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Minimal generators of the defining ideal of the Rees Algebra associated to a rational plane parametrization with $\mu = 2$

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MEGA 2013 - Frankfurt

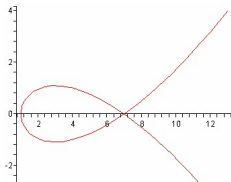
arXiv:1301.6286



Rational Plane Parametrizations

$$\begin{aligned}\phi : \quad \mathbb{P}^1 &\rightarrow \mathbb{P}^2 \\ (t_0 : t_1) &\mapsto (a(t_0, t_1) : b(t_0, t_1) : c(t_0, t_1))\end{aligned}$$

- $a, b, c \in \mathbb{K}[T_0, T_1]$, homogeneous of the same degree $d \geq 1$
- $\gcd(a, b, c) = 1$

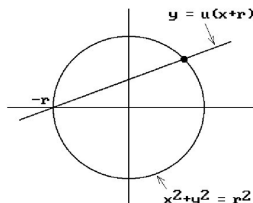


Example

$$d = 2, r > 0$$

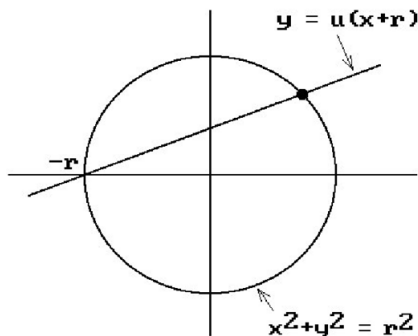
$$\begin{cases} x_0 = a(t_0, t_1) = 2rt_0t_1 \\ x_1 = b(t_0, t_1) = r(t_0^2 - t_1^2) \\ x_2 = c(t_0, t_1) = t_0^2 + t_1^2 \end{cases}$$

$$x_0^2 + x_1^2 - r^2 x_2^2 = 0$$



From parametric to implicit

$$\begin{aligned} \text{Res}_{\underline{T}}(X_2 \cdot a(\underline{T}) - X_0 \cdot c(\underline{T}), X_2 \cdot b(\underline{T}) - X_1 \cdot c(\underline{T})) \\ = \\ -4X_2r^2(X_2r^2 - X_0^2 - X_1^2) \end{aligned}$$



The Sylvester Resultant

$$X_2 a(\underline{T}) - X_0 c(\underline{T}) = -X_0 T_0^2 + 2X_2 r T_0 T_1 - X_0 T_1^2$$

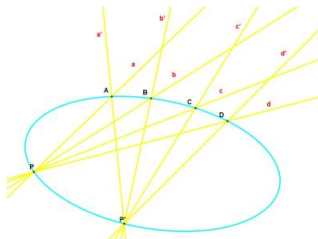
$$X_2 b(\underline{T}) - X_1 c(\underline{T}) = (X_2 r - X_1) T_0^2 + 0 T_0 T_1 - (X_2 r + X_1) T_1^2$$

$$\begin{aligned} & \text{Res}_{\underline{T}}(X_2 \cdot a(\underline{T}) - X_0 \cdot c(\underline{T}), X_2 \cdot b(\underline{T}) - X_1 \cdot c(\underline{T})) \\ &= \\ & \det \begin{pmatrix} -X_0 & 2X_2 r & -X_0 & 0 \\ 0 & -X_0 & 2X_2 r & -X_0 \\ X_2 r - X_1 & 0 & -(X_2 r + X_1) & 0 \\ 0 & X_2 r - X_1 & 0 & -(X_2 r + X_1) \end{pmatrix} \end{aligned}$$

