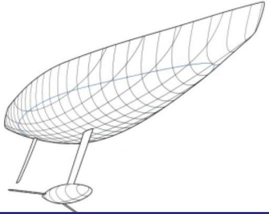
Relating geometric singularities of parametric curves and surfaces with algebraic moving ideals

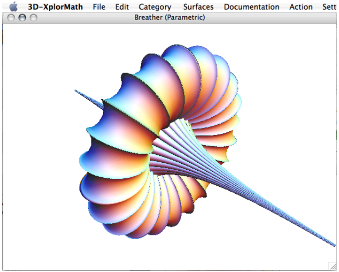
Carlos D'Andrea

ARCADES Workshop

UNIVERSITAT DE BARCELONA



Curves and Surfaces “on screen”



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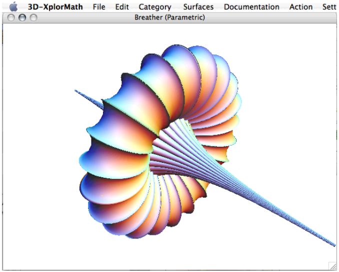
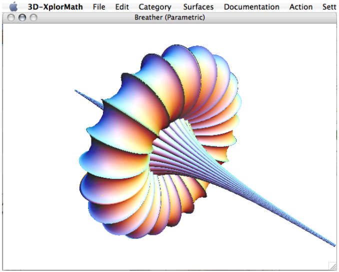


- Affine Geometry

Curves and Surfaces "on screen"



- Affine Geometry
- Projective Geometry

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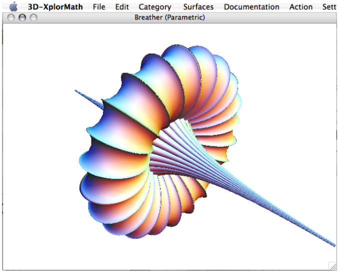

- Affine Geometry
- Projective Geometry
- Real Topology

Geometry $\leftrightarrow$ Algebra

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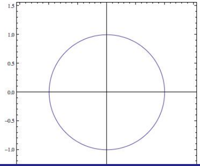
Parametric and *Implicit*
representations of curves and surfaces

Geometry $\leftrightarrow$ Algebra

Parametric and *Implicit*
representations of curves and surfaces

$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R}^2 \\ t & \longmapsto & \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \end{array} \qquad x^2 + y^2 - 1 = 0$$



Implicit and parametric forms

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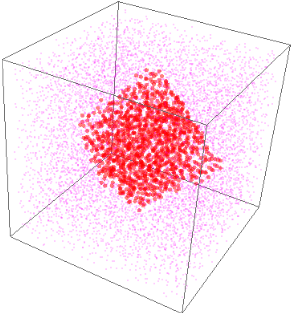
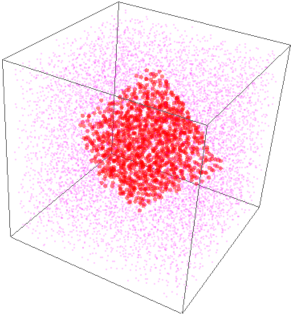


- <u>Parametric:</u> "plot" points

Implicit and parametric forms



- Parametric: "plot" points
- Implicit: "split" regions

Our Project..

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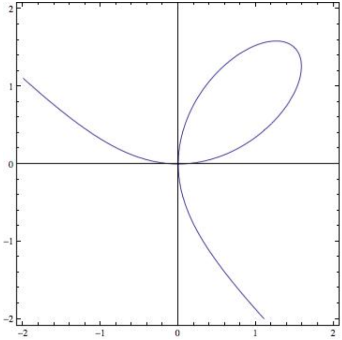
implicitization of curves and surfaces



From affine to projective

$$\begin{array}{ccc} \mathbb{K} & \dashrightarrow & \mathbb{K}^2 \\ t & \longmapsto & \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \end{array}$$

