Moving curve ideals of rational plane parametrizations

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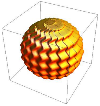


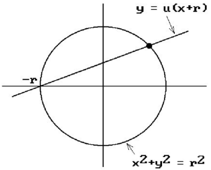
Computational Algebra,Algebraic Geometry & Applications

A Conference in honor of Alicia Dickenstein
Buenos Aires, Argentina, August 1–3 2016
http://mate.dm.uba.ar/~coalaga/

Parametrization of the circle

$$\begin{array}{ccc} \mathbb{K} & \dashrightarrow & \mathbb{K}^2 \\ t & \longmapsto & \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \end{array}$$

$$\begin{array}{cccc} \varphi : & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ & (t_0 : t_1) & \longmapsto & (t_0^2 + t_1^2 : t_0^2 - t_1^2 : 2t_0t_1) \end{array}$$



Implicitization via Sylvester Resultants

$$F = X_2(T_0^2 + T_1^2) - X_0(2T_0T_1) \quad = \quad X_2T_0^2 - 2X_0T_0T_1 + X_2T_1^2$$

$$G = X_2(T_0^2 - T_1^2) - X_1(2T_0T_1) \quad = \quad X_2T_0^2 - 2X_1T_0T_1 - X_2T_1^2$$

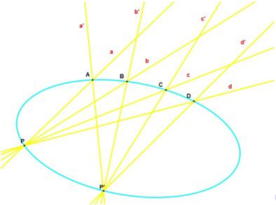
$$\mathrm{Res}_{\underline{T}}(F,G) = \begin{vmatrix} X_2 & -2X_0 & X_2 & 0 \\ 0 & X_2 & -2X_0 & X_2 \\ X_2 & -2X_1 & -X_2 & 0 \\ 0 & X_2 & -2X_1 & -X_2 \end{vmatrix} = 4X_2^2(X_1^2 + X_2^2 - X_0^2)$$

Rows of the Sylvester matrix encode moving lines

$$\mathcal{L}(T_0, T_1, X_0, X_1, X_2) = v_0(\underline{T})X_0 + v_1(\underline{T})X_1 + v_2(\underline{T})X_2$$

such that

$$\mathcal{L}(T_0, T_1, u_0(\underline{T}), u_1(\underline{T}), u_2(\underline{T})) = 0$$



In our example...

$$
\begin{aligned}
\mathcal{L}_1(\underline{T},\underline{X}) &= -2T_0^2T_1X_0 + 0X_1 + (T_0^3 + T_0T_1^2)X_2 \\
\mathcal{L}_2(\underline{T},\underline{X}) &= -2T_0T_1^2X_0 + 0X_1 + (T_0^2T_1 + T_1^3)X_2 \\
\mathcal{L}_3(\underline{T},\underline{X}) &= 0X_0 - 2T_0^2T_1X_1 + (T_0^3 - T_0T_1^2)X_2 \\
\mathcal{L}_4(\underline{T},\underline{X}) &= 0X_0 - 2T_0T_1^2X_1 + (T_0^2T_1 - T_1^3)X_2
\end{aligned}
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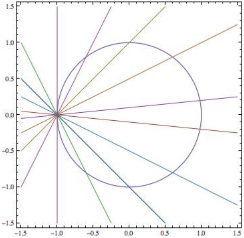
\end{aligned}
$$

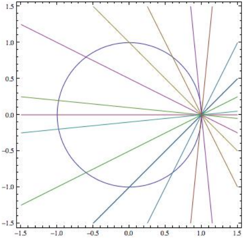
$$
\begin{pmatrix}
X_2 & -2X_0 & X_2 & 0\\
0 & X_2 & -2X_0 & X_2\\
X_2 & -2X_1 & -X_2 & 0\\
0 & X_2 & -2X_1 & -X_2
\end{pmatrix}
$$

Can you get a smaller determinant?