The μ basis of a parametrization

There exist $\mu \leq \frac{d}{2}$ and two other parametrizations $\varphi_\mu(t_0, t_1)$, $\varphi_{d-\mu}(t_0, t_1)$ of degrees μ , $d - \mu$ such that

$$\phi(t_0, t_1)'' = \varphi_\mu(t_0, t_1) \wedge \varphi_{d-\mu}(t_0, t_1)$$



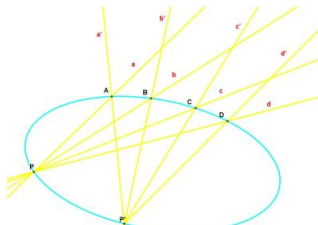
The μ basis of a parametrization

The set $\{\varphi_\mu(\underline{T}), \varphi_{d-\mu}(\underline{T})\}$ is called a μ -**basis** of ϕ .

$$\varphi_j(\underline{T}) = (a_j(\underline{T}), b_j(\underline{T}), c_j(\underline{T}))$$

$$a_j(\underline{T})X_0 + b_j(\underline{T})X_1 + c_j(\underline{T})X_2$$

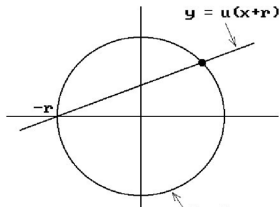
is a **moving line** which follows the curve



A μ basis for the parametrization of the circle

$$\begin{aligned}\varphi_1(\underline{T}) &= (-T_0, T_1, rT_1) \\ \tilde{\varphi}_1(\underline{T}) &= (-T_1, -T_0, rT_0)\end{aligned}$$

$$\begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \\ -T_0 & T_1 & rT_1 \\ -T_1 & -T_0 & rT_0 \end{vmatrix} = (2rT_0T_1, r(T_0^2 - T_1^2), T_0^2 + T_1^2)$$



μ bases and Hilbert's Syzygy Theorem

The homogeneous polynomial ideal
 $I = (a(\underline{T}), b(\underline{T}), c(\underline{T})) \subset \mathbb{K}[\underline{T}_0, \underline{T}_1]$ has a
Hilbert-Burch resolution of the type

$$0 \rightarrow \mathbb{K}[\underline{T}]^2 \xrightarrow{(\varphi_\mu, \varphi_{d-\mu})^t} \mathbb{K}[\underline{T}]^3 \xrightarrow{(a,b,c)} \mathbb{K}[\underline{T}]$$

A μ basis of the parametrization is a basis of $\text{Syz}(I)$
as a $\mathbb{K}[\underline{T}]$ -module

Why do we care about μ bases?

Implicit equation

=

$$\text{Res}_{\underline{T}}(\varphi_{\mu}(\underline{T}) \cdot (X_0, X_1, X_2), \varphi_{d-\mu}(\underline{T}) \cdot (X_0, X_1, X_2))$$

For the circle:

$$\varphi_1(\underline{T}) \cdot (X_0, X_1, X_2) = -X_0 T_0 + (X_1 + rX_2) T_1$$

$$\tilde{\varphi}_1(\underline{T}) \cdot (X_0, X_1, X_2) = (-X_1 + rX_2) T_0 - X_0 T_1$$

$$\begin{vmatrix} -X_0 & X_1 + rX_2 \\ -X_1 + rX_2 & -X_0 \end{vmatrix} = X_0^2 + X_1^2 - r^2 X_2^2$$

A bit of history

- Sederberg, Saito, Qi, Klimaszewski. (1994), **Curve implicitization using moving lines**, Computer Aided Geometric Design 11, 687–706
- Sederberg, Chen. **Implicitization using moving curves and surfaces**. Proceedings of SIGGRAPH 1995, 301–308.
- Sederberg, Goldman, Du. (1997), **Implicitizing rational curves by the method of moving algebraic curves**, J. Symbolic Comp. 23, 153–175
- Cox., Sederberg, Chen. (1998), **The moving line ideal basis for planar rational curves**, Computer Aided Geometric Design 15, 803–827
- ...

