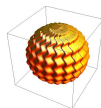


The Rees Algebra of some monomial parametrizations

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Applications of Computer Algebra
New York - July 2014

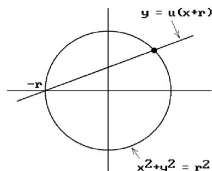
joint with Teresa Cortadellas (U. Barcelona)



Parametrization of the circle

$$\begin{array}{ccc} \mathbb{K} & \dashrightarrow & \mathbb{K}^2 \\ t & \mapsto & \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \end{array}$$

$$\begin{array}{ccc} \varphi : \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ (t_0 : t_1) & \longmapsto & (t_0^2 + t_1^2 : t_0^2 - t_1^2 : 2t_0 t_1) \end{array}$$



The Sylvester Resultant

$$F = X_2(T_0^2 + T_1^2) - X_0(2T_0T_1) = X_2T_0^2 - 2X_0T_0T_1 + X_2T_1^2$$

$$G = X_2(T_0^2 - T_1^2) - X_1(2T_0T_1) = X_2T_0^2 - 2X_1T_0T_1 - X_2T_1^2$$

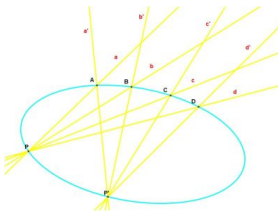
$$\text{Res}_T(F, G) = \begin{vmatrix} X_2 & -2X_0 & X_2 & 0 \\ 0 & X_2 & -2X_0 & X_2 \\ X_2 & -2X_1 & -X_2 & 0 \\ 0 & X_2 & -2X_1 & -X_2 \end{vmatrix} = 4X_2^2(X_1^2 + X_2^2 - X_0^2)$$

Rows of the Sylvester matrix encode **moving lines**

$$\mathcal{L}(T_0, T_1, X_0, X_1, X_2) = v_0(\underline{T})X_0 + v_1(\underline{T})X_1 + v_2(\underline{T})X_2$$

such that

$$\mathcal{L}(T_0, T_1, u_0(\underline{T}), u_1(\underline{T}), u_2(\underline{T})) = 0$$



In our example...

$$\mathcal{L}_1(\underline{T}, \underline{X}) = -2T_0^2 T_1 X_0 + 0X_1 + (T_0^3 + T_0 T_1^2) X_2$$

$$\mathcal{L}_2(\underline{T}, \underline{X}) = -2T_0 T_1^2 X_0 + 0X_1 + (T_0^2 T_1 + T_1^3) X_2$$

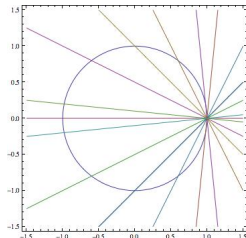
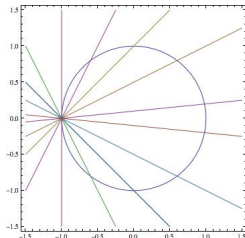
$$\mathcal{L}_3(\underline{T}, \underline{X}) = 0X_0 - 2T_0^2 T_1 X_1 + (T_0^3 - T_0 T_1^2) X_2$$

$$\mathcal{L}_4(\underline{T}, \underline{X}) = 0X_0 - 2T_0 T_1^2 X_1 + (T_0^2 T_1 - T_1^3) X_2$$

$$\begin{pmatrix} X_2 & -2X_0 & X_2 & 0 \\ 0 & X_2 & -2X_0 & X_2 \\ X_2 & -2X_1 & -X_2 & 0 \\ 0 & X_2 & -2X_1 & -X_2 \end{pmatrix}$$

Can you get a smaller determinant?

$$\begin{array}{rcl} \mathcal{L}_1(\underline{T}, \underline{X}) & = & \begin{array}{ccc} X_2 & T_0 & -(X_0 + X_1) \\ & & T_1 \end{array} \\ \mathcal{L}_2(\underline{T}, \underline{X}) & = & \begin{array}{ccc} (-X_0 + X_1) & T_0 & +X_2 \\ & & T_1 \end{array} \end{array}$$



$$\begin{vmatrix} X_2 & -X_0 - X_1 \\ -X_0 + X_1 & X_2 \end{vmatrix} = X_1^2 + X_2^2 - X_0^2$$

