XIV Encuentro de Álgebra Computacional y Aplicaciones

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Moving surfaces ideals of rational parametrizations

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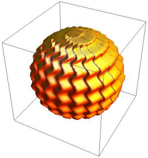
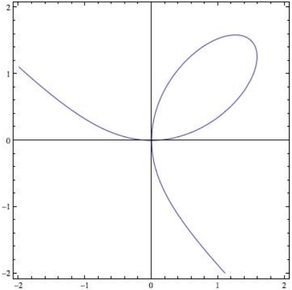
Computer Algebra and Polynomials
Linz - November 2013



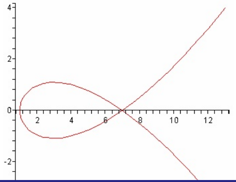
Episode 1 : Curves



Rational Plane Curves

$$\begin{array}{rccc} \phi: & \mathbb{P}^1 & \to & \mathbb{P}^2 \\ & (t_0 : t_1) & \mapsto & (a(t_0,t_1) : b(t_0,t_1) : c(t_0,t_1)) \end{array}$$

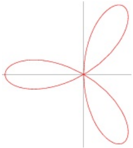
- $a, b, c \in \mathbb{K}[T_0, T_1]$, homogeneous of the same degree $d \geq 1$
- $\gcd(a, b, c) = 1$



Rational Plane Curves

The image of the map is a **rational plane curve**

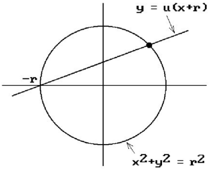
- It has degree d if ϕ is injective "almost everywhere"
- It has genus 0, which means "maximal" number of multiple points $\frac{(d-1)(d-2)}{2}$
- Computing the "implicit equation" is relatively easy from ϕ



From affine to projective

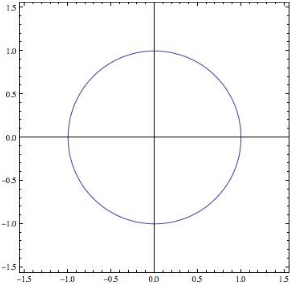
$$\begin{array}{ccc} \mathbb{K} & \dashrightarrow & \mathbb{K}^2 \\ t & \longmapsto & \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \end{array}$$

$$\begin{array}{cccc} \phi: & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ & (t_0 : t_1) & \longmapsto & (t_0^2 + t_1^2 : t_0^2 - t_1^2 : 2t_0t_1) \end{array}$$



From parametric to implicit

$$\mathrm{Res}_{\underline{T}}\big(X_2\cdot a(\underline{T}) - X_0\cdot c(\underline{T}), X_2\cdot b(\underline{T}) - X_1\cdot c(\underline{T})\big)$$
$$=$$
$$-4X_2^2(X_0^2 - X_1^2 - X_2^2)$$



The Sylvester Resultant

$$X_2 a(\underline{T}) - X_0 c(\underline{T}) = X_2 T_0^2 - 2X_0 T_0 T_1 + X_2 T_1^2$$

$$X_2 b(\underline{T}) - X_1 c(\underline{T}) = X_2 T_0^2 - 2X_1 T_0 T_1 - X_2 T_1^2$$

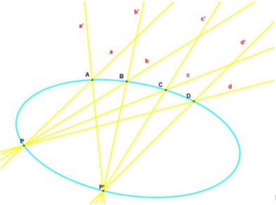
$$\mathrm{Res}_{\underline{T}}(X_2 \cdot a(\underline{T}) - X_0 \cdot c(\underline{T}), X_2 \cdot b(\underline{T}) - X_1 \cdot c(\underline{T})) = \det \begin{pmatrix} X_2 & -2X_0 & X_2 & 0 \\ 0 & X_2 & -2X_0 & X_2 \\ X_2 & -2X_1 & -X_2 & 0 \\ 0 & X_2 & -2X_1 & -X_2 \end{pmatrix}$$

The Sylvester matrix is a matrix of moving lines

$$\mathcal{L}(T_0, T_1, X_0, X_1, X_2) = v_0(\underline{T})X_0 + v_1(\underline{T})X_1 + v_2(\underline{T})X_2$$

such that

$$\mathcal{L}(T_0, T_1, u_0(\underline{T}), u_1(\underline{T}), u_2(\underline{T})) = 0$$



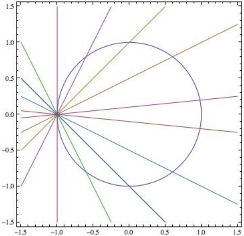
In our example...

