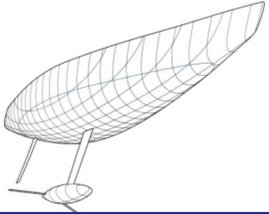
Moving curve ideals of rational plane parametrizations

Carlos D'Andrea

Georgia State University - December 2015



From Wikiversity

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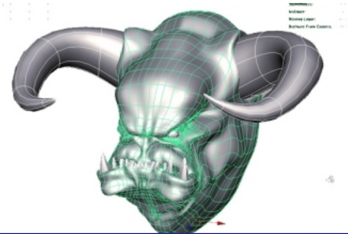
Computer-aided geometric design deals with the mathematical description of shape for use in computer graphics, numerical analysis, approximation theory, data structures, and computer algebra.



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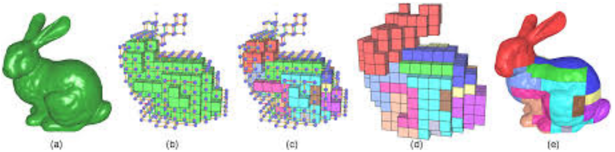
While this field may be mathematical in nature, it is specifically geared toward use in computer science and in engineering fields, making it a field that stretches across several disciplines.



Concepts

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- Primitives (lines,points,vectors,etc.)

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- Bézier Curves

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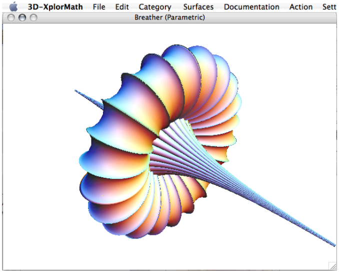
- Tensor-product surfaces
- Interpolation

Curves and Surfaces “on screen”



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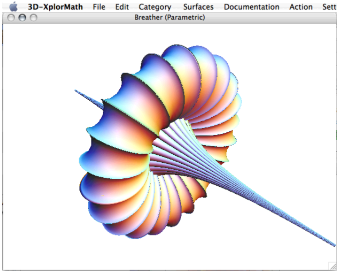
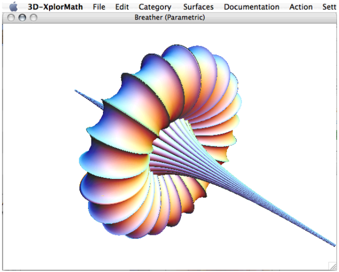


- Affine Geometry

Curves and Surfaces "on screen"



- Affine Geometry
- Projective Geometry

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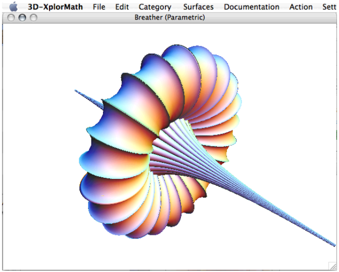

- Affine Geometry
- Projective Geometry
- Real Topology

Geometry $\leftrightarrow$ Algebra

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Parametric and *Implicit*
representations of curves and surfaces

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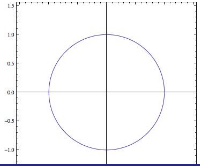
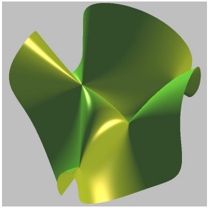
Parametric and *Implicit*
representations of curves and surfaces

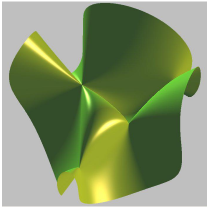
$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R}^2 \\ t & \longmapsto & \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \end{array} \qquad x^2 + y^2 - 1 = 0$$



Algebraic Curves and Surfaces

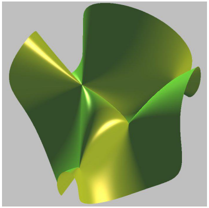


Algebraic Curves and Surfaces



- Input, Process and Output in CAGD must be arithmetically “finite”

Algebraic Curves and Surfaces



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- Finite and "short" codification (polynomials / rational functions of very low degree)

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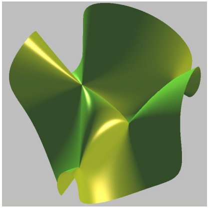

- Input, Process and Output in CAGD must be arithmetically "finite"
- Finite and "short" codification (polynomials / rational functions of very low degree)
- Precision is achieved by "glueing" patches (splines, etc.)