Can you get a smaller determinant?

$$\begin{aligned}
\mathcal{L}_1(\underline{T},\underline{X}) &= X_2 \quad T_0 \; -(X_0+X_1) \; T_1\\
\mathcal{L}_2(\underline{T},\underline{X}) &= (-X_0+X_1) \; T_0 \; +X_2 \; T_1
\end{aligned}$$





$$\begin{vmatrix} X_2 & -X_0-X_1 \\ -X_0+X_1 & X_2 \end{vmatrix} = X_1^2+X_2^2-X_0^2$$

A bit of history on the CAGD side

- Sederberg, Saito, Qi, Klimaszewski. (1994), **Curve implicitization using moving lines**, Computer Aided Geometric Design 11, 687–706
- Sederberg, Chen. **Implicitization using moving curves and surfaces.** Proceedings of SIGGRAPH 1995, 301–308.
- Sederberg, Goldman, Du. (1997), **Implicitizing rational curves by the method of moving algebraic curves,** J. Symbolic Comp. 23, 153–175
- Cox., Sederberg, Chen. (1998), **The moving line ideal basis for planar rational curves**, Computer Aided Geometric Design 15, 803–827
- ...

Moving conics, Moving cubics,...

$$a_j(\underline{T})X_0^2 + b_j(\underline{T})X_0X_1 + c_j(\underline{T})X_0X_2 + d_j(\underline{T})X_1^2 + e_j(\underline{T})X_1X_2 + f_j(\underline{T})X_2^2$$

is a **moving conic** which follows the parametrization if

$$a_j(\underline{T})a(\underline{T})^2 + b_j(\underline{T})a(\underline{T})b(\underline{T}) + c_j(\underline{T})a(\underline{T})c(\underline{T}) + d_j(\underline{T})b(\underline{T})^2 + e_j(\underline{T})b(\underline{T})c(\underline{T}) + f_j(\underline{T})c(\underline{T})^2 = 0$$

The method of moving curves for implicitization

The implicit equation may be computed as a **small** determinant of

$$\begin{vmatrix} \text{some moving lines} \\ \text{some moving conics} \\ \text{some moving cubics} \\ \ldots \end{vmatrix}$$

The **more** singular the curve, the **smaller** the determinant

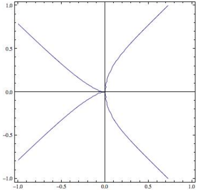
One of those theorems

(Sederberg-Chen 1995)

The implicit equation of a quartic curve with no base points can be written as a 2×2 determinant. If the curve doesn't have a triple point, then each element of the determinant is a quadratic; otherwise one row is linear and one row is cubic

A quartic with triple point

$$\varphi(t_0, t_1) = (t_0^4 - t_1^4 : -t_0^2 t_1^2 : t_0 t_1^3)$$
$$F(X_0, X_1, X_2) = X_2^4 - X_1^4 - X_0 X_1 X_2^2$$

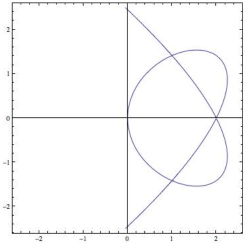


$$\begin{array}{rcl} \mathcal{L}_{1,1}(\underline{T}, \underline{X}) & = & T_0 X_2 + T_1 X_1 \\ \mathcal{L}_{1,3}(\underline{T}, \underline{X}) & = & T_0 (X_1^3 + X_0 X_2^2) + T_1 X_2^3 \end{array}$$

$$\begin{pmatrix} X_2 & X_1 \\ X_1^3 + X_0 X_2^2 & X_2^3 \end{pmatrix}$$

A quartic without a triple point

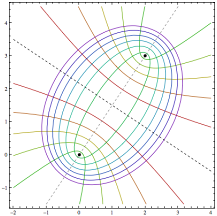
$$\varphi(t_0 : t_1) = (t_0^4 : 6t_0^2t_1^2 - 4t_1^4 : 4t_0^3t_1 - 4t_0t_1^3)$$
$$F(\underline{X}) = X_2^4 + 4X_0X_1^3 + 2X_0X_1X_2^2 - 16X_0^2X_1^2 - 6X_0^2X_2^2 + 16X_0^3X_1$$



$$\mathcal{L}_{1,2}(\underline{T},\underline{X}) = T_0(X_1X_2 - X_0X_2) + T_1(-X_2^2 - 2X_0X_1 + 4X_0^2)$$
$$\tilde{\mathcal{L}}_{1,2}(\underline{T},\underline{X}) = T_0(X_1^2 + \tfrac{1}{2}X_2^2 - 2X_0X_1) + T_1(X_0X_2 - X_1X_2)$$

For large d, we do not know...

which moving lines?
which moving conics?
which moving cubics?
$\ldots$



The Rees Algebra associated to the parametrization

Cox, D. **The moving curve ideal and the Rees algebra.** Theoret. Comput. Sci. 392 (2008), no. 1–3.