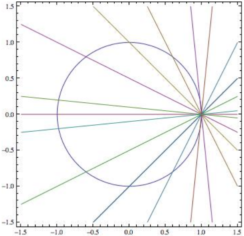
$$
\begin{aligned}
\mathcal{L}_1(\underline{T},\underline{X}) &= -2T_0^2T_1X_0 + 0X_1 + (T_0^3 + T_0T_1^2)X_2 \\
\mathcal{L}_2(\underline{T},\underline{X}) &= -2T_0T_1^2X_0 + 0X_1 + (T_0^2T_1 + T_1^3)X_2 \\
\mathcal{L}_3(\underline{T},\underline{X}) &= 0X_0 - 2T_0^2T_1X_1 + (T_0^3 - T_0T_1^2)X_2 \\
\mathcal{L}_4(\underline{T},\underline{X}) &= 0X_0 - 2T_0T_1^2X_1 + (T_0^2T_1 - T_1^3)X_2
\end{aligned}
$$

$$
\begin{pmatrix}
X_2 & -2X_0 & X_2 & 0 \\
0 & X_2 & -2X_0 & X_2 \\
X_2 & -2X_1 & -X_2 & 0 \\
0 & X_2 & -2X_1 & -X_2
\end{pmatrix}
$$

Can you do it in degree 1?

$$\begin{array}{rcllll} \mathcal{L}_1(\underline{T},\underline{X}) & = & X_2 & T_0 & -(X_0+X_1) & T_1 \\ \mathcal{L}_2(\underline{T},\underline{X}) & = & (-X_0+X_1) & T_0 & +X_2 & T_1 \end{array}$$





$$\det\begin{pmatrix} X_2 & -X_0-X_1 \\ -X_0+X_1 & X_2 \end{pmatrix} = X_1^2 + X_2^2 - X_0^2$$

The module of moving lines following ϕ

(Hilbert)

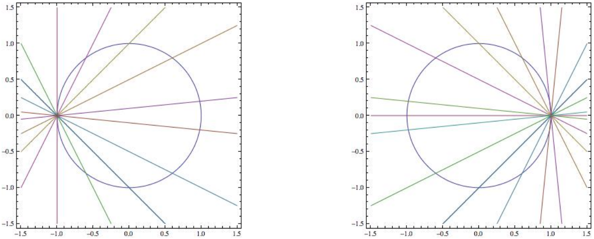
There are $\mu \leq \frac{d}{2}$ and $\mathcal{P}_\mu(\underline{T},\underline{X}),\ \mathcal{Q}_{d-\mu}(\underline{T},\underline{X})$ independent moving lines following ϕ such that every other moving line $\mathcal{L}_\delta(\underline{T},\underline{X})$ following ϕ is of the form

$$p_{\delta-\mu}(\underline{T})\mathcal{P}_\mu(\underline{T},\underline{X}) + q_{\delta-d+\mu}(\underline{T})\mathcal{P}_{d-\mu}(\underline{T},\underline{X})$$

Geometric version

There exist $\mu \leq \frac{d}{2}$ and two other plane parametrizations $\varphi_\mu(t_0, t_1)$, $\psi_{d-\mu}(t_0, t_1)$ of degrees μ, $d-\mu$ such that

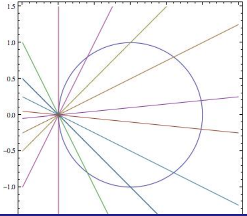
$$\boxed{\phi(t_0, t_1) = \varphi_\mu(t_0, t_1) \wedge \psi_{d-\mu}(t_0, t_1)}$$

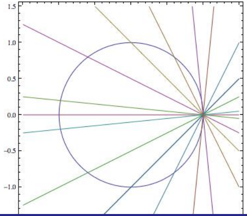


For the unit circle

$$\begin{aligned}\varphi_1(t_0:t_1) &= (-t_1:-t_1:t_0)\\ \psi_1(t_0:t_1) &= (-t_0:t_0:t_1)\end{aligned}$$

$$\begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \\ -t_1 & -t_1 & t_0 \\ -t_0 & t_0 & t_1 \end{vmatrix} = \big(-t_0^2 - t_1^2, t_1^2 - t_0^2, -2t_0t_1\big)$$





μ bases and Hilbert's Syzygy Theorem

The homogeneous polynomial ideal $I = \big(a(\underline{T}), b(\underline{T}), c(\underline{T})\big) \subset \mathbb{K}[T_0, T_1]$ has a **Hilbert-Burch resolution** of the type

$$0 \to \mathbb{K}[\underline{T}]^2 \stackrel{(\varphi_\mu, \psi_{d-\mu})^{\mathbf{t}}}{\longrightarrow} \mathbb{K}[\underline{T}]^3 \stackrel{(a,b,c)}{\longrightarrow} \mathbb{K}[\underline{T}]$$

A μ basis of the parametrization is a basis of $\mathrm{Syz}(I)$ as a $\mathbb{K}[\underline{T}]$-module

Why do we care about μ bases?