A bit of history on the CAGD side

- Sederberg, Saito, Qi, Klimaszewski. (1994), **Curve implicitization using moving lines**, Computer Aided Geometric Design 11, 687–706
- Sederberg, Chen. **Implicitization using moving curves and surfaces**. Proceedings of SIGGRAPH 1995, 301–308.
- Sederberg, Goldman, Du. (1997), **Implicitizing rational curves by the method of moving algebraic curves**, J. Symbolic Comp. 23, 153–175
- Cox., Sederberg, Chen. (1998), **The moving line ideal basis for planar rational curves**, Computer Aided Geometric Design 15, 803–827
- ...

Moving conics, Moving cubics,...

$$a_j(\underline{T})X_0^2 + b_j(\underline{T})X_0X_1 + c_j(\underline{T})X_0X_2 + d_j(\underline{T})X_1^2 + \\ e_j(\underline{T})X_1X_2 + f_j(\underline{T})X_2^2$$

is a **moving conic** which follows the
parametrization if

$$a_j(\underline{T})a(\underline{T})^2 + b_j(\underline{T})a(\underline{T})b(\underline{T}) + c_j(\underline{T})a(\underline{T})c(\underline{T}) + \\ d_j(\underline{T})b(\underline{T})^2 + e_j(\underline{T})b(\underline{T})c(\underline{T}) + f_j(\underline{T})c(\underline{T})^2 = 0$$

The method of moving curves for implicitization

The implicit equation may be computed as a **small** determinant of

some moving lines
some moving conics
some moving cubics
...

the **more** singular the curve is, the **smaller** the determinant is

One of those theorems

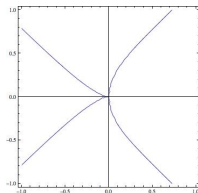
(Sederberg-Chen 1995)

The implicit equation of a quartic curve with no base points can be written as a 2×2 determinant. If the curve doesn't have a triple point, then each element of the determinant is a quadratic; otherwise one row is linear and one row is cubic

A quartic with triple point

$$\varphi(t_0, t_1) = (t_0^4 - t_1^4 : -t_0^2 t_1^2 : t_0 t_1^3)$$

$$F(X_0, X_1, X_2) = X_2^4 - X_1^4 - X_0 X_1 X_2^2$$

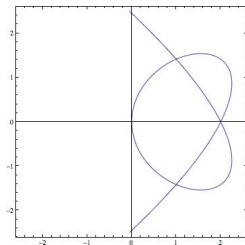


$$\begin{aligned} \mathcal{L}_{1,1}(\underline{T}, \underline{X}) &= T_0 X_2 + T_1 X_1 \\ \mathcal{L}_{1,3}(\underline{T}, \underline{X}) &= T_0 (X_1^3 + X_0 X_2^2) + T_1 X_2^3 \\ &\quad \begin{pmatrix} X_2 & X_1 \\ X_1^3 + X_0 X_2^2 & X_2^3 \end{pmatrix} \end{aligned}$$

A quartic without a triple point

$$\varphi(t_0 : t_1) = (t_0^4 : 6t_0^2t_1^2 - 4t_1^4 : 4t_0^3t_1 - 4t_0t_1^3)$$

$$F(\underline{X}) = X_2^4 + 4X_0X_1^3 + 2X_0X_1X_2^2 - 16X_0^2X_1^2 - 6X_0^2X_2^2 + 16X_0^3X_1$$



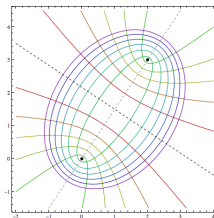
$$\mathcal{L}_{1,2}(\underline{T}, \underline{X}) = T_0(X_1X_2 - X_0X_2) + T_1(-X_2^2 - 2X_0X_1 + 4X_0^2)$$

$$\tilde{\mathcal{L}}_{1,2}(\underline{T}, \underline{X}) = T_0(X_1^2 + \frac{1}{2}X_2^2 - 2X_0X_1) + T_1(X_0X_2 - X_1X_2)$$

For large d , we do not know...

which moving lines?
which moving conics?
which moving cubics?

...



The Rees Algebra associated to the parametrization
 Cox, D. **The moving curve ideal and the Rees algebra**. Theoret. Comput. Sci. 392 (2008), no. 1–3.