$\mathcal{K}_\varphi :=$ {moving curves following φ} $=$ kernel of

$$
\begin{array}{ccc}
\mathbb{K}[T_0, T_1, X_0, X_1, X_2] & \stackrel{\varphi_R}{\to} & \mathbb{K}[T_0, T_1, s] \\
T_i & \mapsto & T_i \\
X_0 & \mapsto & a(\underline{T})s \\
X_1 & \mapsto & b(\underline{T})s \\
X_2 & \mapsto & c(\underline{T})s
\end{array}
$$

"The defining ideal of the Rees Algebra associated to φ"

Understanding $\mathcal{K}_\varphi$ should be a sort of "Rossetta Stone" for the method of moving curves

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For instance, experiments suggest that the more singular the curve, the simpler the description of $\mathcal{K}_\varphi$

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- Is there a **generic** value for $n_0(\mathcal{K}_\varphi)$? Is this the maximal value?
- Where is $n_0(\mathcal{K}_\varphi)$ constant?

Known so far

- $\mu = 1$ (Hong-Simis-Vasconcelos, Cox-Hoffmann-Wang, Busé, Cortadellas-**D**)
- $\mu = 2$(Busé, Cortadellas-**D**, Kustin-Polini-Ulrich)
- $(\mathcal{K}_{\varphi})_{(1,2)} \neq 0$ (Cortadellas- **D**)
- $d = 6$(Kustin-Polini-Ulrich)
- Monomial plane parametrizations (Cortadellas-**D**)

Monomial plane parametrizations

(joint with Teresa Cortadellas)

$$\begin{array}{rccc} \varphi_{\mu,d}: & \mathbb{P}^1 & \to & \mathbb{P}^2 \\ & (t_0 : t_1) & \mapsto & (t_0^d : t_0^{d-\mu} t_1^{\mu} : t_1^d) \end{array}$$

- $1 \leq \mu < d/2$, $\gcd(\mu, d) = 1$
- The implicit equation is $x_1^d - x_0^{d-\mu} x_2^{\mu} = 0$
- Two singular points: $(1:0:0)$ and $(0:0:1)$ of multiplicities μ and $d-\mu$ respectively

A Minimal resolution of $\mathcal{K}_{\varphi_{\mu,d}}$ (Cortadellas-D)

(J. Symbolic Comput. 70, 2015)

$$0 \to S^{q-1} \to S^{2q} \to S^{q+2} \to S \stackrel{\varphi_{\mu,d,R}}{\to} \mathbb{K}[T_0, T_1, s] \to 0$$

- $S = \mathbb{K}[T_0, T_1, X_0, X_1, X_2]$
- It is a resolution of S-modules
- q and the maps of the resolution depend on the complexity of the Euclidean algorithm to compute $\gcd(\mu, d)$

A Minimal *bigraded* resolution

$$0 \to \oplus_{n=1}^{q-1} S(-(b_n, |\sigma_n - \tau_n| + 2|\sigma_{m_{\ell(n)}} - \tau_{m_{\ell(n)}}|)) \to$$

$$\stackrel{\varphi_{\mu,d,3}}{\to} \oplus_{n=1}^{q} S(-(b_n, |\sigma_n - \tau_n| + |\sigma_{m_{\ell(n)}} - \tau_{m_{\ell(n)}}|))^2 \to$$

$$\stackrel{\varphi_{\mu,d,2}}{\to} \oplus_{n=1}^{q+2} S(-(b_n, |\sigma_n - \tau_n|)) \stackrel{\varphi_{\mu,d,1}}{\to} S \stackrel{\varphi_{\mu,d,R}}{\to} \mathbb{K}[T_0, T_1, s]$$

The theorem on an example

$\mu = 3,\ d = 10$

$$\begin{array}{ccc} \mathbb{P}^1 & \stackrel{\varphi_{3,10}}{\rightarrow} & \mathbb{P}^2 \\ (t_0 : t_1) & \mapsto & (t_0^{10} : t_0^7 t_1^3 : t_1^{10}) \end{array}$$

$$\begin{array}{ccc} \mathbb{K}[T_0, T_1, X_0, X_1, X_2] & \stackrel{\varphi_{3,10,R}}{\rightarrow} & \mathbb{K}[T_0, T_1, s] \\ T_i & \mapsto & T_i \\ X_0 & \mapsto & T_0^{10} s \\ X_1 & \mapsto & T_0^7 T_1^3 s \\ X_2 & \mapsto & T_1^{10} s \end{array}$$

