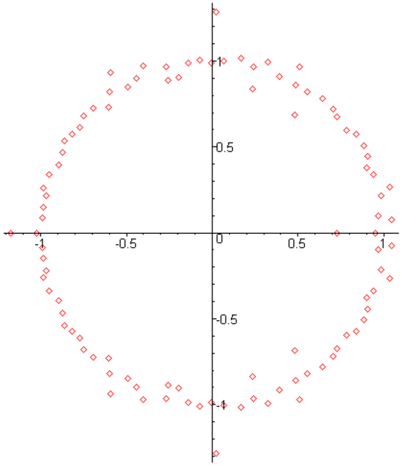


Now set $d = 30$ and

$$\begin{aligned} f \;=\; & x^{30} - x^{29} - x^{28} + x^{26} + x^{25} - x^{24} - x^{23} - x^{22} \\ & + x^{21} - x^{20} + x^{19} + x^{18} + x^{16} + x^{15} - x^{14} \\ & + x^{13} + x^{12} + x^{10} + x^{9} - x^{6} + x^{5} - 1 \end{aligned}$$



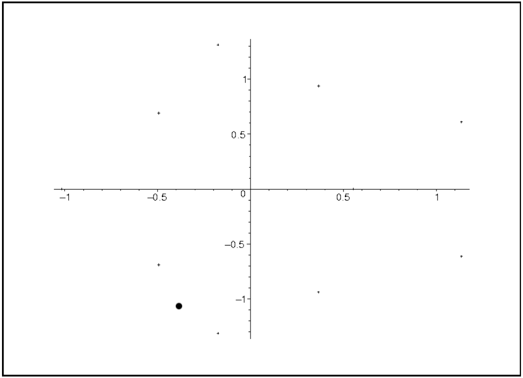
$d = 100$ and
$f = -x^{100} - x^{98} + x^{96} + x^{94} - x^{93} + x^{92} - x^{91} - x^{90} + \cdot\cdot$



A more ambitious experiment

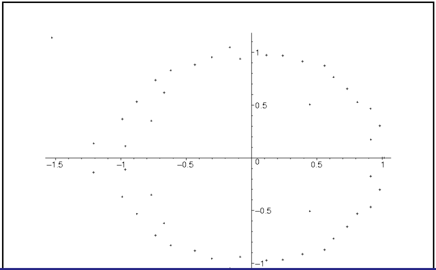
Let us say now that f has degree $d \gg 0$ with integer coefficients between $-d$ and d. What happens now?

$d = 10$ and $f = -6 + 8x - x^2 + 10x^3 - 3x^4 + 8x^5 + 4x^6 - 9x^7 + 9x^8 - 6x^9 + 5x^{10}$



$d = 50$ and

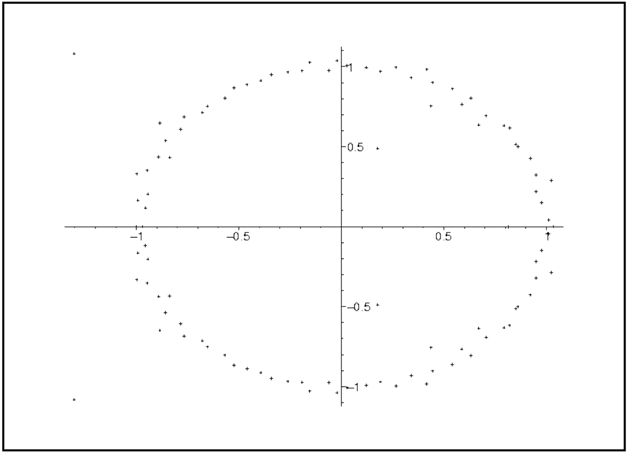
$$f = -24 + 12x - 44x^{48} - 48x^{49} - 42x^{28} + 15x^{29} + 34x^{26} + 22x^{27} - 24x^{24} + 29x^{25} + 14x^{2} - 40x^{3} - 48x^{4} + 35x^{5} + 24x^{6} + 27x^{7} - 3x^{8} - 15x^{9} - 21x^{10} + 12x^{14} - 15x^{50} - 14x^{33} + 38x^{34} + 10x^{35} - 23x^{36} + 48x^{37} + 30x^{38} - 23x^{39} - 31x^{40} + 2x^{41} + 24x^{42} + 9x^{43} - 15x^{44} - 29x^{45} + 45x^{46} + 40x^{47} + 40x^{31} - 40x^{32} + 38x^{11} + 8x^{12} - 16x^{13} - 39x^{15} + 2x^{16} - 38x^{17} - x^{18} + 16x^{19} - 44x^{20} - 20x^{21} + 22x^{22} + 28x^{23} + 32x^{30}$$