$\mathcal{K}_\varphi := \{\text{moving curves following } \varphi\} = \text{kernel of}$

$$\begin{array}{ccc} \mathbb{K}[T_0, T_1, X_0, X_1, X_2] & \xrightarrow{\varphi_R} & \mathbb{K}[T_0, T_1, s] \\ T_i & \mapsto & T_i \\ X_0 & \mapsto & a(\underline{T})s \\ X_1 & \mapsto & b(\underline{T})s \\ X_2 & \mapsto & c(\underline{T})s \end{array}$$

“The defining ideal of the Rees Algebra associated to
 φ ”

The implicit equation may be obtained as the determinant of a very small matrix:

$$\begin{vmatrix} \dots \\ \text{some minimal generators of } \mathcal{K}_\varphi \\ \dots \end{vmatrix}$$

The more singular the curve, the simpler the description of $\mathcal{K}_\varphi$

Main Problem

- Compute $n_0(\mathcal{K}_\varphi)$, the number of minimal generators of $\mathcal{K}_\varphi$
- Show that if φ is “more singular” than φ' then $n_0(\mathcal{K}_\varphi) \leq n_0(\mathcal{K}_{\varphi'})$
- Describe **all** the possible values and the parameters of the function $n_0(\mathcal{K}_\varphi)$
- Is there a **generic** value for $n_0(\mathcal{K}_\varphi)$? Is this the maximal value?
- Where is $n_0(\mathcal{K}_\varphi)$ constant?

Known so far

- $\mu = 1$ (Hong-Simis-Vasconcelos, Cox-Hoffmann-Wang, Busé, Cortadellas-**D**)
- $\mu = 2$ (Busé, Cortadellas-**D**, Kustin-Polini-Ulrich)
- $(\mathcal{K}_\varphi)_{(1,2)} \neq 0$ (Cortadellas- **D**)
- Monomial plane parametrizations (Cortadellas-**D**)

Monomial plane parametrizations

$$\varphi_{u,d} : \mathbb{P}_{\mathbb{K}}^1 \rightarrow \mathbb{P}_{\mathbb{K}}^2 \\ (t_0 : t_1) \mapsto (t_0^d : t_0^{d-u} t_1^u : t_1^d)$$

- $1 \leq u < d/2$, $\gcd(u, d) = 1$.
- The implicit equation is $x_1^d - x_0^{d-u} x_2^u = 0$
- Two singular points: $(1 : 0 : 0)$ and $(0 : 0 : 1)$ of multiplicities u and $d - u$ respectively

A Minimal resolution of $\mathcal{K}_{\varphi_{u,d}}$ (Cortadellas-D)

(arxiv.1311.5488)

$$0 \rightarrow S^{q-1} \rightarrow S^{2q} \rightarrow S^{q+2} \rightarrow S \xrightarrow{\varphi_{u,d,R}} \mathbb{K}[T_0, T_1, s] \rightarrow 0$$

- $S = \mathbb{K}[T_0, T_1, X_0, X_1, X_2]$
- It is a resolution of S -modules
- q and the maps of the resolution depend on the arithmetics of $\gcd(u, d)$

A Minimal *bigraded* resolution

(arxiv.1311.5488)

$$0 \rightarrow \bigoplus_{n=1}^{q-1} S(-(b_n, |\sigma_n - \tau_n| + 2|\sigma_{m_{\ell(n)}} - \tau_{m_{\ell(n)}}|)) \rightarrow$$

$$\xrightarrow{\varphi_{u,d,3}} \bigoplus_{n=1}^q S(-(b_n, |\sigma_n - \tau_n| + |\sigma_{m_{\ell(n)}} - \tau_{m_{\ell(n)}}|))^2 \rightarrow$$

$$\xrightarrow{\varphi_{u,d,2}} \bigoplus_{n=1}^{q+2} S(-(b_n, |\sigma_n - \tau_n|)) \xrightarrow{\varphi_{u,d,1}} S \xrightarrow{\varphi_{u,d,R}} \mathbb{K}[T_0, T_1, s]$$

