Rational Plane Curves with $\mu = 2$

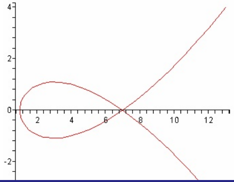
Carlos D'Andrea

Algebraic Geometry and Geometric Modeling
Banff - January 2013

Rational Plane Curves

$$\begin{array}{cccc} \phi : & \mathbb{P}^1 & \to & \mathbb{P}^2 \\ & (t_0 : t_1) & \mapsto & (a(t_0,t_1) : b(t_0,t_1) : c(t_0,t_1)) \end{array}$$

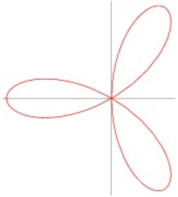
- $a, b, c \in \mathbb{K}[T_0, T_1]$, homogeneous of the same degree $d \geq 1$
- $\gcd(a, b, c) = 1$



Rational Plane Curves

The image of the map is a **rational plane curve**

- The curve has degree d if ϕ is injective (proper)
- It has genus 0, which means "maximal" number of multiple points $\frac{(d-1)(d-2)}{2}$
- Computing the "implicit equation" is relatively easy from ϕ

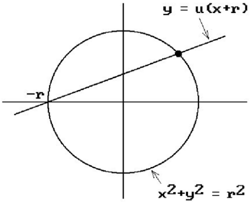


Example

$$d = 2, \; r > 0$$

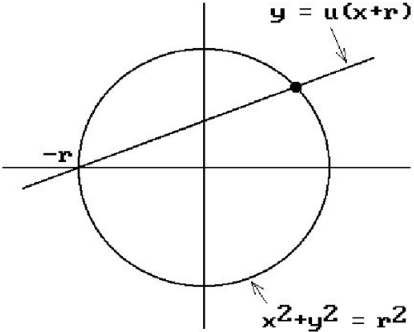
$$\begin{cases} x_0 = a(t_0, t_1) = 2rt_0t_1 \\ x_1 = b(t_0, t_1) = r(t_0^2 - t_1^2) \\ x_2 = c(t_0, t_1) = t_0^2 + t_1^2 \end{cases}$$

$$\boxed{x_0^2 + x_1^2 - r^2x_2^2 = 0}$$



From parametric to implicit

$$\mathrm{Res}_{\underline{T}}\left(X_2 \cdot a(\underline{T}) - X_0 \cdot c(\underline{T}), X_2 \cdot b(\underline{T}) - X_1 \cdot c(\underline{T})\right)$$
$$=$$
$$-4X_2 r^2 (X_2 r^2 - X_0^2 - X_1^2)$$



The Sylvester Resultant

$$X_2 a(\underline{T}) - X_0 c(\underline{T}) = -X_0 T_0^2 + 2X_2 r T_0 T_1 - X_0 T_1^2$$

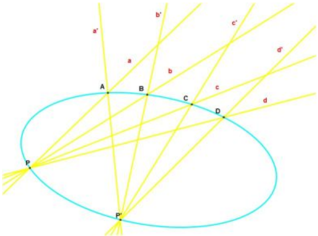
$$X_2 b(\underline{T}) - X_1 c(\underline{T}) = (X_2 r - X_1) T_0^2 + 0\, T_0 T_1 - (X_2 r + X_1) T_1^2$$

$$\mathrm{Res}_{\underline{T}}(X_2 \cdot a(\underline{T}) - X_0 \cdot c(\underline{T}), X_2 \cdot b(\underline{T}) - X_1 \cdot c(\underline{T})) \\ = \\ \det \begin{pmatrix} -X_0 & 2X_2 r & -X_0 & 0 \\ 0 & -X_0 & 2X_2 r & -X_0 \\ X_2 r - X_1 & 0 & -(X_2 r + X_1) & 0 \\ 0 & X_2 r - X_1 & 0 & -(X_2 r + X_1) \end{pmatrix}$$

The μ basis of a parametrization

There exist $\mu \leq \frac{d}{2}$ and two other parametrizations $\varphi_\mu(t_0, t_1)$, $\varphi_{d-\mu}(t_0, t_1)$ of degrees μ, $d-\mu$ such that

$$\boxed{\phi(t_0, t_1) = \varphi_\mu(t_0, t_1) \wedge \varphi_{d-\mu}(t_0, t_1)}$$