Implicit equation

$$=$$

$$\mathrm{Res}_{\underline{T}}(\mathcal{P}_{\mu}(\underline{T},\underline{X}),\ \mathcal{Q}_{d-\mu}(\underline{T},\underline{X}))$$

Busé-D (2012)

If B denotes a Bézout matrix and S a Sylvester matrix then,

$$X_2\, S(\mathcal{P}_{\mu}(\underline{T},\underline{X}),\ \mathcal{Q}_{d-\mu}(\underline{T},\underline{X})) = M\, B(aX_2 - cX_0, bX_2 - cX_1),$$

with $M \in \mathbb{K}^{d\times d}$ invertible

A bit of history on the CAGD side

- Sederberg, Saito, Qi, Klimaszewski. (1994), **Curve implicitization using moving lines**, Computer Aided Geometric Design 11, 687–706
- Sederberg, Chen. **Implicitization using moving curves and surfaces.** Proceedings of SIGGRAPH 1995, 301–308.
- Sederberg, Goldman, Du. (1997), **Implicitizing rational curves by the method of moving algebraic curves,** J. Symbolic Comp. 23, 153–175
- Cox., Sederberg, Chen. (1998), **The moving line ideal basis for planar rational curves**, Computer Aided Geometric Design 15, 803–827
- . . .

Moving conics, Moving cubics,...

$$a_j(\underline{T})X_0^2 + b_j(\underline{T})X_0X_1 + c_j(\underline{T})X_0X_2 + d_j(\underline{T})X_1^2 + e_j(\underline{T})X_1X_2 + f_j(\underline{T})X_2^2$$

is a **moving conic** which follows the parametrization if

$$a_j(\underline{T})a(\underline{T})^2 + b_j(\underline{T})a(\underline{T})b(\underline{T}) + c_j(\underline{T})a(\underline{T})c(\underline{T}) + d_j(\underline{T})b(\underline{T})^2 + e_j(\underline{T})b(\underline{T})c(\underline{T}) + f_j(\underline{T})c(\underline{T})^2 = 0$$

The method of moving curves for implicitization

The implicit equation may be computed as a **small** determinant of

$$\left| \begin{array}{c} \text{some moving lines} \\ \text{some moving conics} \\ \text{some moving cubics} \\ \ldots \end{array} \right|$$

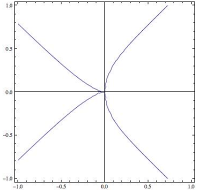
the **more** singular the curve is, the **smaller** the determinant is

One of those theorems (Sederberg & Chen 1995)

The implicit equation of a quartic curve with no base points can be written as a 2×2 determinant. If the curve doesn't have a triple point, then each element of the determinant is a quadratic; otherwise one row is linear and one row is cubic

A quartic with triple point

$$\phi(t_0, t_1) = (t_0^4 - t_1^4 : -t_0^2 t_1^2 : t_0 t_1^3)$$
$$F(X_0, X_1, X_2) = X_2^4 - X_1^4 - X_0 X_1 X_2^2$$

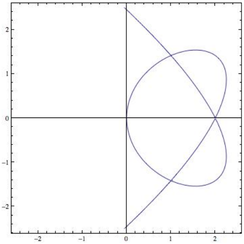


$$\begin{array}{rcl} \mathcal{L}_{1,1}(\underline{T}, \underline{X}) & = & T_0 X_2 + T_1 X_1 \\ \mathcal{L}_{1,3}(\underline{T}, \underline{X}) & = & T_0 (X_1^3 + X_0 X_2^2) + T_1 X_2^3 \end{array}$$