Implicit and parametric forms

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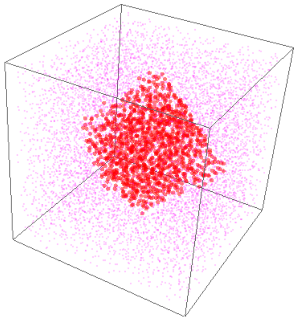
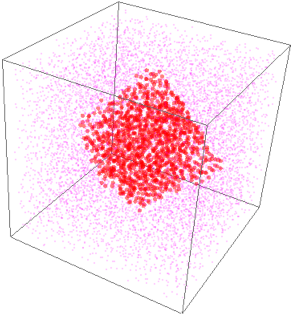


- <u>Parametric:</u> “plot” points

Implicit and parametric forms



- Parametric: "plot" points
- Implicit: "split" regions

Passing from one form to the other

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is important

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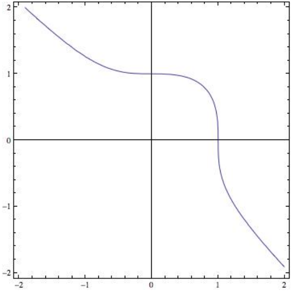
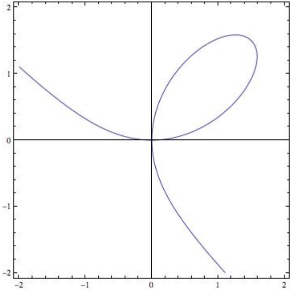
is important although not always possible





From now on..

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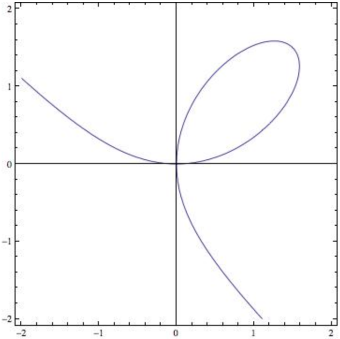
implicitization of curves in the plane



From affine to projective

$$\begin{array}{ccc} \mathbb{K} & \dashrightarrow & \mathbb{K}^2 \\ t & \longmapsto & \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \end{array}$$

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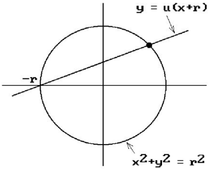
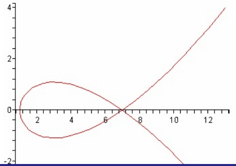
$$\begin{array}{cccc} \phi: & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ & (t_0 : t_1) & \longmapsto & (t_0^2 + t_1^2 : t_0^2 - t_1^2 : 2t_0t_1) \end{array}$$



Parametrization of Plane Curves

$$\begin{array}{cccc} \phi : & \mathbb{P}^1 & \rightarrow & \mathbb{P}^2 \\ & (t_0 : t_1) & \mapsto & (a(t_0,t_1) : b(t_0,t_1) : c(t_0,t_1)) \end{array}$$

- $a,\ b,\ c \in \mathbb{K}[T_0,\ T_1]$, homogeneous of the same degree $d \geq 1$
- $\gcd(a,b,c) = 1$



Rational Curves in the plane

The image of ϕ is a **rational plane curve**

Rational Curves in the plane

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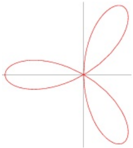

- It has degree d if ϕ is "generically" injective

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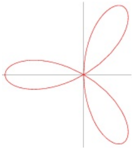

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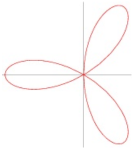

- It has degree d if ϕ is "generically" injective
- it has genus 0, which means the maximal number of multiple points $\frac{(d-1)(d-2)}{2}$
- Computing its implicit equation is relatively easy from ϕ