$d = 100$ and

$f=$

$$30-45x-91x^{74}-33x^{75}+4x^{73}-59x^{79}+35x^{92}-57x^{48}+49x^{49}+2x^{93}-87x^{28}-16x^{29}-78x^{26}-31x^{27}+19x^{50}-73x^{24}-63x^{25}+98x^{2}+29x^{3}-97x^{4}+47x^{5}+46x^{6}-88x^{7}-74x^{8}-60x^{9}-62x^{10}-27x^{81}-82x^{80}-92x^{78}-50x^{77}-41x^{76}-21x^{95}+8x^{66}-7x^{67}+75x^{64}-19x^{94}-48x^{63}+92x^{65}-18x^{60}+53x^{61}+84x^{59}-15x^{57}-13x^{58}-64x^{91}+84x^{90}-54x^{89}+67x^{55}-81x^{56}-27x^{54}-61x^{88}+43x^{87}+49x^{86}+51x^{84}-12x^{85}-64x^{83}+52x^{82}+43x^{70}-91x^{71}-97x^{72}+76x^{68}+14x^{69}+73x^{99}-56x^{97}+41x^{98}+73x^{96}+44x^{100}+2x^{51}-79x^{52}+87x^{53}-43x^{14}+39x^{62}+50x^{33}+53x^{34}+64x^{35}+57x^{36}-57x^{37}-31x^{38}+85x^{39}+30x^{40}-49x^{41}+6x^{42}-82x^{43}+34x^{44}+59x^{45}+7x^{46}+91x^{47}+59x^{31}+58x^{32}-4x^{11}-71x^{12}-68x^{13}+74x^{15}+60x^{16}-3x^{17}+23x^{18}-55x^{19}+80x^{20}-32x^{21}+17x^{22}-14x^{23}-69x^{30}$$



1
0.5
−1
−0.5
0.5
1
−0.5
−1

What is going on???

The Erdös-Turán theorem

Let $f(x) = a_d x^d + \cdots + a_0 = a_d(x - \rho_1 \, e^{i\,\theta_1}) \cdots (x - \rho_d \, e^{i\,\theta_d})$

Definition

The *angle discrepancy* of f is

$$\Delta_\theta(f) := \sup_{0 \le \alpha < \beta < 2\pi} \left| \frac{\#\{k : \alpha \le \theta_k < \beta\}}{d} - \frac{\beta - \alpha}{2\pi} \right|$$

The ε*-radius discrepancy* of f is

$$\Delta_{\mathrm{r}}(f;\varepsilon) := \frac{1}{d} \, \# \left\{ k : 1 - \varepsilon < \rho_k < \frac{1}{1-\varepsilon} \right\}$$

Also set $||f|| := \sup_{|z|=1} |f(z)|$

Theorem [Erdös-Turán 1948], [Hughes-Nikeghbali 2008]

$$\Delta_\theta(f) \le c\sqrt{\frac{1}{d}\log\left(\frac{||f||}{\sqrt{|a_0 a_d|}}\right)} \quad , \quad 1-\Delta_{\mathrm{r}}(f;\varepsilon) \le \frac{2}{\varepsilon d}\log\left(\frac{||f||}{\sqrt{|a_0 a_d|}}\right)$$

Here $\sqrt{2} \le c \le 2{,}5619$ [Amoroso-Mignotte 1996]