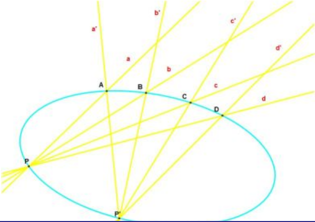
The μ basis of a parametrization

The set $\{\varphi_\mu(\underline{T}), \varphi_{d-\mu}(\underline{T})\}$ is called a μ-**basis** of ϕ.

$$\varphi_j(\underline{T}) = \big(a_j(\underline{T}), b_j(\underline{T}), c_j(\underline{T})\big)$$

$$\boxed{a_j(\underline{T})X_0 + b_j(\underline{T})X_1 + c_j(\underline{T})X_2}$$

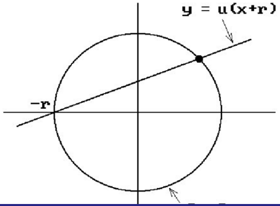
is a **moving line** which follows the curve



A μ basis for the parametrization of the circle

$$\begin{array}{rcl} \varphi_1(\underline{T}) & = & (-T_0, T_1, rT_1) \\ \tilde{\varphi}_1(\underline{T}) & = & (-T_1, -T_0, rT_0) \end{array}$$

$$\begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \\ -T_0 & T_1 & rT_1 \\ -T_1 & -T_0 & rT_0 \end{vmatrix} = \left(2rT_0T_1, r(T_0^2 - T_1^2), T_0^2 + T_1^2\right)$$



μ bases and Hilbert's Syzygy Theorem

The homogeneous polynomial ideal $I = \big(a(\underline{T}), b(\underline{T}), c(\underline{T})\big) \subset \mathbb{K}[T_0, T_1]$ has a **Hilbert-Burch resolution** of the type

$$0 \to \mathbb{K}[\underline{T}]^2 \xrightarrow{(\varphi_\mu, \varphi_{d-\mu})^{\mathbf{t}}} \mathbb{K}[\underline{T}]^3 \xrightarrow{(a,b,c)} \mathbb{K}[\underline{T}]$$

A μ basis of the parametrization is a basis of $\mathrm{Syz}(I)$ as a $\mathbb{K}[\underline{T}]$-module

How to compute a μ basis?

Wait for the next talk

Why do we care about μ bases?

Implicit equation

$$=$$

$$\mathrm{Res}_{\underline{T}}\big(\varphi_\mu(\underline{T})\cdot(X_0,X_1,X_2),\ \varphi_{d-\mu}(\underline{T})\cdot(X_0,X_1,X_2)\big)$$

For the circle:

$$\begin{aligned}\varphi_1(\underline{T})\cdot(X_0,X_1,X_2) &= -X_0T_0+(X_1+rX_2)T_1\\ \tilde{\varphi}_1(\underline{T})\cdot(X_0,X_1,X_2) &= (-X_1+rX_2)T_0-X_0T_1\end{aligned}$$

$$\begin{vmatrix} -X_0 & X_1+rX_2 \\ -X_1+rX_2 & -X_0 \end{vmatrix} = X_0^2+X_1^2-r^2X_2^2$$

Compact formulas for computing resultants

- Sylvester $\sim 2D$
- Hybrid
- Bézout $\sim D$

Busé-D (2012)

If B denotes a Bézout matrix and S a Sylvester matrix then,

$$X_2\, S(\varphi_\mu \cdot \underline{X}, \varphi_{d-\mu} \cdot \underline{X}) = M\, B(aX_2 - cX_0, bX_2 - cX_1),$$

with $M \in \mathbb{K}^{d \times d}$ invertible

A bit of history

- Sederberg, Saito, Qi, Klimaszewski. (1994), **Curve implicitization using moving lines**, Computer Aided Geometric Design 11, 687–706
- Sederberg, Chen. **Implicitization using moving curves and surfaces.** Proceedings of SIGGRAPH 1995, 301–308.
- Sederberg, Goldman, Du. (1997), *Implicitizing rational curves by the method of moving algebraic curves*, J. Symbolic Comp. 23, 153–175
- Cox., Sederberg, Chen. (1998), **The moving line ideal basis for planar rational curves**, Computer Aided Geometric Design 15, 803–827
- ...