Classic Euclidean remainder sequences

Applied to the pair $(d-\mu,\mu)=(7,3)$

$$\begin{aligned} 7 &= 2\cdot 3+1 \\ 3 &= 3\cdot 1+0, \end{aligned}$$

$$\boxed{q=2+3=5}$$

$$0\to S^{q-1}\to S^{2q}\to S^{q+2}\to S\to \mathbb{K}[T_0,T_1,s]\to 0$$

specializes to

$$0\to S^{4}\to S^{10}\to S^{7}\to S\to \mathbb{K}[T_0,T_1,s]\to 0$$

Slow Euclidean remainder sequences

$$
\begin{array}{ccccccccc}
7 & = & 1 & \cdot & 7 & + & 0 & \cdot & 3 \\
3 & = & 0 & \cdot & 7 & + & 1 & \cdot & 3 \\
4 & = & 1 & \cdot & 7 & + & (-1) & \cdot & 3 \\
1 & = & 1 & \cdot & 7 & + & (-2) & \cdot & 3 \\
2 & = & -1 & \cdot & 7 & + & 3 & \cdot & 3 \\
1 & = & -2 & \cdot & 7 & + & 5 & \cdot & 3 \\
0 & = & -3 & \cdot & 7 & + & 7 & \cdot & 3
\end{array}
$$

There are $\boxed{q+2}$ elements in the sequence

The minimal generators of $\mathcal{K}_{\varphi_{3,10}}$ are

$$\begin{aligned}
F_7(\underline{T},\underline{X}) &= T_0^7X_2 - T_1^7X_1\\
F_3(\underline{T},\underline{X}) &= T_0^3X_1 - T_1^3X_0\\
F_1(\underline{T},\underline{X}) &= T_0X_0^2X_2 - T_1X_1^3\\
F_4(\underline{T},\underline{X}) &= T_0^4X_0X_2 - T_1^4X_1^2\\
F_2(\underline{T},\underline{X}) &= T_0^2X_1^4 - T_1^2X_0^3X_2\\
F_1^*(\underline{T},\underline{X}) &= T_0X_1^7 - T_1X_0^5X_2^2\\
F_0(\underline{T},\underline{X}) &= X_0^7X_2^3 - X_1^{10}
\end{aligned}$$

Theorem (Cortadellas-D)

This family of generators is minimal, and also a *Gröbner basis* of $\mathcal{K}_{\varphi_{u,d}}$ for the lexicographic order

The construction of the other maps of the resolution can also be made explicit via the SERS

Geometric features

A curve C_0 is **adjoint** to another curve C if $m_p(\mathrm{C}_0) \geq m_p(\mathrm{C}) - 1$ for all $p \in \mathrm{C}$

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Conjecture (Cox 2008)

Any $T_0\,\mathcal{A}(X_0, X_1, X_2) + T_1\,\mathcal{B}(X_0, X_1, X_2) \in (\mathcal{K}_\varphi)_{(1,\ell)}$ is a **pencil of adjoints for** $\ell \geq d - 2$

False in general (Busé, Jia, Cortadellas-**D**,...)

Measuring the difference

Theorem (Cortadellas - **D**)

For $\ell \geq d-2$, the number

$$\dim_{\mathbb{K}}\left(\left(\mathcal{K}_{\varphi_{\mu,d}}\right)_{1,\ell}/\{\text{pencils of adjoints}\}\cap\left(\mathcal{K}_{\varphi_{\mu,d}}\right)_{1,\ell}\right)$$

depends only on (μ, d), and grows quadratically with d

Work in Progress

Monomial curves in "arithmetic progression"

(joint with Teresa Cortadellas)

$$\begin{array}{ccc} \mathbb{P}^1 & \to & \mathbb{P}^k \\ (t_0 : t_1) & \mapsto & (t_0^{a+kb} : t_0^{a+(k-1)b} t_1^b : \ldots : t_1^{a+kb}) \end{array}$$

$$\gcd(a, b) = 1$$

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The Betti numbers and the generators depend on

b_n, σ_n, τ_n and their values modulo $k - 1$

Thanks!