Corollary: the equidistribution

Let $f_d(x)$ of degree d such that $\log\left(\frac{||f_d||}{\sqrt{|a_{d,0}a_{d,d}|}}\right) = o(d)$, then

$$\lim_{d\to\infty}\frac{1}{d}\#\left\{k : \alpha \le \theta_{dk} < \beta\right\} = \frac{\beta-\alpha}{2\pi}$$

$$\lim_{d\to\infty}\frac{1}{d}\#\left\{k : 1-\varepsilon < \rho_{dk} < \frac{1}{1-\varepsilon}\right\} = 1$$

Some consequences

1. The number of real roots of f is $\leq 51\sqrt{d \log\left(\frac{||f||}{\sqrt{|a_0 a_d|}}\right)}$ [Erhardt-Schur-Szego]

2. If $g(z) = 1 + b_1 z + b_2 z^2 + \ldots$ converges on the unit disk, then the zeros of its d-partial sums distribute uniformely on the unit circle as $d \to \infty$ [Jentzsch-Szego]

Equidistribution in several variables (joint work with Martin Sombra & André Galligo)

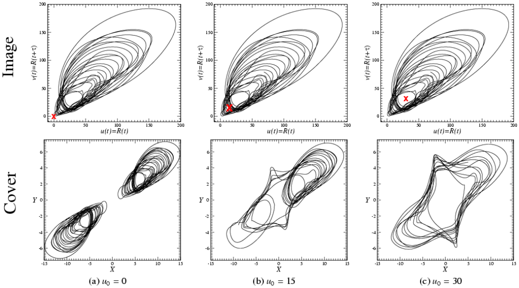
* For a finite sequence of points $\mathbf{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_m\} \subset (\mathbb{C}^\times)^n$, we can define $\Delta_\theta(\mathbf{P})$ and $\Delta_{\mathrm{r}}(\mathbf{P}, \varepsilon)$

* Every such set $\mathbf{P}$ is the solution set of a complete intersection $\boldsymbol{f} = 0$ with $\boldsymbol{f} = (f_1, \ldots, f_n)$ Laurent Polynomials in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

Problem

* Estimate $\Delta_\theta(\mathbf{P})$ and $\Delta_{\mathrm{r}}(\mathbf{P}, \varepsilon)$ in terms of $\boldsymbol{f}$
* Which is the analogue of $\frac{||f||}{\sqrt{|a_0 a_d|}}$ in several variables?
* Equidistribution theorems

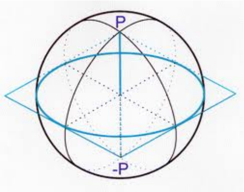
Some Evidence



Singularities of families of algebraic plane curves with "controlled" coefficients tend to the equidistribution
[Diaconis-Galligo]

More Evidence: equidistribution of algebraic points

A sequence of algebraic points $\{\mathbf{p}_k\}_{k\in\mathbb{N}} \subset (\mathbb{C}^*)^n$ such that $\deg(\mathbf{p}_k) = k$ and $\lim_{k\to\infty} h(\mathbf{p}_k) = 0$ "equidistributes" in $S^1 \times S^1 \times \ldots \times S^1$
[Bilu 1997]



The multivariate setting

For $f_1, \ldots, f_n \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ consider

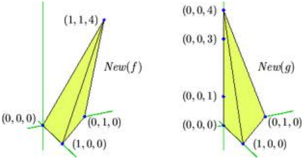
$$V(f_1, \ldots, f_n) = \{\xi \in (\mathbb{C}^\times)^n : f_1(\xi) = \cdots = f_n(\xi) = 0\} \subset (\mathbb{C}^\times)^n$$

and V_0 the subset of isolated points Set $Q_i := \mathrm{N}(f_i) \subset \mathbb{R}^n$ the

Newton polytope, then

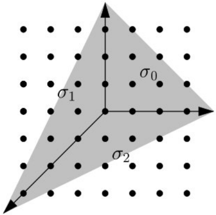
$$\# V_0 \leq \mathrm{MV}_n(Q_1, \ldots, Q_n) =: D \quad \text{[BKK]}$$

From now on, we will assume $\# V_0 = D$, in particular $V(\boldsymbol{f}) = V_0$.



