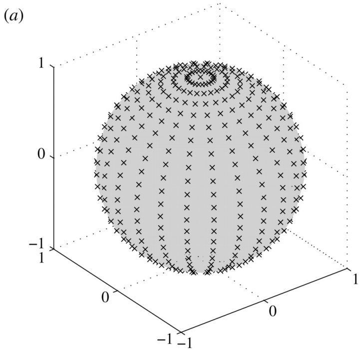
Quantitative equidistribution of algebraic points in the N-dimensional torus

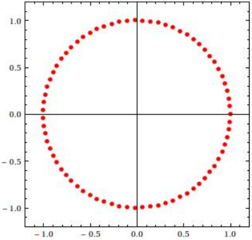
Carlos D'Andrea, Marta Narváez-Clauss, Martín Sombra

February 4, 2014

Equidistribution vs Distribution at Random

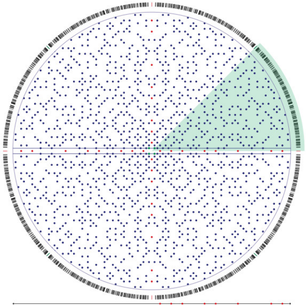


Equidistribution on the 1-dimensional torus (S^1)



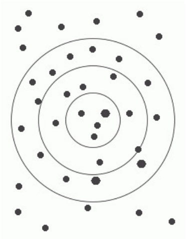
$$\mathbf{P} = \{p_1, \ldots, p_d\} \subset \mathbb{C}^\times, \quad p_j = \rho_j \, e^{i\theta_j}$$

Angular Discrepancy



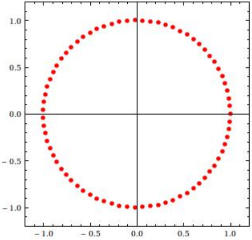
$$\Delta_\theta(\mathbf{P}) := \sup_{0\leq\alpha<\beta<2\pi} \left| \frac{\#\{k : \alpha \leq \theta_k < \beta\}}{\#\boldsymbol{P}} - \frac{\beta-\alpha}{2\pi} \right|$$

Radius Discrepancy



$$\Delta_{\mathrm{r}}(\mathbf{P};\varepsilon) := 1 - \frac{\#\left\{k : 1-\varepsilon < \rho_k < \frac{1}{1-\varepsilon}\right\}}{\#\boldsymbol{P}}$$

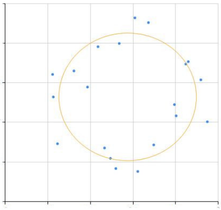
$$(0 < \varepsilon < 1)$$



There is equidistribution when

$$\Delta_\theta(\mathbf{P}),\ \Delta_{\mathrm{r}}(\mathbf{P};\varepsilon) \ll 1\ \forall\varepsilon$$

Another way of measuring equidistribution



- $\delta_{\mathbf{P}} = \frac{1}{d} \sum_{k=1}^{d} \delta_{p_k}$

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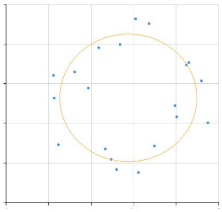

- $\delta_{\mathbf{P}} = \frac{1}{d}\sum_{k=1}^{d}\delta_{p_k}$
- μ_{S^1} : the Haar measure supported on $S^1 \subset \mathbb{C}^\times$

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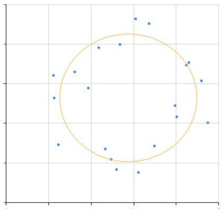

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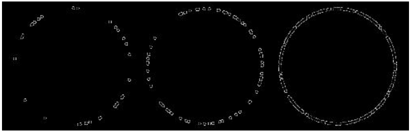
- μ_{S^1} : the Haar measure supported on $S^1 \subset \mathbb{C}^\times$

$$\boxed{\text{Equidistribution} \quad \Longleftrightarrow \quad \left| \int_{\mathbb{C}^\times} F\, \mathrm{d}\delta_{\mathbf{P}} - \int_{\mathbb{C}^\times} F d\mu_{S^1} \right| \ll 1}$$

Quantitative equidistribution



How to estimate

$\Delta_\theta(\mathbf{P})$, $\Delta_{\mathrm{r}}(\mathbf{P};\varepsilon)$, and

$$\left|\int_{\mathbb{C}^\times} F\,\mathrm{d}\delta_{\mathbf{P}} - \int_{\mathbb{C}^\times} F d\mu_{S^1}\right|$$

as functions of $\mathbf{P}$ ε, F?