The theorem on an example

$$u = 3, d = 10$$

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\varphi_{3,10}} & \mathbb{P}^2 \\ (t_0 : t_1) & \mapsto & (t_0^{10} : t_0^7 t_1^3 : t_1^{10}) \end{array}$$

$$\begin{array}{ccc} \mathbb{K}[T_0, T_1, X_0, X_1, X_2] & \xrightarrow{\varphi_{3,10,R}} & \mathbb{K}[T_0, T_1, s] \\ T_i & \mapsto & T_i \\ X_0 & \mapsto & T_0^{10} s \\ X_1 & \mapsto & T_0^7 T_1^3 s \\ X_2 & \mapsto & T_1^{10} s \end{array}$$

Classic Euclidean remainder sequences

Applied to the pair $(d - u, u) = (7, 3)$

$$7 = 2 \cdot 3 + 1$$

$$3 = 3 \cdot 1 + 0,$$

$$q = 2 + 3 = 5$$

$$0 \rightarrow S^{q-1} \rightarrow S^{2q} \rightarrow S^{q+2} \rightarrow S \xrightarrow{\varphi_{u,d,R}} \mathbb{K}[T_0, T_1, s] \rightarrow 0$$

specializes to

$$0 \rightarrow S^4 \rightarrow S^{10} \rightarrow S^7 \rightarrow S \xrightarrow{\varphi_{3,10,R}} \mathbb{K}[T_0, T_1, s] \rightarrow 0$$

Slow Euclidean remainder sequences

$$\begin{array}{rclclcl} 7 & = & 1 & \cdot & 7 & + & 0 & \cdot & 3 \\ 3 & = & 0 & \cdot & 7 & + & 1 & \cdot & 3 \\ 4 & = & 1 & \cdot & 7 & + & (-1) & \cdot & 3 \\ 1 & = & 1 & \cdot & 7 & + & (-2) & \cdot & 3 \\ 2 & = & -1 & \cdot & 7 & + & 3 & \cdot & 3 \\ 1 & = & -2 & \cdot & 7 & + & 5 & \cdot & 3 \\ 0 & = & -3 & \cdot & 7 & + & 7 & \cdot & 3 \end{array}$$

There are $q + 2$ elements in the sequence

The minimal generators of $\mathcal{K}_{\varphi_{3,10}}$ are

$$\begin{aligned}F_7(\underline{T}, \underline{X}) &= T_0^7 X_2 - T_1^7 X_1 \\F_3(\underline{T}, \underline{X}) &= T_0^3 X_1 - T_1^3 X_0 \\F_1(\underline{T}, \underline{X}) &= T_0 X_0^2 X_2 - T_1 X_1^3 \\F_4(\underline{T}, \underline{X}) &= T_0^4 X_0 X_2 - T_1^4 X_1^2 \\F_2(\underline{T}, \underline{X}) &= T_0^2 X_1^4 - T_1^2 X_0^3 X_2 \\F_1^*(\underline{T}, \underline{X}) &= T_0 X_1^7 - T_1 X_0^5 X_2^2 \\F_0(\underline{T}, \underline{X}) &= X_0^7 X_2^3 - X_1^{10}\end{aligned}$$

Theorem (Cortadellas-D)

This family of generators is minimal,
and also a *Gröbner basis* of $\mathcal{K}_{\varphi_{u,d}}$ for
the lexicographic order

The construction of the other maps of the resolution
can also be made explicit via the SERS
(arxiv.1311.5488)

Geometric features

A curve C_0 is **adjoint** to another curve C if
 $m_p(C_0) \geq m_p(C) - 1$ for all $p \in C$

Cox's conjecture (2008)

Any $T_0 \mathcal{A}(X_0, X_1, X_2) + T_1 \mathcal{B}(X_0, X_1, X_2) \in (\mathcal{K}_\varphi)_{(1,\ell)}$
is a **pencil of adjoints** for $\ell \geq d - 2$

False in general (Busé, Jia, Cortadellas-D,...)

Measuring the difference

Theorem (Cortadellas - D)

For $\ell \geq d - 2$, the number

$$\dim_{\mathbb{K}} \left((\mathcal{K}_{\varphi_{u,d}})_{1,\ell} / \{\text{pencils of adjoints}\} \cap (\mathcal{K}_{\varphi_{u,d}})_{1,\ell} \right)$$

depends only on (u, d) , and grows quadratically with d

(arxiv.1311.5488)