Sylvester's resultant

$$X_2 a(\underline{T}) - X_0 c(\underline{T}) = X_2 T_0^2 - 2X_0 T_0 T_1 + X_2 T_1^2$$

$$X_2 b(\underline{T}) - X_1 c(\underline{T}) = X_2 T_0^2 - 2X_1 T_0 T_1 - X_2 T_1^2$$

$$\mathrm{Res}_{\underline{T}}\left(X_2 \cdot a(\underline{T}) - X_0 \cdot c(\underline{T}), X_2 \cdot b(\underline{T}) - X_1 \cdot c(\underline{T})\right) = \det \begin{pmatrix} X_2 & -2X_0 & X_2 & 0 \\ 0 & X_2 & -2X_0 & X_2 \\ X_2 & -2X_1 & -X_2 & 0 \\ 0 & X_2 & -2X_1 & -X_2 \end{pmatrix}$$

From parametric to implicit

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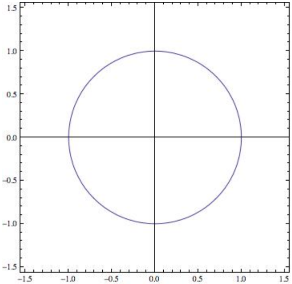
$$\mathrm{Res}_{\underline{T}}\left(X_2 \cdot a(\underline{T}) - X_0 \cdot c(\underline{T}), X_2 \cdot b(\underline{T}) - X_1 \cdot c(\underline{T})\right)$$
$$=$$
$$-4X_2^2(X_0^2 - X_1^2 - X_2^2)$$

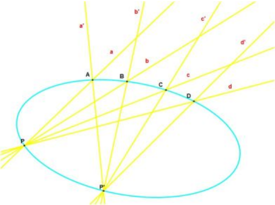


Moving lines

$$\mathcal{L}(T_0, T_1, X_0, X_1, X_2) = v_0(\underline{T})X_0 + v_1(\underline{T})X_1 + v_2(\underline{T})X_2$$

such that

$$\mathcal{L}(T_0, T_1, a(\underline{T}), b(\underline{T}), c(\underline{T})) = 0$$



In our example...

$$\begin{aligned}
\mathcal{L}_1(\underline{T},\underline{X}) &= -2T_0^2T_1X_0 + 0X_1 + (T_0^3 + T_0T_1^2)X_2\\
\mathcal{L}_2(\underline{T},\underline{X}) &= -2T_0T_1^2X_0 + 0X_1 + (T_0^2T_1 + T_1^3)X_2\\
\mathcal{L}_3(\underline{T},\underline{X}) &= 0X_0 - 2T_0^2T_1X_1 + (T_0^3 - T_0T_1^2)X_2\\
\mathcal{L}_4(\underline{T},\underline{X}) &= 0X_0 - 2T_0T_1^2X_1 + (T_0^2T_1 - T_1^3)X_2
\end{aligned}$$

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\mathcal{L}_3(\underline{T},\underline{X}) &= \quad 0X_0 - 2T_0^2T_1X_1 + (T_0^3 - T_0T_1^2)X_2 \\
\mathcal{L}_4(\underline{T},\underline{X}) &= \quad 0X_0 - 2T_0T_1^2X_1 + (T_0^2T_1 - T_1^3)X_2
\end{aligned}
$$

$$
\begin{pmatrix}
X_2 & -2X_0 & X_2 & 0 \\
0 & X_2 & -2X_0 & X_2 \\
X_2 & -2X_1 & -X_2 & 0 \\
0 & X_2 & -2X_1 & -X_2
\end{pmatrix}
$$

In general

The determinant of a "matrix of moving lines" is a multiple of the implicit equation

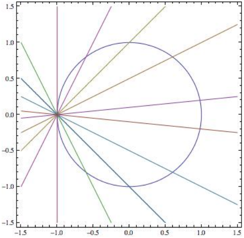
$$\begin{pmatrix} L_{11}(\underline{X}) & L_{12}(\underline{X}) & \dots & L_{1k}(\underline{X}) \\ L_{21}(\underline{X}) & L_{22}(\underline{X}) & \dots & L_{2k}(\underline{X}) \\ \vdots & \vdots & \dots & \vdots \\ L_{k1}(\underline{X}) & L_{k2}(\underline{X}) & \dots & L_{kk}(\underline{X}) \end{pmatrix}$$

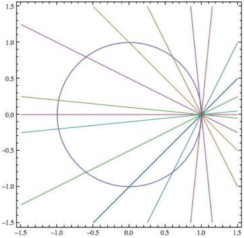
How small can the matrix be?