A toric variety in the background

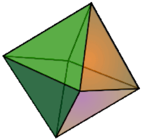
$\#V_0 = D$ is equivalent to the fact that the system $f_1 = 0, \ldots, f_n = 0$ does not have solutions in the toric variety associated to the polytope $Q_1 + Q_2 + \ldots + Q_n$ [Bernstein 1975], [Huber-Sturmfels 1995]



Can be tested with resultants “at infinity”!

A multivariate Erdös-Turán measure

f	$\leftrightarrow$	"multidirectional" Chow forms
$a_0,\ a_d$	$\leftrightarrow$	facet resultants



$$E_{\boldsymbol{f},\mathbf{a}}(z) = \mathrm{Res}_{\{\mathbf{0},\mathbf{a}\},\mathcal{A}_1,\dots,\mathcal{A}_n}(z - \boldsymbol{x}^{\mathbf{a}}, f_1, \dots, f_n)$$

$$\eta(\boldsymbol{f}) = \sup_{\mathbf{a}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} \frac{1}{D\|\mathbf{a}\|} \log\left(\frac{\|E_{\boldsymbol{f},\mathbf{a}}(z)\|}{\prod_v |\mathrm{Res}_{\mathcal{A}_1^v,\dots,\mathcal{A}_n^v}(f_1^v,\dots,f_n^v)|^{\frac{|\langle v,\mathbf{a}\rangle|}{2}}}\right)$$

Theorem (D-Galligo-Sombra)

- $\eta(\boldsymbol{f}) < +\infty$
- For $n = 1, \eta(f)$ coincides with the Erdös-Turán measure $\frac{\|f\|}{\sqrt{|a_0 a_D|}}$
- If $f_1, \ldots, f_n \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $\boldsymbol{f} = 0$ has $D > 0$ zeroes, then

$$\Delta_\theta(\boldsymbol{f}) \leq c(n)\eta(\boldsymbol{f})^{\frac{1}{3}} \log^+\left(\frac{1}{\eta(\boldsymbol{f})}\right) \quad , \quad 1-\Delta_{\mathrm{r}}(f;\varepsilon) \leq c(n)\,\eta(\boldsymbol{f})$$

with $c(n) \leq 2^{3n} n^{\frac{n+1}{2}}$

Corollary (D-Galligo-Sombra)

The number of real roots of a sparse system $\boldsymbol{f} = 0$ with $f_1, \ldots, f_n \in \mathbb{R}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is bounded above by

$$D\, c'(n) \eta(\boldsymbol{f})^{\frac{1}{3}} \log^+ \left(\frac{1}{\eta(\boldsymbol{f})} \right)$$

with $c'(n) \leq 2^{4n} n^{\frac{n+1}{2}}$

Estimates on $\eta(\boldsymbol{f})$

Suppose $Q_i \subset d_i\Delta + \mathbf{a}_i$, with Δ being the fundamental simplex of $\mathbb{R}^n$. Then

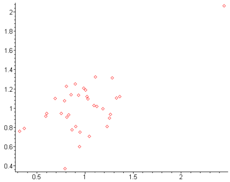
$$\eta(\boldsymbol{f}) \quad < \quad \frac{1}{D}\Bigg(2nd_1 \ldots d_n \sum_{j=1}^{n} \frac{\log \|f_j\|_{\sup}}{d_j} + \\ \frac{1}{2}\sum_{\boldsymbol{v}} \|\boldsymbol{v}\| \log^+ |\operatorname{Res}_{\mathcal{A}_1^v,\ldots,\mathcal{A}_n^v}(f_1^v,\ldots,f_n^v)^{-1}|\Bigg)$$