Moving conics, Moving cubics,...

$$a_j(\underline{T})X_0^2 + b_j(\underline{T})X_0X_1 + c_j(\underline{T})X_0X_2 + d_j(\underline{T})X_1^2 + e_j(\underline{T})X_1X_2 + f_j(\underline{T})X_2^2$$

is a **moving conic** which follows the parametrization if

$$a_j(\underline{T})a(\underline{T})^2 + b_j(\underline{T})a(\underline{T})b(\underline{T}) + c_j(\underline{T})a(\underline{T})c(\underline{T}) + d_j(\underline{T})b(\underline{T})^2 + e_j(\underline{T})b(\underline{T})c(\underline{T}) + f_j(\underline{T})c(\underline{T})^2 = 0$$

The method of moving curves for implicitization

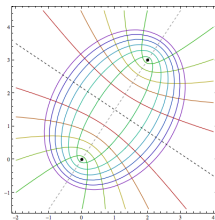
The implicit equation may be computed as a **small** determinant of

some moving lines
some moving conics
some moving cubics
...

the **more** singular the curve, the **smaller** the determinant

which moving lines?
which moving conics?
which moving cubics?

...



The Rees Algebra associated to the parametrization
 Cox, D. **The moving curve ideal and the Rees algebra**. Theoret. Comput. Sci. 392 (2008), no. 1–3, 23–36.

$\mathcal{K} := \{\text{moving curves following } \phi\} = \text{kernel of}$

$$\begin{array}{ccc}
 \mathbb{K}[T_0, T_1, X_0, X_1, X_2] & \rightarrow & \mathbb{K}[T_0, T_1, s] \\
 T_i & \mapsto & T_i \\
 X_0 & \mapsto & a(\underline{T})s \\
 X_1 & \mapsto & b(\underline{T})s \\
 X_2 & \mapsto & c(\underline{T})s
 \end{array}$$

“The defining ideal of the Rees Algebra associated to
 ϕ ”

The implicit equation may be obtained as the determinant of a very small matrix:

$$\begin{vmatrix} \dots \\ \text{some minimal generators of } \mathcal{K} \\ \dots \end{vmatrix}$$

The more singular the curve, the simpler the description of \mathcal{K}

Computing minimal generators of \mathcal{K}

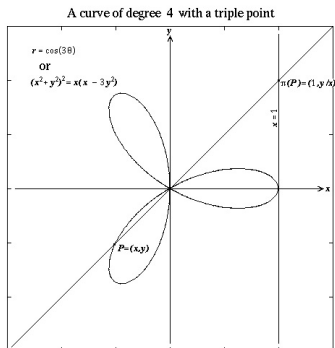
The case $\mu = 1$

- Hong, J.; Simis; Vasconcelos. **On the homology of two- dimensional elimination.** J. Symbolic Comput. 43 (2008)
- Cox, Hoffman, Wang, H. **Syzygies and the Rees algebra.** J. Pure Appl. Algebra 212 (2008)
- Busé. **On the equations of the moving curve ideal of a rational algebraic plane curve.** J. Algebra 321 (2009)
- Cortadellas, D. **Minimal generators of the defining ideal of the Rees Algebra associated to monoid parametrizations.** Computer Aided Geometric Design, 27 (6) 2010



$$\mu = 1$$

$\mu = 1 \iff$ there is a point of multiplicity $d - 1$



The Pappas Project (pappas@geom.umn.edu)
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Monoid curves if $d \geq 2$

Minimal generators for the space of moving curves ($\mu = 1$)

One can always assume

$$\varphi_1 \cdot (X_0, X_1, X_2) = T_0 X_1 - T_1 X_0$$

A complete set of minimal generators can be obtained by “**reducing**” the other element of the μ basis via this form