$$\mathcal{L}_{1,1}(\underline{T},\underline{X}) = X_2 \; T_0 \; -(X_0 + X_1) \; T_1$$
$$\mathcal{L}'_{1,1}(\underline{T},\underline{X}) = (-X_0 + X_1) \; T_0 \; +X_2 \; T_1$$

How small can the matrix be?

$$\mathcal{L}_{1,1}(\underline{T},\underline{X}) = X_2 \, T_0 - (X_0 + X_1) \, T_1$$
$$\mathcal{L}'_{1,1}(\underline{T},\underline{X}) = (-X_0 + X_1) \, T_0 + X_2 \, T_1$$





$$\det \begin{pmatrix} X_2 & -X_0 - X_1 \\ -X_0 + X_1 & X_2 \end{pmatrix} = X_1^2 + X_2^2 - X_0^2$$

The (free) module of moving lines

(Hilbert (1890))

There exists $\mu \leq \frac{d}{2}$ and

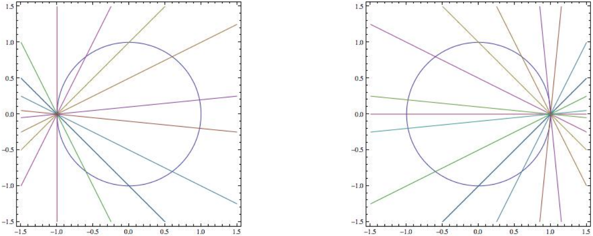
$\mathcal{P}_\mu(\underline{T}, \underline{X}),\ \mathcal{Q}_{d-\mu}(\underline{T}, \underline{X})$ moving lines following ϕ such that any other $\mathcal{L}_\delta(\underline{T}, \underline{X})$ following ϕ is of the form

$$p_{\delta-\mu}(\underline{T})\mathcal{P}_\mu(\underline{T}, \underline{X}) + q_{\delta-d+\mu}(\underline{T})\mathcal{P}_{d-\mu}(\underline{T}, \underline{X})$$

Geometric version

There exist $\mu \leq \frac{d}{2}$ and two other parametrizations $\varphi_\mu(t_0, t_1)$, $\psi_{d-\mu}(t_0, t_1)$ of degrees μ, $d - \mu$ such that

$$\boxed{\phi(t_0, t_1) = \varphi_\mu(t_0, t_1) \wedge \psi_{d-\mu}(t_0, t_1)}$$



For the unit circle...

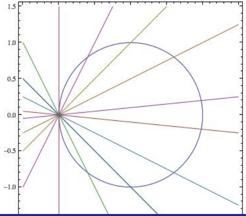
For the unit circle...

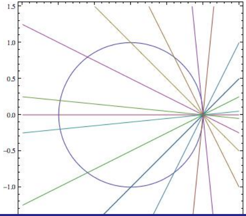
$$\begin{aligned} \varphi_1(t_0 : t_1) &= (-t_1 : -t_1 : t_0) \\ \psi_1(t_0 : t_1) &= (-t_0 : t_0 : t_1) \end{aligned}$$

For the unit circle...

$$\boxed{\begin{array}{rcl} \varphi_1(t_0:t_1) & = & (-t_1:-t_1:t_0) \\ \psi_1(t_0:t_1) & = & (-t_0:t_0:t_1) \end{array}}$$

$$\begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \\ -t_1 & -t_1 & t_0 \\ -t_0 & t_0 & t_1 \end{vmatrix} = \left(-t_0^2 - t_1^2, t_1^2 - t_0^2, -2t_0t_1\right)$$





Hilbert's Syzygy Theorem

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The homogeneous ideal
$I = (a(\underline{T}), b(\underline{T}), c(\underline{T})) \subset \mathbb{K}[T_0, T_1]$ has a
Hilbert-Burch resolution of the type

$$0 \to \mathbb{K}[\underline{T}]^2 \xrightarrow{(\varphi_\mu, \psi_{d-\mu})^{\mathbf{t}}} \mathbb{K}[\underline{T}]^3 \xrightarrow{(a,b,c)} \mathbb{K}[\underline{T}]$$

Hilbert's Syzygy Theorem

The homogeneous ideal $I = \left(a(\underline{T}), b(\underline{T}), c(\underline{T})\right) \subset \mathbb{K}[T_0, T_1]$ has a **Hilbert-Burch resolution** of the type

$$0 \to \mathbb{K}[\underline{T}]^2 \xrightarrow{(\varphi_\mu, \psi_{d-\mu})^{\mathbf{t}}} \mathbb{K}[\underline{T}]^3 \xrightarrow{(a,b,c)} \mathbb{K}[\underline{T}]$$

A μ-basis of the parametrization is a basis of $\mathrm{Syz}(I)$ as a $\mathbb{K}[\underline{T}]$-module

Why do we care about these bases?