Quantitative equidistribution for polynomials

$\mathbf{P} = Z(f)$, $f(x) = a_d x^d + \cdots + a_0 \in \mathbb{C}[x]$

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Theorem

[Erdös-Turán 1948], [Hughes-Nikeghbali 2008]

$$\Delta_\theta(\boldsymbol{P}) \leq c \sqrt{\frac{1}{d} \log\left(\frac{||f||}{\sqrt{|a_0 a_d|}}\right)}$$

$$\Delta_{\mathrm{r}}(\boldsymbol{P}; \varepsilon) \leq \frac{2}{\varepsilon d} \log\left(\frac{||f||}{\sqrt{|a_0 a_d|}}\right)$$

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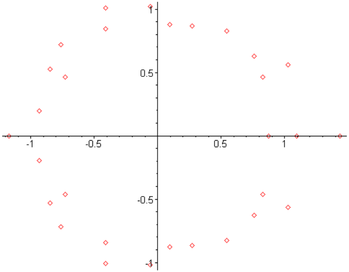
with $||f|| := \max|a_k|,\ c \le 10$

Example

$d = 30$ and

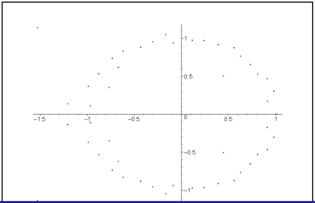
$$\begin{aligned} f \;=\; & x^{30} - x^{29} - x^{28} + x^{26} + x^{25} - x^{24} - x^{23} - x^{22} \\ & + x^{21} - x^{20} + x^{19} + x^{18} + x^{16} + x^{15} - x^{14} \\ & + x^{13} + x^{12} + x^{10} + x^{9} - x^{6} + x^{5} - 1 \end{aligned}$$



Example 2

$d = 50$ and

$$f = -24 + 12x - 44x^{48} - 48x^{49} - 42x^{28} + 15x^{29} + 34x^{26} + 22x^{27} - 24x^{24} + 29x^{25} + 14x^{2} - 40x^{3} - 48x^{4} + 35x^{5} + 24x^{6} + 27x^{7} - 3x^{8} - 15x^{9} - 21x^{10} + 12x^{14} - 15x^{50} - 14x^{33} + 38x^{34} + 10x^{35} - 23x^{36} + 48x^{37} + 30x^{38} - 23x^{39} - 31x^{40} + 2x^{41} + 24x^{42} + 9x^{43} - 15x^{44} - 29x^{45} + 45x^{46} + 40x^{47} + 40x^{31} - 40x^{32} + 38x^{11} + 8x^{12} - 16x^{13} - 39x^{15} + 2x^{16} - 38x^{17} - x^{18} + 16x^{19} - 44x^{20} - 20x^{21} + 22x^{22} + 28x^{23} + 32x^{30}$$



For $f \in \mathbb{C}[x]$, its *Mahler Measure* is

$$M(f) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta\right)$$

Quantitative equidistribution of algebraic numbers

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$$M(f) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta\right)$$

Let $p \in \bar{\mathbb{Q}}^{\times}$, and $g_p \in \mathbb{Z}[x]$ its irreducible defining polynomial. The *height* of p is

$$\mathrm{h}(p) = \frac{\log M(g_p)}{\deg(g_p)}$$

Quantitative equidistribution

$p \in \mathbb{Q}^{\times}$, $\mathbf{P} = \mathcal{O}(p)$

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$p \in \mathbb{Q}^\times$, $\mathbf{P} = \mathcal{O}(p)$

Theorem [Petsche 2003]