Moving conics, Moving cubics,...

$$a_j(\underline{T})X_0^2 + b_j(\underline{T})X_0X_1 + c_j(\underline{T})X_0X_2 + d_j(\underline{T})X_1^2 + e_j(\underline{T})X_1X_2 + f_j(\underline{T})X_2^2$$

is a **moving conic** which follows the parametrization if

$$a_j(\underline{T})a(\underline{T})^2 + b_j(\underline{T})a(\underline{T})b(\underline{T}) + c_j(\underline{T})a(\underline{T})c(\underline{T}) + d_j(\underline{T})b(\underline{T})^2 + e_j(\underline{T})b(\underline{T})c(\underline{T}) + f_j(\underline{T})c(\underline{T})^2 = 0$$

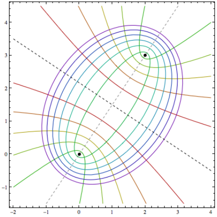
The method of moving curves for implicitization

The implicit equation may be computed as a **small** determinant of

$$\left|\begin{array}{c}\text{some moving lines}\\ \text{some moving conics}\\ \text{some moving cubics}\\ \ldots\end{array}\right|$$

the **more** singular the curve is, the **smaller** the determinant is

which moving lines?
which moving conics?
which moving cubics?
...



The Rees Algebra associated to the parametrization

Cox, D. **The moving curve ideal and the Rees algebra.** Theoret. Comput. Sci. 392 (2008), no. 1–3, 23–36.

$\mathcal{K}$:= {moving curves following ϕ} = kernel of

$$\begin{array}{ccc} \mathbb{K}[T_0, T_1, X_0, X_1, X_2] & \to & \mathbb{K}[T_0, T_1, s] \\ T_i & \mapsto & T_i \\ X_0 & \mapsto & a(\underline{T})s \\ X_1 & \mapsto & b(\underline{T})s \\ X_2 & \mapsto & c(\underline{T})s \end{array}$$

"The defining ideal of the Rees Algebra associated to ϕ"

The implicit equation may be obtained as the determinant of a very small matrix:

$$\left| \begin{array}{c} \cdots \\ \text{some minimal generators of } \mathcal{K} \\ \cdots \end{array} \right|$$

The more singular the curve, the simpler the description of $\mathcal{K}$

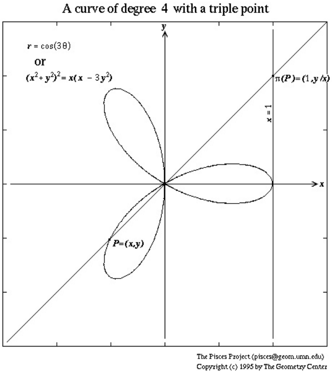
Computing minimal generators of $\mathcal{K}$

The case $\mu = 1$

- Hong, J.; Simis; Vasconcelos. **On the homology of two- dimensional elimination.** J. Symbolic Comput. 43 (2008)
- Cox, Hoffman, Wang, H. **Syzygies and the Rees algebra.** J. Pure Appl. Algebra 212 (2008)
- Busé. **On the equations of the moving curve ideal of a rational algebraic plane curve.** J. Algebra 321 (2009)
- Cortadellas, **D. Minimal generators of the defining ideal of the Rees Algebra associated to monoid parametrizations.** Computer Aided Geometric Design, 27 (6) 2010

$\mu = 1$

$\mu = 1 \iff$ there is a point of multiplicity $d - 1$



Monoid curves if $d > 2$

Minimal generators for the space of moving curves $(\mu = 1)$

One can always assume

$$\varphi_1 \cdot (X_0, X_1, X_2) = T_0 X_1 - T_1 X_0$$

A complete set of minimal generators can be obtained by "**reducing**" the other element of the μ basis via this form

Example

$$\varphi_1 \cdot (X_0, X_1, X_2) = T_0 X_1 - T_1 X_0$$
$$\varphi_3 \cdot (X_0, X_1, X_2) = (T_0^3 + T_0^2 T_1) X_0 + T_1^3 X_1 + T_0 T_1^2 X_3$$

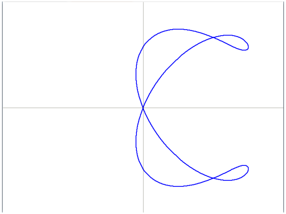
$$F_{2,2} = (T_0^2 X_0 + T_0^2 X_1) X_0 + T_1^2 X_1 X_1 + T_0 T_1 X_1 X_3$$
$$F_{1,3} = (T_0 X_0^2 + T_0 X_0 X_1) X_0 + T_1 X_1^2 X_1 + T_0 X_1^2 X_3$$
$$F_{0,4} = (X_0^3 + X_0^2 X_1) X_0 + X_1^3 X_1 + X_0 X_1^2 X_3$$