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Implicit equation

$$=$$

$$\mathrm{Res}_{\underline{T}}\left(\mathcal{P}_{\mu}(\underline{T},\underline{X}),\ \mathcal{Q}_{d-\mu}(\underline{T},\underline{X})\right)$$

Why do we care about these bases?

Implicit equation

$$=$$

$$\mathrm{Res}_{\underline{T}}(\mathcal{P}_\mu(\underline{T},\underline{X}),\ \mathcal{Q}_{d-\mu}(\underline{T},\underline{X}))$$

Busé-D (2012)

If B is a Bézout matrix, and S one of Sylester type, then

$$X_2\, S(\mathcal{P}_\mu(\underline{T},\underline{X}),\ \mathcal{Q}_{d-\mu}(\underline{T},\underline{X})) = M\cdot B(aX_2 - cX_0, bX_2 - cX_1),$$

with $M \in \mathbb{K}^{d\times d}$ invertible

A bit of history

- Sederberg, Saito, Qi, Klimaszewski. (1994), **Curve implicitization using moving lines**, Computer Aided Geometric Design 11, 687–706
- Sederberg, Chen. **Implicitization using moving curves and surfaces.** Proceedings of SIGGRAPH 1995, 301–308.
- Sederberg, Goldman, Du. (1997), **Implicitizing rational curves by the method of moving algebraic curves,** J. Symbolic Comp. 23, 153–175
- Cox, Sederberg, Chen. (1998), **The moving line ideal basis for planar rational curves**, Computer Aided Geometric Design 15, 803–827
- ...

Moving conics, moving cubics,...

Moving conics, moving cubics,...

$$o(\underline{T})X_0^2 + p(\underline{T})X_0X_1 + q(\underline{T})X_0X_2 + r(\underline{T})X_1^2 + s(\underline{T})X_1X_2 + t(\underline{T})X_2^2$$

is a **moving conic** following the parametrization if

Moving conics, moving cubics,...

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is a **moving conic** following the parametrization if

$$o(\underline{T})a(\underline{T})^2 + p(\underline{T})a(\underline{T})b(\underline{T}) + q(\underline{T})a(\underline{T})c(\underline{T}) + r(\underline{T})b(\underline{T})^2 + s(\underline{T})b(\underline{T})c(\underline{T}) + t(\underline{T})c(\underline{T})^2 = 0$$

The method of moving curves

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The implicit equation can be computed as the determinant of a **small** matrix with entries

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$$\begin{vmatrix} \text{some moving lines} \\ \text{some moving conics} \\ \text{some moving cubics} \\ \ldots \end{vmatrix}$$

The method of moving curves

The implicit equation can be computed as the determinant of a **small** matrix with entries

$$\begin{vmatrix} \text{some moving lines} \\ \text{some moving conics} \\ \text{some moving cubics} \\ \ldots \end{vmatrix}$$

the more **singular** the curve, the **simpler** the description of the determinant

Example (Sederberg & Chen 1995)

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The implicit equation of a quartic can be computed as a 2×2 determinant.

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If the curve has a triple point, then one row is linear and the other is cubic.

Example (Sederberg & Chen 1995)

The implicit equation of a quartic can be computed as a 2×2 determinant.
If the curve has a triple point, then one row is linear and the other is cubic.
Otherwise, both rows are quadratic.