For $\varepsilon > 0$ and an *admissible* function $F : \mathbb{C}^\times \longrightarrow \mathbb{C}$, there is a constant $c(F,\varepsilon) > 0$ such that

$$\left| \int_{\mathbb{C}^\times} F \mathrm{d}\delta_{\mathbf{P}} - \int_{\mathbb{C}^\times} F \mathrm{d}\mu_{S^1} \right| \leq c(F,\varepsilon) \mathrm{h}(\xi)^{\frac{1}{3}-\varepsilon}$$

Quantitative equidistribution 2

$p \in \overline{\mathbb{Q}}^{\times}$, $\mathbf{P} = \mathcal{O}(p)$, $\#\mathbf{P}$=d

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Theorem [Favre - Rivera L. 2006]

$\exists C > 0$ such that $\forall F \in \mathcal{C}^1(\mathbb{P}^1(\mathbb{C}), \mathbb{R})$

$$\left| \int_{\mathbb{P}^1(\mathbb{C})} F \mathrm{d}\delta_{\mathbf{P}} - \int_{\mathbb{P}^1(\mathbb{C})} F \mathrm{d}\mu_{S^1} \right| \leq \mathrm{Lip}(F) \left[\frac{1}{d} + \left(\mathrm{h}(p) + C \frac{\log d}{d} \right)^{\frac{1}{2}} \right]$$

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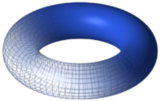
$$\mathrm{Lip}(F) = \sup_{x \neq y} \frac{|F(x) - F(y)|}{d(x,y)}$$

Equidistribution in the N-dimensional torus



$$\overbrace{S^1 \times \ldots \times S^1}^{N}$$

$$\mathbf{P} = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d\} \subset (\mathbb{C}^\times)^N$$

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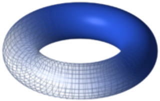


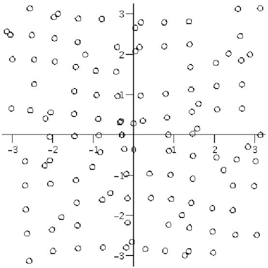
$$\overbrace{S^1 \times \ldots \times S^1}^{N}$$

$$\mathbf{P} = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d\} \subset (\mathbb{C}^\times)^N$$

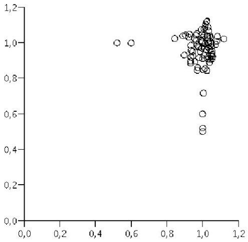
$$\boldsymbol{p}_k = (p_{k1}, \ldots, p_{kN}) = (\rho_{k1} e^{i\theta_{k1}}, \ldots, \rho_{kN} e^{i\theta_{kN}})$$

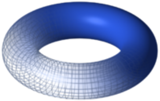
Radius and Angular Discrepancy

$\Delta_\theta(\mathbf{P})$



$\Delta_{\mathrm{r}}(\mathbf{P}, \varepsilon)$





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as functions of $\mathbf{P}$, ε, F?

Evidence

Theorem [Bilu, 1997]

Let $\{\boldsymbol{p}_k\}_{k\in\mathbb{N}} \subset (\bar{\mathbb{Q}}^\times)^N$ be a *strict sequence* with $\lim_{k\to\infty} \mathrm{h}(\boldsymbol{p}_k) = 0$. Then, for $F \in \mathcal{C}_0((\mathbb{C}^\times)^N, \mathbb{C})$

$$\lim_{k\to\infty} \left| \int_{(\mathbb{C}^\times)^N} F \mathrm{d}\delta_{\boldsymbol{P}_k} - \int_{(\mathbb{C}^\times)^N} F \mathrm{d}\mu_{(S^1)^N} \right| = 0$$

Quantitative equidistribution for polynomials

$$\mathbf{P} = Z(f_1, \ldots, f_N),\ f_j \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}],\ 1 \leq j \leq N$$

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Theorem [D-Galligó-Sombra, 2014]

$$\Delta_\theta(\boldsymbol{P}) \leq c\,\eta(\boldsymbol{f})^{\frac{1}{3}} \log^+\left(\frac{1}{\eta(\boldsymbol{f})}\right),\ \Delta_{\mathrm{r}}(\boldsymbol{P};\varepsilon) \leq c\,\eta(\boldsymbol{f})$$

with $c \leq 2^{3n} n^{\frac{n+1}{2}}$ and $\eta(f) = \frac{\|f\|}{\sqrt{|a_0 a_d|}}$ if $N = 1$

