Sparse resultants: initial forms, vanishing coefficients, homogeneities and generalized Macaulay formulas

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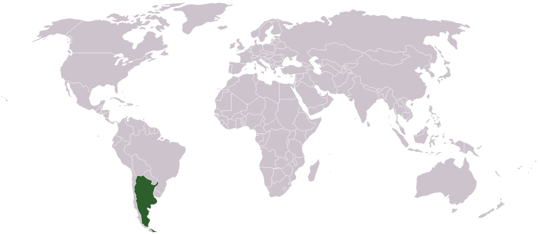
Computational Algebra, Algebraic Geometry & Applications



A Conference in honor of Alicia Dickenstein

Buenos Aires, Argentina, August 1–3 2016

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Back to Sylvester's times...

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$$
\begin{aligned}
f &= a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m \\
g &= b_0 + b_1 x + b_2 x^2 + \ldots + b_k x^k
\end{aligned}
$$

What is the resultant of f and g?

Algebraic Definitions

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- The determinant of the "Sylvester" Matrix (Sylvester)
- One of the (two) generators of the ideal

$$\langle f,\ g\rangle \cap \mathbb{Z}[a_0,\ldots,a_m,\ b_0,\ldots,b_k]$$

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\downarrow & & \downarrow \pi \\
\pi(W) & \subset & \mathbb{P}^m \times \mathbb{P}^k
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the (up to $\pm$) primitive defining equation of $\pi(W)$

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- One of its terms is $a_m^k b_0^n$

Algebra meets Geometry

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$$\mathrm{Res}_{m,k}(f,g)=0$$

$$\Updownarrow$$

$$\exists p\in\mathbb{P}^1 \,|\, f(p)=g(p)=0$$

Poisson's formula and additivity

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$$\mathrm{Res}_{m_1+m_2,k}(f_1 \cdot f_2, g) = \mathrm{Res}_{m_1,k}(f_1, g) \cdot \mathrm{Res}_{m_2,k}(f_2, g)$$

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- The defining equation of

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- $\pi(W)$ does not have codimension 1

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- $\langle f_0, f_1, f_2\rangle \cap \mathbb{Z}[c_{ij}]$ is generated by the determinant of the coefficients
- $\pi(W)$ has codimension 1 **but** $\pi|_W$ **is not birational anymore**

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- **But you do not get neither satisfactory Poisson nor additivity formulae**

A Geometric Sparse Resultant

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(**D**-Sombra), PLMS 2015

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The sparse resultant of $f_0, \ldots, f_n$ is the defining equation of the direct image $\pi_* W$

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With this definition, Poisson's formula works!

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- Classically: $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2} = \det(c_{ij})$
- With the new definition, $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2} = \det(c_{ij})^4$

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- $\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n} = \prod_v \mathrm{Res}_{\mathcal{A}_{1,v},\ldots,\mathcal{A}_{n,v}}^{-h_{\mathcal{A}_0}(v)} \cdot \left(\prod_\xi f_0(\xi)^{m_\xi}\right)$

$$\xi \in V(f_1,\ldots,f_n)$$

Algebra meets Geometry

(if $\pi(W)$ has codimension one)

$$\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}(f_0,\ldots,,f_n)=0$$
$$\Updownarrow$$
$$\exists p\in X_{\mathcal{A}}|\, f_0(p)=\ldots=f_n(p)=0$$

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$$\exists p\in X_{\mathcal{A}}|\; f_0(p)=\ldots=f_n(p)=0$$

$X_{\mathcal{A}}$ is the toric variety defined by $\mathcal{A}_0,\ldots,\mathcal{A}_n$ with respect to $(\mathbb{C}^{\times})^n$

More properties

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- $\deg_{c_{i,a}}(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}) = MV(Q_0, \ldots, \widehat{Q}_i, \ldots, Q_n)$

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  $$Q_i = \mathrm{chull}(\mathcal{A}_i)$$
- "hidden variables":
  $$\mathrm{Res}^{x_n}_{\mathcal{A}_1,\ldots,\mathcal{A}_n}(f_1,\ldots,f_n) = \pm x_n^d \, \mathrm{Res}_{\{\mathbf{0},\mathbf{e_n}\},\mathcal{A}_\mathbf{1},\ldots,\mathcal{A}_\mathbf{n}}(z - x_n, f_1,\ldots,f_n)\big|_{z=x_n}$$
  for some $d \in \mathbb{Z}$

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(**D**-Jeronimo-Sombra)

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- "Determinantal" formulae

of $\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}$

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the product is over all primitive $\mathbf{v} \in \mathbb{Z}^{n+1}$ inner normals to a facet of the lower envelope of the "lifted" polytopes $Q_{0,\omega_0} + \cdots + Q_{n,\omega_n}$

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$$= \pm \mathrm{Res}_{\tilde{\mathcal{A}}_0,\ldots,\tilde{\mathcal{A}}_n} \cdot \prod_{\mathbf{v}} \mathrm{Res}_{\mathcal{A}_{0,\mathbf{v}},\ldots,\mathcal{A}_{n,\mathbf{v}}}$$

iff the CMD is "relevant", otherwise it vanishes identically

More Homogeneities

$$\boxed{\begin{array}{c}\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}\big(f_0(\lambda_1 x_1,\dots,\lambda_n x_n),\dots,f_n(\lambda_1 x_1,\dots,\lambda_n x_n)\big) \\ = \\ \lambda_1^{A_1}\cdots\lambda_n^{A_n}\cdot \mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}\end{array}}$$

More Homogeneities

$$\boxed{\begin{gathered}\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}\big(f_0(\lambda_1 x_1,\ldots,\lambda_n x_n),\ldots,f_n(\lambda_1 x_1,\ldots,\lambda_n x_n)\big)\\ = \\ \lambda_1^{A_1}\cdots\lambda_n^{A_n}\cdot \mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}\end{gathered}}$$

where

$$\boxed{\begin{aligned}&A_i = MV_{\mathbb{Z}^{n+1}}(Q_{0,i,\mu_0},\ldots,Q_{n,i,\mu_n})\\ &+\textstyle\sum_{j=0}^n \mu_j MV_{\mathbb{Z}^n}(Q_0,\ldots,Q_{i-1},Q_{i+1},\ldots,Q_n)\end{aligned}}$$

with $\mu_j \ll 0$, and

$$Q_{j,i,\mu_j} = \mathrm{chull}\big(\{(\mathbf{x},\, x_i),(\mathbf{x},\, \mu_i)\mid \mathbf{x}\in Q_j\}\big)\subset\mathbb{R}^{n+1}$$

“Determinantal” formulae

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Work in Progress!

$$\mathcal{R}(f_0, f_1, f_2) = \det \begin{bmatrix} -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 & 0 \\ -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 \\ 0 & -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 \\ -c_4 & -c_5 & -c_8 & 0 & 0 & 0 & a_1 & 0 & a_3 & 0 \\ -c_1 & -c_3 & -c_7 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ -c_0 & -c_2 & -c_6 & 0 & 0 & 0 & 0 & a_0 & 0 & a_2 \\ 0 & 0 & 0 & -c_4 & -c_5 & -c_8 & b_1 & 0 & b_3 & 0 \\ 0 & 0 & 0 & -c_1 & -c_3 & -c_7 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & -c_0 & -c_2 & -c_6 & 0 & b_0 & 0 & b_2 \end{bmatrix}$$

Thanks!

http://mate.dm.uba.ar/~coalaga/