Example

$$\varphi_1 \cdot (X_0, X_1, X_2) = T_0 X_1 - T_1 X_0$$

$$\varphi_3 \cdot (X_0, X_1, X_2) = (T_0^3 + T_0^2 T_1) X_0 + T_1^3 X_1 + T_0 T_1^2 X_3$$

$$F_{2,2} = (T_0^2 X_0 + T_0^2 X_1) X_0 + T_1^2 X_1 X_1 + T_0 T_1 X_1 X_3$$

$$F_{1,3} = (T_0 X_0^2 + T_0 X_0 X_1) X_0 + T_1 X_1^2 X_1 + T_0 X_1^2 X_3$$

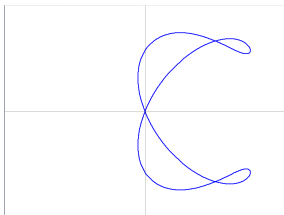
$$F_{0,4} = (X_0^3 + X_0^2 X_1) X_0 + X_1^3 X_1 + X_0 X_1^2 X_3$$

The ideal of moving curves has $d + 1$ minimal generators

$$\mu = 2$$

The curve has either

- one point of multiplicity $d - 2$ and the rest double
- only double points



$\mu = 2$ with only double points

Busé, J. Algebra 321 (2009)

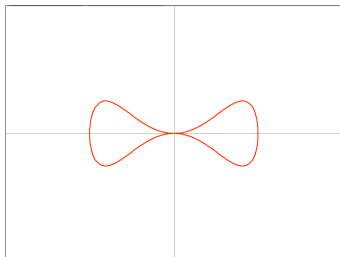
Constructs an explicit family of $\frac{(d+1)(d-4)}{2} + 6$ generators of the moving curve ideal

(Cortadellas-D arXiv:1301.6286): remove the “jacobian”, and you will get a family of minimal generators

$\mu = 2$ with a very singular point

(Cortadellas-D arXiv:1301.6286)

Compute an explicit family of minimal generators of $\mathcal{O}(\frac{d}{2})$ elements, via reduction



How does it work?

(Cortadellas-D arXiv:1301.6286)

$$\varphi_1 \cdot (X_0, X_1, X_2) = T_0^2 X_1 - T_1^2 X_0$$

$$\varphi_{2k-3} \cdot (X_0, X_1, X_2) = T_1^{2k-3} X_1 - T_0^{2k-3} X_2$$

$$F_{2k-5,2} = T_1^{2k-5} X_1 X_1 - T_0^{2k-5} X_0 X_2$$

$$F_{2k-7,3} = T_1^{2k-7} X_1^2 X_1 - T_0^{2k-7} X_0^2 X_2$$

...

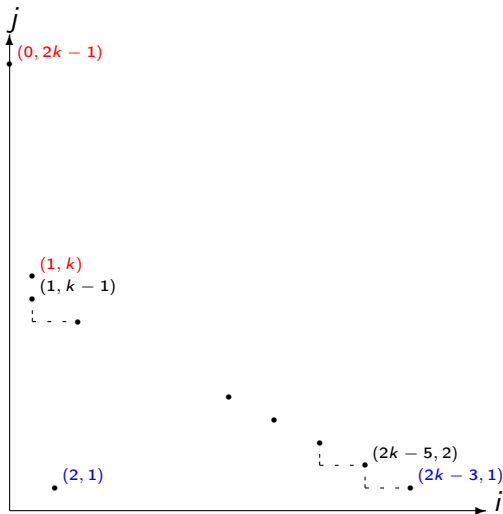
$$F_{1,k-1} = T_1 X_1^{k-2} X_1 - T_0 X_0^{k-2} X_2$$

How does it work?