From affine to projective

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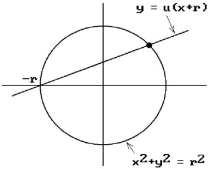
$$\begin{array}{cccc} \phi : & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ & (t_0 : t_1) & \longmapsto & (t_0^2 + t_1^2 : t_0^2 - t_1^2 : 2t_0t_1) \end{array}$$



Sylvester's resultant

$$X_2 a(\underline{T}) - X_0 c(\underline{T}) = X_2 T_0^2 - 2X_0 T_0 T_1 + X_2 T_1^2$$

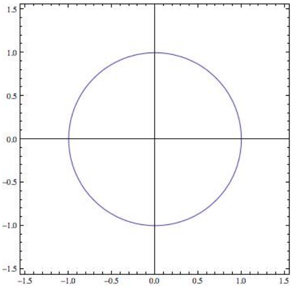
$$X_2 b(\underline{T}) - X_1 c(\underline{T}) = X_2 T_0^2 - 2X_1 T_0 T_1 - X_2 T_1^2$$

$$\mathrm{Res}_{\underline{T}}(X_2 \cdot a(\underline{T}) - X_0 \cdot c(\underline{T}), X_2 \cdot b(\underline{T}) - X_1 \cdot c(\underline{T})) = \det \begin{pmatrix} X_2 & -2X_0 & X_2 & 0 \\ 0 & X_2 & -2X_0 & X_2 \\ X_2 & -2X_1 & -X_2 & 0 \\ 0 & X_2 & -2X_1 & -X_2 \end{pmatrix}$$

From parametric to implicit

From parametric to implicit

$$\mathrm{Res}_{\underline{T}}\left(X_2 \cdot a(\underline{T}) - X_0 \cdot c(\underline{T}), X_2 \cdot b(\underline{T}) - X_1 \cdot c(\underline{T})\right)$$
$$=$$
$$-4X_2^2(X_0^2 - X_1^2 - X_2^2)$$

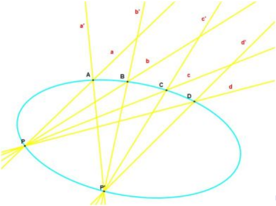


Moving lines

$$\mathcal{L}(T_0, T_1, X_0, X_1, X_2) = v_0(\underline{T})X_0 + v_1(\underline{T})X_1 + v_2(\underline{T})X_2$$

such that

$$\mathcal{L}(T_0, T_1, a(\underline{T}), b(\underline{T}), c(\underline{T})) = 0$$



In our example...

$$
\begin{aligned}
\mathcal{L}_1(\underline{T},\underline{X}) &= -2T_0^2T_1X_0 + 0X_1 + (T_0^3 + T_0T_1^2)X_2\\
\mathcal{L}_2(\underline{T},\underline{X}) &= -2T_0T_1^2X_0 + 0X_1 + (T_0^2T_1 + T_1^3)X_2\\
\mathcal{L}_3(\underline{T},\underline{X}) &= 0X_0 - 2T_0^2T_1X_1 + (T_0^3 - T_0T_1^2)X_2\\
\mathcal{L}_4(\underline{T},\underline{X}) &= 0X_0 - 2T_0T_1^2X_1 + (T_0^2T_1 - T_1^3)X_2
\end{aligned}
$$

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\end{aligned}
$$

$$
\begin{pmatrix}
X_2 & -2X_0 & X_2 & 0 \\
0 & X_2 & -2X_0 & X_2 \\
X_2 & -2X_1 & -X_2 & 0 \\
0 & X_2 & -2X_1 & -X_2
\end{pmatrix}
$$

In general

The determinant of a "matrix of moving lines" is a multiple of the implicit equation

$$\begin{pmatrix} L_{11}(\underline{X}) & L_{12}(\underline{X}) & \dots & L_{1k}(\underline{X}) \\ L_{21}(\underline{X}) & L_{22}(\underline{X}) & \dots & L_{2k}(\underline{X}) \\ \vdots & \vdots & \dots & \vdots \\ L_{k1}(\underline{X}) & L_{k2}(\underline{X}) & \dots & L_{kk}(\underline{X}) \end{pmatrix}$$

