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Formulas in Interpolation

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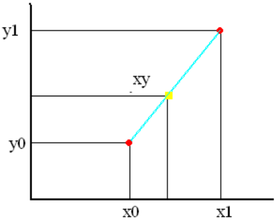
Interpolation in Mathematics

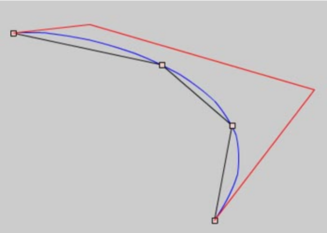
- 03C40 Model theory – $\boxed{\textit{Interpolation}}$, preservation, definability
- 30E05 Functions of a complex variable – Moment problems, $\boxed{\textit{interpolation}}$ problems
- 41A05 Approximations and expansions – $\boxed{\textit{Interpolation}}$
- 42A15 Harmonic analysis on Euclidean spaces – Trigonometric $\boxed{\textit{interpolation}}$
- ...



Interpolation in Computational Algebra and Algebraic Geometry?

Interpolation has been a very effective tool for

- Computational issues
- Theoretical "closed" formulae
- Applications



An example: resultants and subresultants

$$f(x) = f_m x^m + \ldots + f_1 x + f_0 = f_m(x - a_1)\ldots(x - a_m)$$
$$g(x) = g_n x^n + \ldots + g_1 x + g_0 = g_n(x - b_1)\ldots(x - b_n)$$

$$\boxed{\mathrm{Res}(f,g) = \det\left(Sylv(f,g)\right) = f_m^n g_n^m \prod (a_i - b_j)}$$

$$\begin{pmatrix}
f_m & \cdots & f_0 & & & \\
 & f_m & \cdots & f_0 & & \\
 & & \ddots & & \ddots & \\
 & & & f_m & \cdots & f_0 \\
g_n & \cdots & g_0 & & & \\
 & g_n & \cdots & g_0 & & \\
 & & \ddots & & \ddots & \\
 & & & g_n & \cdots & g_0
\end{pmatrix}$$

Subresultants

For $0 \leq t < \min\{m, n\}$,

$$\mathrm{Sres}_t(f,g) = $$

$$\det \begin{array}{c} m+n-2t \\ \left|\begin{array}{ccccc} f_m & \cdots & \cdots & f_{t+1-(n-t-1)} & x^{n-t-1}f(x) \\ & \ddots & & \vdots & \vdots \\ & f_m & \cdots & a_{t+1} & x^0 f(x) \\ \hline g_n & \cdots & \cdots & n_{t+1-(m-t-1)} & x^{m-t-1}g(x) \\ & \ddots & & \vdots & \vdots \\ & g_n & \cdots & g_{t+1} & x^0 g(x) \end{array}\right| \end{array} \begin{array}{c} \\ n-t \\ \\ \\ m-t \\ \end{array}$$

Subresultants in Roots (J.J. Sylvester, 1853)

For any p, q, $p+q=t$, $\mathrm{Sres}_t(f,g)$ is proportional to

$$\sum_{\substack{A\subset\{1,\dots,m\},\, B\subset\{1,\dots,n\}\\ |A|=p,\,|B|=q}} \prod(x-a_i)\prod(x-b_j)\,\frac{\prod(a_i-b_j)\prod(a_{i'}-a_{j'})}{\prod(a_i-a_{i'})\prod(b_j-b_{j'})}$$

with $a_i \in A$, $b_j \in B$, $a_{i'} \notin A$, $b_{j'} \notin B$

No known formula for multiple roots!

Some attempts in

- D-Krick-Szanto, Linear Algebra and its Applications 438 (2013)
- Roy- Szpirglas (divided differences)

Are these formulas useful?

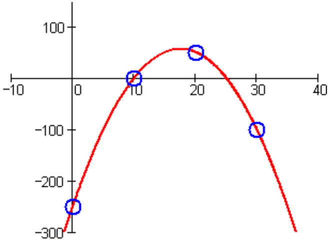
(Sylvester 1853)

Connection between Sturm sequences and Hermite's method for computing real roots of a polynomial via formulas in roots for subresultants

Subresultants and interpolating polynomial

For $m < n$,

$$\mathrm{Sres}_{m-1}(f,g) = \sum_{i=1}^{m} g(a_i) \left(\prod_{j \neq i} \frac{x - a_j}{a_i - a_j} \right),$$

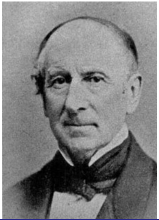


Cauchy Rational Interpolation

Given $\{(x_0, y_0)\ldots,(x_\ell, y_\ell)\}$, $a + b = \ell$ determine –if possible– polynomials $A(x)$, $B(x)$ with $\deg(A) \leq a$, $\deg(B) \leq b$, and

$$\frac{A}{B}(x_i) = y_i,\ \ 0 \leq i \leq \ell$$

(Cauchy, 1841)



Explicit formulas for the rational interpolation

The solution -when exists- has the form

$$A(x) = \sum_{J \subset \{0,\dots,\ell\}, |J|=a} \frac{\prod_j (x-x_j) \left(\prod_{j'} y_{j'}\right)}{\prod (x_{j'}-x_j)} \qquad j \in J,\, j' \notin J$$

$$B(x) = \sum_{J \subset \{0,\dots,\ell\}, |J|=b} \frac{\prod_j (x-x_j) \left(\prod_{j} y_{j}\right)}{\prod (x_{j'}-x_j)} \qquad j \in J,\, j' \notin J$$

(Teresa Krick's talk this afternoon)

Formulas for the *osculatory* rational interpolation?

Subresultants and rational interpolation

- $f(x) = \prod_{j=0}^{\ell}(x - x_j)$
- $g(x)$ such that $g(x_j) = y_j$, $j = 0, \ldots, \ell$.
- $d \leq a$ with some conditions
- $\mathrm{Sres}_d(f, g,) = F_d \cdot f + G_d \cdot g$

$$\boxed{\frac{A(x)}{B(x)} = \frac{\mathrm{Sres}_d(f, g)}{G_d(x)}}$$

(D-Krick-Szanto 13)

Works with multiplicities also!

Matrix/determinantal formulas

$$A(x) = \det \begin{array}{cc} \scriptstyle \ell+1 & \scriptstyle 1 \\ \left|\begin{array}{c|c} V_{d+1}(\overline{X}) & \begin{matrix} 1 \\ \vdots \\ x^d \end{matrix} \\ \hline U_{\ell-d+1}(\overline{X}, Y) & \mathbf{0} \end{array}\right| \end{array}, \quad B(x) = \det \begin{array}{cc} \scriptstyle \ell+1 & \scriptstyle 1 \\ \left|\begin{array}{c|c} V_{d+1}(\overline{X}) & \mathbf{0} \\ \hline U_{\ell-d+1}(\overline{X}, Y) & \begin{matrix} 1 \\ \vdots \\ x^{\ell-d} \end{matrix} \end{array}\right| \end{array}$$

(Teresa Krick's talk this afternoon)

Hermite-Birkhoff Interpolation

- Laureano González-Vega (this afternoon)
- Agnes Szanto's talk (Sunday)

More interpolation in this minisymposia

- Interpolation and implicitization, Christos Konaxis
- Evaluation and interpolation, Joris van der Hoeven
- Interpolation on the Hilbert scheme, Bernard Mourrain
- Interpolation with non standard bases, George Labahn
- Interpolation of linear systems in Algebraic Geometry, Rick Miranda & Giorgio Ottaviani

Thanks!