$$\begin{pmatrix} X_2 & X_1 \\ X_1^3 + X_0 X_2^2 & X_2^3 \end{pmatrix}$$

A quartic without a triple point

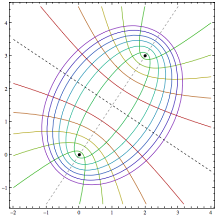
$$\phi(t_0 : t_1) = (t_0^4 : 6t_0^2 t_1^2 - 4t_1^4 : 4t_0^3 t_1 - 4t_0 t_1^3)$$
$$F(\underline{X}) = X_2^4 + 4X_0X_1^3 + 2X_0X_1X_2^2 - 16X_0^2X_1^2 - 6X_0^2X_2^2 + 16X_0^3X_1$$



$$\mathcal{L}_{1,2}(\underline{T}, \underline{X}) = T_0(X_1X_2 - X_0X_2) + T_1(-X_2^2 - 2X_0X_1 + 4X_0^2)$$
$$\tilde{\mathcal{L}}_{1,2}(\underline{T}, \underline{X}) = T_0(X_1^2 + \frac{1}{2}X_2^2 - 2X_0X_1) + T_1(X_0X_2 - X_1X_2)$$

For large d, we do not know...

which moving lines?
which moving conics?
which moving cubics?
...



The Rees Algebra associated to the parametrization

Cox, D. **The moving curve ideal and the Rees algebra.** Theoret. Comput. Sci. 392 (2008), no. 1–3, 23–36.

$\mathcal{K}_\phi := \{$moving curves following $\phi\} =$ kernel of

$$\begin{array}{ccc} \mathbb{K}[T_0, T_1, X_0, X_1, X_2] & \to & \mathbb{K}[T_0, T_1, s] \\ T_i & \mapsto & T_i \\ X_0 & \mapsto & a(\underline{T})s \\ X_1 & \mapsto & b(\underline{T})s \\ X_2 & \mapsto & c(\underline{T})s \end{array}$$

"The defining ideal of the Rees Algebra associated to ϕ"

The implicit equation may be obtained as the determinant of a very small matrix:

$$\left| \begin{array}{c} \cdots \\ \text{some minimal generators of } \mathcal{K}_\phi \\ \cdots \end{array} \right|$$

The more singular the curve, the simpler the description of $\mathcal{K}_\phi$

Main Problem

Compute a minimal set of generators of $\mathcal{K}_\phi$ for **any** ϕ

Known for:

- $\mu = 1$ (Hong-Simis-Vasconcelos, Cox-Hoffmann-Wang, Busé, Cortadellas-**D**)
- $\mu = 2$(Busé, Cortadellas-**D**, Kustin-Polini-Ulrich)
- $(\mathcal{K}_\phi)_{(1,2)} \neq 0$ (Cortadellas- **D**)
- Monomial plane parametrizations (Cortadellas-**D**)

Coarse problem

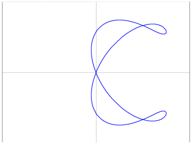
Compute $n_0(\mathcal{K}_\phi)$, the number of minimal generators of $\mathcal{K}_\phi$

Show that if ϕ is "more singular" than ϕ' then $n_0(\mathcal{K}_\phi) \leq n_0(\mathcal{K}_{\phi'})$

Example: $\mu = 2$

The curve has either

- one point of multiplicity $d - 2$
  $n_0 = \mathcal{O}\left(\frac{d}{2}\right)$
  (Cortadellas-**D**, Kustin-Polini-Ulrich)
- only double points
  $n_0 = \mathcal{O}\left(\frac{d^2}{2}\right)$ (Busé)



Related Problems

- Describe **all** the possible values and the parameters of the function $n_0(\mathcal{K}_\phi)$
- Is there a **generic** value for $n_0(\mathcal{K}_\phi)$? Is this the maximal value?
- Where is $n_0(\mathcal{K}_\phi)$ constant?

An example with $\mu = 3$

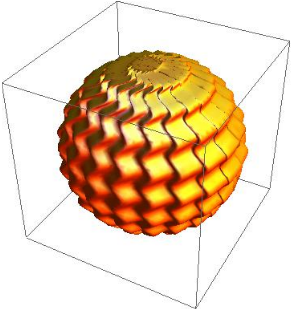
$$\begin{cases} \mathcal{P}_3 = T_0^3 X_0 + (T_1^3 - T_0 T_1^2) X_1 \\ \mathcal{Q}_7 = (T_0^6 T_1 - T_0^2 T_1^5) X_0 + (T_0^4 T_1^3 + T_0^2 T_1^5) X_1 + (T_0^7 + T_1^7) X_2 \end{cases}$$

with minimal generators of bidegree
$(3,1), (7,1), (2,3), (2,3), (4,2), (2,4), \mathbf{(1,6)}, (1,6), (1,6), (0,10)$

$$\begin{cases} \tilde{\mathcal{P}}_3 = (T_0^3 - T_0^2 T_1) X_0 + (T_1^3 + T_0 T_1^2 - T_0 T_1^2) X_1 \\ \tilde{\mathcal{Q}}_7 = (T_0^6 T_1 - T_0^2 T_1^5) X_0 + (T_0^4 T_1^3 + T_0^2 T_1^5) X_1 + (T_0^7 + T_1^7) X_2 \end{cases}$$