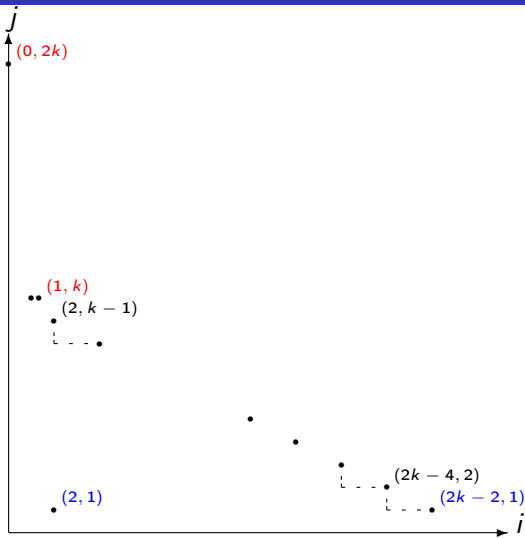
(Cortadellas-D arXiv:1301.6286)

- One can assume that the very singular point is $(0 : 0 : 1)$, i.e. that
$$\varphi_2 \cdot (X_0, X_1, X_2) = Q_0(\underline{T})X_1 - Q_1(\underline{T})X_0$$
- Reduce φ_{d-2} (every polynomial of degree larger than 2 can be written as a polynomial combination of $Q_0(\underline{T}), Q_1(\underline{T})$)
- Add one or two more elements at level $(1, k)$ depending on whether d is odd or even, and also the implicit equation

Minimal generators for $d = 2k - 1$



Minimal generators for $d = 2k$



Adjoints

A curve C_0 is **adjoint** to another curve C if
 $m_p(C_0) \geq m_p(C) - 1$ for all $p \in C$

Busé, J. Algebra 321 (2009)

Any moving curve of the form

$$F_{1,\ell} = T_0 \mathcal{A}(X_0, X_1, X_2) + T_1 \mathcal{B}(X_0, X_1, X_2)$$

is a **pencil of adjoints** ($\mu = 2$, only double points)

Adjoints

(Cortadellas - D arXiv:1301.6286)

If the curve has $\mu = 2$ and a point of multiplicity $d-2$,
for any $\ell \geq d-2$,

$$\dim_{\mathbb{K}} (\mathcal{K}_{1,\ell} / \{\text{pencils of adjoints}\} \cap \mathcal{K}_{1,\ell}) \geq \begin{cases} (k-2)^2 & d = 2k-1 \\ (k-1)(k-2) & d = 2k \end{cases}$$

The equality holds for generic curves in this class

What about $\mu \geq 3$?

Busé's method does not generalize to $\mu = 3$ Ours either!

$$\begin{cases} \varphi_3 = T_0^3 X_0 + (T_1^3 - T_0 T_1^2) X_1 \\ \varphi_7 = (T_0^6 T_1 - T_0^2 T_1^5) X_0 + (T_0^4 T_1^3 + T_0^2 T_1^5) X_1 + (T_0^7 + T_1^7) X_2 \end{cases}$$

with minimal generators of bidegree

$(3, 1), (7, 1), (2, 3), (2, 3), (4, 2), (2, 4), (1, 6), (1, 6), (1, 6), (0, 10)$

$$\begin{cases} \tilde{\varphi}_3 = (T_0^3 - T_0^2 T_1) X_0 + (T_1^3 + T_0 T_1^2 - T_0 T_1^2) X_1 \\ \tilde{\varphi}_7 = (T_0^6 T_1 - T_0^2 T_1^5) X_0 + (T_0^4 T_1^3 + T_0^2 T_1^5) X_1 + (T_0^7 + T_1^7) X_2 \end{cases}$$

with minimal generators of bidegree

$(3, 1), (7, 1), (2, 3), (2, 3), (4, 2), (2, 4), (1, 5), (1, 6), (1, 6), (0, 10)$

Current related work

- Cox, Kustin, Polini. **A study of singularities on rational curves via syzygies**. Memoirs AMS 1045, 2013
- Cortadellas, D. **Rational plane curves parameterizable by conics**. J. Algebra 373 (2013)
- Hassanzadeh, Simis. **Implicitization of the Jonquières parametrizations**. arXiv:1205.1083
- Kustin, Polini, Ulrich. **The bi-graded structure of Symmetric Algebras with applications to Rees rings**. arXiv:1301.7106
- ...

Thanks!



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