A quartic with a triple point

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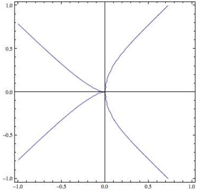
$$\phi(t_0, t_1) = (t_0^4 - t_1^4 : -t_0^2 t_1^2 : t_0 t_1^3)$$
$$F(X_0, X_1, X_2) = X_2^4 - X_1^4 - X_0 X_1 X_2^2$$

A quartic with a triple point

$$\phi(t_0, t_1) = (t_0^4 - t_1^4 : -t_0^2 t_1^2 : t_0 t_1^3)$$
$$F(X_0, X_1, X_2) = X_2^4 - X_1^4 - X_0 X_1 X_2^2$$



A quartic with a triple point

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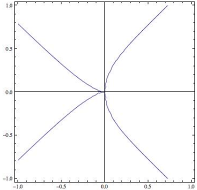

$$\begin{aligned}
\mathcal{L}_{1,1}(\underline{T},\underline{X}) &= T_0 X_2 + T_1 X_1 \\
\mathcal{L}_{1,3}(\underline{T},\underline{X}) &= T_0(X_1^3 + X_0 X_2^2) + T_1\, X_2^3
\end{aligned}$$

$$\begin{pmatrix} X_2 & X_1 \\ X_1^3 + X_0 X_2^2 & X_2^3 \end{pmatrix}$$

A quartic without triple points

$$\phi(t_0 : t_1) = (t_0^4 : 6t_0^2t_1^2 - 4t_1^4 : 4t_0^3t_1 - 4t_0t_1^3)$$
$$F(\underline{X}) = X_2^4 + 4X_0X_1^3 + 2X_0X_1X_2^2 - 16X_0^2X_1^2 - 6X_0^2X_2^2 + 16X_0^3X_1$$

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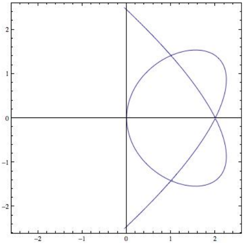

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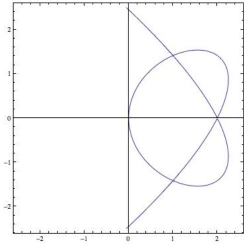
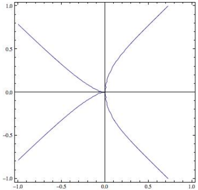


$$\mathcal{L}_{1,2}(\underline{T},\underline{X}) = T_0(X_1X_2 - X_0X_2) + T_1(-X_2^2 - 2X_0X_1 + 4X_0^2)$$
$$\tilde{\mathcal{L}}_{1,2}(\underline{T},\underline{X}) = T_0(X_1^2 + \frac{1}{2}X_2^2 - 2X_0X_1) + T_1(X_0X_2 - X_1X_2)$$

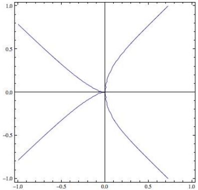
Very concentrated singularities

Very concentrated singularities



If the curve has a point of multiplicity $d - 1$

Very concentrated singularities



If the curve has a point of multiplicity $d-1$
the implicit equation is always a 2×2 determinant

$$\begin{vmatrix} \mathcal{L}_{1,1}(\underline{X}) & \mathcal{L}'_{1,1}(\underline{X}) \\ \mathcal{L}_{1,d-1}(\underline{X}) & \mathcal{L}'_{1,d-1}(\underline{X}) \end{vmatrix}$$

In general, we do not know..

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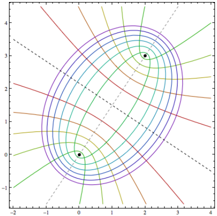
which moving lines?
which moving conics?
which moving cubics?
...



The Rees Algebra associated to the parametrization

Cox, D. **The moving curve ideal and the Rees algebra.** Theoret. Comput. Sci. 392 (2008), no. 1–3, 23–36

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$\mathcal{K}_\phi :=$ {Moving curves following ϕ} = homogeneous elements in the kernel of

$$\begin{array}{ccc} \mathbb{K}[T_0, T_1, X_0, X_1, X_2] & \to & \mathbb{K}[T_0, T_1, s] \\ T_i & \mapsto & T_i \\ X_0 & \mapsto & a(\underline{T})s \\ X_1 & \mapsto & b(\underline{T})s \\ X_2 & \mapsto & c(\underline{T})s \end{array}$$

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"The ideal of moving curves following ϕ"

Method of moving curves revisited

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Method of moving curves revisited

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The more singular the curve, the simpler the description of $\mathcal{K}_\phi$

New Problem

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Compute a minimal system of generators of $\mathcal{K}_\phi$