The ideal of moving curves has $d+1$ minimal generators

$\mu = 2$

The curve has either

- one point of multiplicity $d-2$ and the rest double
- only double points



$\mu = 2$ with only double points

Busé, J. Algebra 321 (2009)

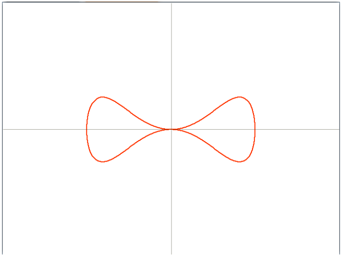
Constructs an explicit family of $\frac{(d+1)(d-4)}{2} + 6$ generators of the moving curve ideal

(Cortadellas-**D** `arXiv:1301.6286`): remove the "jacobian", and you will get a family of minimal generators

$\mu = 2$ with a very singular point

(Cortadellas-**D** `arXiv:1301.6286`)

Compute an explicit family of minimal generators of $\mathcal{O}(\frac{d}{2})$ elements, via reduction



How does it work? (Example)

(Cortadellas-**D** `arXiv:1301.6286`)

$$\varphi_1 \cdot (X_0, X_1, X_2) = T_0^2 X_1 - T_1^2 X_0$$
$$\varphi_{2k-3} \cdot (X_0, X_1, X_2) = T_1^{2k-3} X_1 - T_0^{2k-3} X_2$$

$$F_{2k-5,2} = T_1^{2k-5} X_1 X_1 - T_0^{2k-5} X_0 X_2$$
$$F_{2k-7,3} = T_1^{2k-7} X_1^2 X_1 - T_0^{2k-7} X_0^2 X_2$$
$$\ldots$$
$$F_{1,k-1} = T_1 X_1^{k-2} X_1 - T_0 X_0^{k-2} X_2$$

How does it work?

(Cortadellas-**D** `arXiv:1301.6286`)

- One can assume that the very singular point is $(0:0:1)$, i.e. that $\varphi_2 \cdot (X_0, X_1, X_2) = Q_0(\underline{T})X_1 - Q_1(\underline{T})X_0$
- Reduce φ_{d-2} (every polinomial of degree larger than 2 can be written as a polynomial combination of $Q_0(\underline{T}),\ Q_1(\underline{T})$)
- Add one or two more elements at level $(1, k)$ depending on whether d is odd or even, and also the implicit equation

In our example...

$$\varphi_2 \cdot (X_0, X_1, X_2) = T_0 T_0 X_1 - T_1 T_1 X_0$$
$$\varphi_{2k-3} \cdot (X_0, X_1, X_2) = T_1^{2k-3} X_1 - T_0^{2k-3} X_2$$

$$F_{2k-5,2} = T_1^{2k-5} X_1 X_1 - T_0^{2k-5} X_0 X_2$$
$$F_{2k-7,3} = T_1^{2k-7} X_1^2 X_1 - T_0^{2k-7} X_0^2 X_2$$