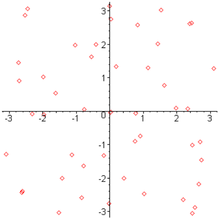
In particular, for $f_1,\ldots,f_n \in \mathbb{Z}[x_1,\ldots,x_n]$, of degrees $d_1,\ldots,d_n$, then

$$\eta(\boldsymbol{f}) \leq 2n\sum_{j=1}^{n} \frac{\log \|f_j\|\sup}{d_j}$$

Dense Example

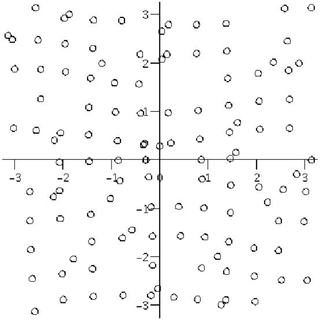
$$
\begin{aligned}
f_1 &= x_1^7 + x_1^6 x_2 + x_1^5 x_2^2 - x_1^4 x_2^3 + x_1^3 x_2^4 + x_1 x_2^6 - x_2^7 - x_1^6 + x_1^4 x_2^2 \\
&\quad - x_1^3 x_2^3 + x_1^2 x_2^4 + x_1 x_2^5 + x_2^6 - x_1^5 - x_1^4 x_2 + x_1 x_2^4 - x_1^4 + x_1^3 x_2 \\
&\quad + x_1 x_2^3 - x_1^3 - x_1^2 x_2 + x_1 x_2^2 + x_1^2 - x_1 x_2 - x_1 - x_2 - 1 \\
f_2 &= -x_1^7 - x_1^5 x_2^2 + x_1^4 x_2^3 + x_1^3 x_2^4 - x_1^2 x_2^5 - x_2^7 + x_1^5 x_2 - x_1 x_2^5 - x_2^6 \\
&\quad + x_1^5 + x_1^4 x_2 - x_1^2 x_2^3 - x_1 x_2^4 + x_1^2 x_2^2 - x_2^4 - x_1^3 - x_1^2 x_2 + x_1 x_2^2 \\
&\quad - x_2^3 + x_1 + x_2 + 1
\end{aligned}
$$

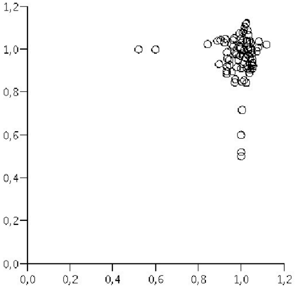




Sparse Example

$$f_1 = x_1^{13} + x_1 x_2^{12} + x_2^{13} + 1, \quad f_2 = x_1^{12} x_2 - x_2^{13} - x_1 x_2 + 1.$$





Sketch of the proof

- For Δ_{r} we apply Erdös-Turán to $E(\boldsymbol{f}, \mathbf{e}_i)$, with $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ the canonical basis of $\mathbb{Z}^n$
- For Δ_θ, we apply E-T to $E(\boldsymbol{f}, \mathbf{a})$ for all $\mathbf{a} \in \mathbb{Z}^n$ to estimate the exponential sums on its roots, then compare it with the equidistribution by tomography *via* Fourier analysis
- In order to bound $\eta(\boldsymbol{f})$, for $E_{\boldsymbol{f},\mathbf{a}}(z) = \mathrm{Res}_{\{\mathbf{0},\mathbf{a}\},\mathcal{A}_1,\ldots,\mathcal{A}_n}(z - \boldsymbol{x}^{\mathbf{a}}, f_1, \ldots, f_n)$ we get

$$\log \|E_{\boldsymbol{f},\mathbf{a}}(z)\| \leq \|\mathbf{a}\| \sum_{j=1}^{n} \mathrm{MV}_{n-1}\left(\pi_{\mathbf{a}}(Q_k) : k \neq j\right) \log \|f_j\|$$

Related results

- Angular equidistribution in terms of fewnomials (Khovanskii)
- Equidistribution of algebraic numbers (Bilu, Petsche, Favre & Rivera-Letelier) and in Berkovich's spaces (Chambert-Loir)
- Equidistribution in $\mathbb{P}^n_{\mathbb{C}}$ by using the Haar measure $\langle f, g\rangle = \int_{S^{2n-1}} f\overline{g}d\mu$ (Shub & Smale, Shiffmann & Zelditch)

UNIVERSITAT DE BARCELONA

Thanks!