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Theorem [D-Narváez-Sombra, 2015]

For $F \in \mathcal{C}_0^{2N+1}((\mathbb{C}^{\times})^N, \mathbb{C})$, $\exists c(F) > 0$ such that

$$\left| \int_{(\mathbb{C}^{\times})^N} F \mathrm{d}\delta_{\mathbf{P}} - \int_{(\mathbb{C}^{\times})^N} F \mathrm{d}\mu_{(S^1)^N} \right| \leq c(F) \left[\frac{1}{d} + \left(\mathrm{h}(\boldsymbol{p}) + C \, \frac{\log\, d}{d} \right)^{\frac{1}{2}} \right]$$

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$d = \min_{\boldsymbol{n} \in \mathbb{N}^N} \|\boldsymbol{n}\| \deg(\boldsymbol{p}^{\boldsymbol{n}})$

C = Favre-Rivera Letelier's constant $(N = 1)$

Corollary: Bilu's Theorem

$$\left|\int_{(\mathbb{C}^\times)^N} F \mathrm{d}\delta_{\boldsymbol{p}_k} - \int_{(\mathbb{C}^\times)^N} F \mathrm{d}\mu_{(S^1)^N}\right| \leq c(F)\left[\frac{1}{d_k} + \left(\mathrm{h}(\boldsymbol{p}_k) + C\,\frac{\log\, d_k}{d_k}\right)^{\frac{1}{2}}\right]$$

Corollary: Bilu's Theorem

$$\left| \int_{(\mathbb{C}^\times)^N} F \mathrm{d}\delta_{\boldsymbol{p}_k} - \int_{(\mathbb{C}^\times)^N} F \mathrm{d}\mu_{(S^1)^N} \right| \leq c(F) \left[\frac{1}{d_k} + \left(\mathrm{h}(\boldsymbol{p}_k) + C \, \frac{\log\, d_k}{d_k} \right)^{\frac{1}{2}} \right]$$

which goes to zero if $h(\boldsymbol{p}_k) \to 0$, and the sequence is strict ($d_k \to \infty$)

The size of the constant

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$$c(F) \;\leq\; \sqrt{2}\,\pi\,\mathrm{Lip}(F) + 2\sum_{l=1}^{N} \left\| \widehat{\frac{\partial F}{\partial \rho_l}} \right\|_{L^1}$$

$$+\frac{1}{\pi}\left(\frac{1}{\log 2} + 37\sqrt{74}\right)\sum_{l=1}^{N} \left\| \widehat{\frac{\partial F}{\partial \theta_l}} \right\|_{L^1}$$

Main ingredient of the Proof

Fourier Analysis on $(\mathbb{C}^\times)^N$

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Fourier Analysis on $(\mathbb{C}^\times)^N$

- Projection via monomial maps

$$\begin{array}{rccc} \chi^{\boldsymbol{n}}: & (\mathbb{C}^\times)^N & \longrightarrow & \mathbb{C}^\times \\ & \boldsymbol{p} & \mapsto & p_1^{n_1}\dots p_N^{n_N} \end{array}$$

Fourier Analysis on $(\mathbb{C}^\times)^N$

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- Apply known results in one dimension

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- Apply known results in one dimension
- Reconstruction via "tomography"

References

- Bilu, *Limit distribution of small points on algebraic tori*, Duke Math. J. 89, (1997)

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- Favre, Rivera-Letelier, *Equidistribution quantitative des points de petite hauteur sur la droite projective*, Math. Ann. 335 (2006)

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- D'Andrea, Galligó, Sombra, *Quantitative equidistribution for the solutions of a system of sparse polynomial equations*, Amer. J. Math 136 (2014)

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- D'Andrea, Galligó, Sombra, *Quantitative equidistribution for the solutions of a system of sparse polynomial equations*, Amer. J. Math 136 (2014)
- D'Andrea, Narváez, Sombra, `arxiv... coming soon!`

Excited about equidistribution of points?

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Tomorrow at 10.30!

José Ignacio Burgos

Essential minimum and equidistribution of small points on toric varieties

Thanks!