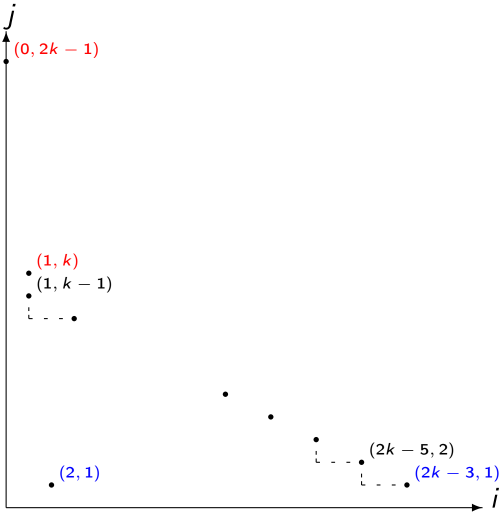
$$\ldots$$

$$F_{1,k-1} = T_1 X_1^{k-2} X_1 - T_0 X_0^{k-2} X_2$$

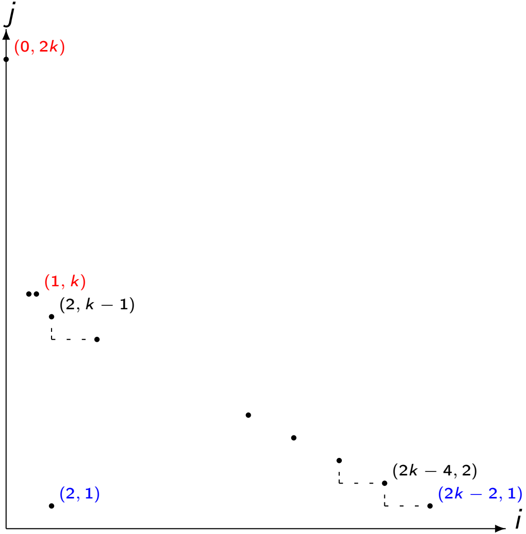
$$\boxed{F_{1,k} = T_0 (X_1^{k-2} X_1) X_1 - T_1 (X_0^{k-2} X_2) X_0}$$

$$\boxed{F_{0,2k-1} = X_1^{2k-1} - X_0^{2k-3} X_2^2}$$

Minimal generators for $d = 2k - 1$



Minimal generators for $d = 2k$



Adjoints

A curve C_0 is **adjoint** to another curve C if $m_p(\mathrm{C}_0) \geq m_p(\mathrm{C}) - 1$ for all $p \in \mathrm{C}$

Busé, J. Algebra 321 (2009)

Any moving curve of the form

$$F_{1,\ell} = T_0\mathcal{A}(X_0, X_1, X_2) + T_1\mathcal{B}(X_0, X_1, X_2)$$

is a **pencil of adjoints** ($\mu = 2$, only double points)

Adjoints

(Cortadellas - D arXiv:1301.6286)

If the curve has $\mu = 2$ and a point of multiplicity $d-2$, for any $\ell \geq d-2$,

$$\dim_{\mathbb{K}}\left(\mathcal{K}_{1,\ell}/\{\text{pencils of adjoints}\} \cap \mathcal{K}_{1,\ell}\right)$$

$$\geq$$

$$\begin{cases} (k-2)^2 & d = 2k-1 \\ (k-1)(k-2) & d = 2k \end{cases}$$

The equality holds for generic curves in this class

What about $\mu \geq 3$?

Busé's method does not work for $\mu = 3$ Ours either!

$$\begin{cases} \varphi_3 = T_0^3 X_0 + (T_1^3 - T_0 T_1^2) X_1 \\ \varphi_7 = (T_0^6 T_1 - T_0^2 T_1^5) X_0 + (T_0^4 T_1^3 + T_0^2 T_1^5) X_1 + (T_0^7 + T_1^7) X_2 \end{cases}$$

with minimal generators of bidegree
$(3,1)$, $(7,1)$, $(2,3)$, $(2,3)$, $(4,2)$, $(2,4)$, $(\mathbf{1},\mathbf{6})$, $(1,6)$, $(1,6)$, $(0,10)$

$$\begin{cases} \tilde{\varphi}_3 = (T_0^3 - T_0^2 T_1) X_0 + (T_1^3 + T_0 T_1^2 - T_0 T_1^2) X_1 \\ \tilde{\varphi}_7 = (T_0^6 T_1 - T_0^2 T_1^5) X_0 + (T_0^4 T_1^3 + T_0^2 T_1^5) X_1 + (T_0^7 + T_1^7) X_2 \end{cases}$$

with minimal generators of bidegree
$(3,1)$, $(7,1)$, $(2,3)$, $(2,3)$, $(4,2)$, $(2,4)$, $(\mathbf{1},\mathbf{5})$, $(1,6)$, $(1,6)$, $(0,10)$

Current related work

- Cox, Kustin, Polini. **A study of singularities on rational curves via syzygies.** To appear in Memoirs of AMS
- Cortadellas, **D. Rational plane curves parameterizable by conics.** J. Algebra 373 (2013)
- Hassanzadeh, Simis. **Implicitization of the Jonquières parametrizations.** arXiv:1205.1083
- Kustin, Polini, Ulrich. **The bi-graded structure of Symmetric Algebras with applications to Rees rings.** arXiv:1301.7106
- Botbol, Busé. **Talk at 11... Stay tuned!**
- ...

UNIVERSITAT DE BARCELONA

Thanks!