New Problem

Compute a minimal system of generators of $\mathcal{K}_\phi$ for **any** ϕ

New Problem

Compute a minimal system of generators of $\mathcal{K}_\phi$ for **any** ϕ

Known for

- $\mu = 1$ (Hong-Simis-Vasconcelos, Cox-Hoffmann-Wang, Busé, Cortadellas-**D**)
- $\mu = 2$ (Busé, Cortadellas-**D**, Kustin-Polini-Ulrich)
- $\left(\mathcal{K}_\phi\right)_{(1,2)} \neq 0$ (Cortadellas- **D**)
- Monomial Parametrizations (Cortadellas-**D**)

A coarser problem

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Compute $n_0(\mathcal{K}_\phi)$, the number of minimal generators of $\mathcal{K}_\phi$

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Compute $n_0(\mathcal{K}_\phi)$, the number of minimal generators of $\mathcal{K}_\phi$

Show that if ϕ is "more singular" than ϕ' then $n_0(\mathcal{K}_\phi) \leq n_0(\mathcal{K}_{\phi'})$

Example: $\mu = 2$

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The curve has either

- one point of multiplicity $d-2$
  $n_0 = \mathcal{O}\left(\frac{d}{2}\right)$
  (Cortadellas-**D**, Kustin-Polini-Ulrich)

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- one point of multiplicity $d-2$
  $n_0 = \mathcal{O}\left(\frac{d}{2}\right)$
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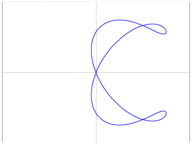
- or only double points
  $n_0 = \mathcal{O}\left(\frac{d^2}{2}\right)$ (Busé)



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- In which "regions" is $n_0(\mathcal{K}_\phi)$ constant?

Breaking News!

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Jeff Madsen

Equations of Rees algebras of ideals in two variables

arXiv:1511.04073

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Gives an algorithm to compute all minimal generators in $\underline{T}$-degree larger than or equal to μ

Ongoing Project

(w/Teresa Cortadellas and David Cox)

- Make explicit these generators and their degrees

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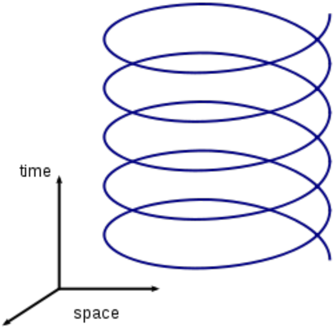
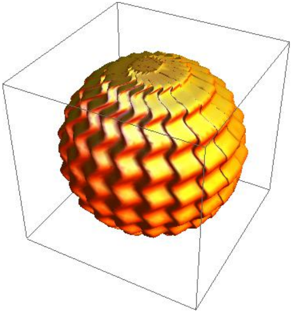
- Make explicit these generators and their degrees
- It seems that $n_0(\mathcal{K}_\varphi)$ depends on the "μ-type of the μ-basis
- Complete the description to a set of **all** minimal generators

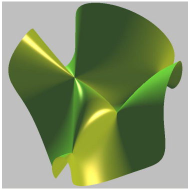
Only curves in the plane?





Rational Surfaces

$$\begin{array}{rccc} \phi_S : & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^3 \\ & \underline{t} = (t_0 : t_1 : t_2) & \longmapsto & \big(a(\underline{t}) : b(\underline{t}) : c(\underline{t}) : d(\underline{t})\big) \end{array}$$



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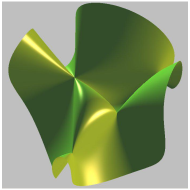

There are base points!

Implicitization via

- Resultants Macaulay, Dixon, Gelfand-Kapranov-Zelevinskii, ...

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Moving planes, moving quadrics,...

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(Sederberg-Chen, Cox-Goldman-Zhang, Busé-Cox, **D**, **D**-Khetan)

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Contrast:

- The module of moving planes is not free

Moving planes, moving quadrics,...

(Sederberg-Chen, Cox-Goldman-Zhang, Busé-Cox, **D**, **D**-Khetan)

Contrast:

- The module of moving planes is not free
- There is a concept of μ-basis given by Chen-Cox-Liu

  Not easy to compute

Some results

Some results

Implicitization

Some results

Implicitization

- Quadratic and cubic surfaces (Chen-Shen-Deng)

Some results

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- Quadratic and cubic surfaces (Chen-Shen-Deng)
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Implicitization

- Quadratic and cubic surfaces (Chen-Shen-Deng)
- Steiner surfaces (Wang-Chen)
- Revolution surfaces (Shi-Goldman)
- . . .