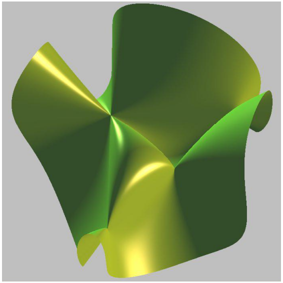
with minimal generators of bidegree
$(3,1), (7,1), (2,3), (2,3), (4,2), (2,4), \mathbf{(1,5)}, (1,6), (1,6), (0,10)$

Episode 2 : Surfaces



Rational Parametrizations

$$\begin{array}{rccc} \phi_S : & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^3 \\ & \underline{t} = (t_0 : t_1 : t_2) & \longmapsto & \big(a(\underline{t}) : b(\underline{t}) : c(\underline{t}) : d(\underline{t})\big) \end{array}$$



Elimination

- multivariate / sparse resultants (Macaulay, Dixon, Gelfand-Kapranov-Zelevinskii), ...
- Determinants of complexes Botbol, Busé, Chardin, Jouanlou, ...

Base Points!

Moving Planes, Quadrics,...

(Sederberg-Chen, Cox-Goldman-Zhang, Busé-Cox, **D**, **D**-Khetan)

Features

- The module of moving planes is not free anymore!
- Definition of μ-basis given by Chen-Cox-Liu. **Not easy to compute**

Some results

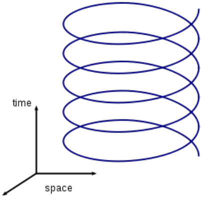
Implicitization

- Quadratic and cubic surfaces (Chen-Shen-Deng)
- Steiner surfaces (Wang-Chen)
- Surfaces of revolution (Shi-Goldman)
- ...

Rees Algebra

- Monoid surfaces (Cortadellas - **D**)
- *de Jonquières* surfaces (Hassanzadeh- Simis)

Episodes 3, 4, . . .



- Curves in space
- Adjoints
- . . .

Adjoints

A curve C_0 is **adjoint** to another curve C if $m_p(\mathrm{C}_0) \geq m_p(\mathrm{C}) - 1$ for all $p \in \mathrm{C}$

Cox's conjecture (2008)

Any moving curve of the form

$$F_{1,\ell} = T_0\mathcal{A}(X_0, X_1, X_2) + T_1\mathcal{B}(X_0, X_1, X_2)$$

is a **pencil of adjoints** if $\ell \geq d - 2$

Cox's conjecture

Works for

- $\mu = 1$ (Cox)
- $\mu = 2$ and only double points (Busé)
- general curves

Fails in general

- $\mu = 2$ and a very singular point (Cortadellas-**D**)
- Monomial plane parametrizations (Cortadellas-**D**)

Conjecture (Cortadellas - D)

$dim_{\mathbb{K}} \left(\mathcal{K}_{1,\ell}/\{\text{pencils of adjoints}\} \cap \mathcal{K}_{1,\ell}\right)$ is independent of ℓ, for $\ell \gg 0$

Current work on the subject

- Cox, Kustin, Polini. **A study of singularities on rational curves via syzygies.** Memoirs of AMS, Volume 222, 2013
- Cortadellas, **D. The Rees Algebra of a monomial plane parametrization** `arXiv:1311.5488` (2013)
- **D. Moving curve ideals of rational plane parametrizations.** Proceedings of this conference, LNCS
- Hassanzadeh, Simis. **Implicitization of the Jonquières parametrizations.**`arXiv:1205.1083`
- Kustin, Polini, Ulrich.**The bi-graded structure of Symmetric Algebras with applications to Rees rings.** `arXiv:1301.7106`
- Iarrobino. **Strata of vector spaces of forms in $k[x, y]$ and of rational curves in $\mathbb{P}^k$.** `arXiv:1306.1282`
- ...

Thanks!

http://www.ub.edu/eaca2014/