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Rees Algebras

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Rees Algebras

- "Monoid" Surfaces (Cortadellas - **D**)

Some results

Implicitization

- Quadratic and cubic surfaces (Chen-Shen-Deng)
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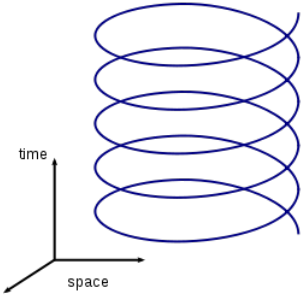
Rees Algebras

- "Monoid" Surfaces (Cortadellas - **D**)
- *de Jonquières* surfaces (Hassanzadeh- Simis)

Similar Results for

Spatial curves

$$\begin{array}{rccc} \phi_C : & \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^3 \\ & \underline{t} = (t_0 : t_1) & \longmapsto & \big(a(\underline{t}) : b(\underline{t}) : c(\underline{t}) : d(\underline{t})\big) \end{array}$$



ARCADES

Algebraic Representations in Computer-Aided Design for complEx Shapes

Marie Sklodowska-Curie European Training Network, 2016 – 2019

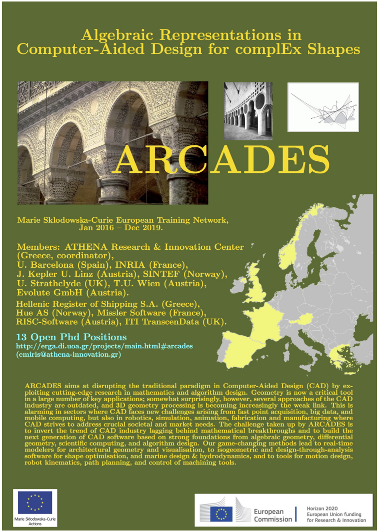
ATHENA Research & Innovation Center, U. Barcelona, INRIA, J. Kepler U. Linz, SINTEF, U. Strathclyde, T.U. Wien, Evolute GmbH

HORIZON2020
Marie Skłodowska-Curie actions

13 Open Phd Positions (2016)

http://erga.di.uoa.gr/projects/main.html#arcades
(emiris@athena-innovation.gr)

Thanks!



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ARCADES
Marie Sklodowska-Curie European Training Network, Jan 2016 – Dec 2019.
Members: ATHENA Research & Innovation Center (Greece, coordinator), U. Barcelona (Spain), INRIA (France), J. Kepler U. Linz (Austria), SINTEF (Norway), U. Strathclyde (UK), T.U. Wien (Austria), Evolute GmbH (Austria).
Hellenic Register of Shipping S.A. (Greece), Hue AS (Norway), Missler Software (France), RISC-Software (Austria), ITI TranscenData (UK).
13 Open Phd Positions
http://erga.di.uoa.gr/projects/main.html#arcades
(emiris@athena-innovation.gr)
ARCADES aims at disrupting the traditional paradigm in Computer-Aided Design (CAD) by exploiting cutting-edge research in mathematics and algorithm design. Geometry is now a critical tool in a large number of key applications; somewhat surprisingly, however, several approaches of the CAD industry are outdated, and 3D geometry processing is becoming increasingly the weak link. This is alarming in sectors where CAD faces new challenges arising from fast point acquisition, big data, and mobile computing, but also in robotics, simulation, animation, fabrication and manufacturing where CAD strives to address crucial societal and market needs. The challenge taken up by ARCADES is to invert the trend of CAD industry lagging behind mathematical breakthroughs and to build the next generation of CAD software based on strong foundations from algebraic geometry, differential geometry, scientific computing, and algorithm design. Our game-changing methods lead to real-time modelers for architectural geometry and visualisation, to isogeometric and design-through-analysis software for shape optimisation, and marine design & hydrodynamics, and to tools for motion design, robot kinematics, path planning, and control of machining tools.
Marie Skłodowska-Curie Actions
European Commission
Horizon 2020
European Union funding for Research & Innovation