How small can the matrix be?

$$
\begin{aligned}
\mathcal{L}_{1,1}(\underline{T},\underline{X}) &= X_2\, T_0 - (X_0 + X_1)\, T_1 \\
\mathcal{L}'_{1,1}(\underline{T},\underline{X}) &= (-X_0 + X_1)\, T_0 + X_2\, T_1
\end{aligned}
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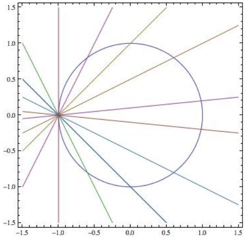

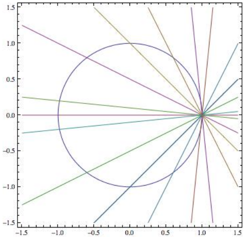

$$
\det\begin{pmatrix} X_2 & -X_0-X_1\\ -X_0+X_1 & X_2 \end{pmatrix} = X_1^2+X_2^2-X_0^2
$$

The (free) module of moving lines

(Hilbert (1890))

There exists $\mu \leq \frac{d}{2}$ and $\mathcal{P}_\mu(\underline{T},\underline{X})$, $\mathcal{Q}_{d-\mu}(\underline{T},\underline{X})$ moving lines following ϕ such that any other $\mathcal{L}_\delta(\underline{T},\underline{X})$ following ϕ is of the form

$$p_{\delta-\mu}(\underline{T})\mathcal{P}_\mu(\underline{T},\underline{X}) + q_{\delta-d+\mu}(\underline{T})\mathcal{P}_{d-\mu}(\underline{T},\underline{X})$$

Why do we care about these bases?

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Implicit equation

$=$

$$\mathrm{Res}_{\underline{T}}\left(\mathcal{P}_{\mu}(\underline{T},\underline{X}),\ \mathcal{Q}_{d-\mu}(\underline{T},\underline{X})\right)$$

Why do we care about these bases?

Implicit equation

$$=$$

$$\mathrm{Res}_{\underline{T}}(\mathcal{P}_\mu(\underline{T},\underline{X}),\ \mathcal{Q}_{d-\mu}(\underline{T},\underline{X}))$$

Busé-D (2012)

If B is a Bézout matrix, and S one of Sylester type, then

$$X_2\, S(\mathcal{P}_\mu(\underline{T},\underline{X}),\ \mathcal{Q}_{d-\mu}(\underline{T},\underline{X})) = M \cdot B(aX_2 - cX_0, bX_2 - cX_1),$$

with $M \in \mathbb{K}^{d\times d}$ invertible

Moving conics, moving cubics,...

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$$o(\underline{T})X_0^2 + p(\underline{T})X_0X_1 + q(\underline{T})X_0X_2 + r(\underline{T})X_1^2 + s(\underline{T})X_1X_2 + t(\underline{T})X_2^2$$

is a **moving conic** following the parametrization if

Moving conics, moving cubics,...

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is a **moving conic** following the parametrization if

$$o(\underline{T})a(\underline{T})^2 + p(\underline{T})a(\underline{T})b(\underline{T}) + q(\underline{T})a(\underline{T})c(\underline{T}) + r(\underline{T})b(\underline{T})^2 + s(\underline{T})b(\underline{T})c(\underline{T}) + t(\underline{T})c(\underline{T})^2 = 0$$

The method of moving curves

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The implicit equation can be computed as the determinant of a **small** matrix with entries

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The method of moving curves

The implicit equation can be computed as the determinant of a **small** matrix with entries

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the more **singular** the curve, the **simpler** the description of the determinant

Example (Sederberg & Chen 1995)

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The implicit equation of a quartic can be computed as a 2×2 determinant.

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The implicit equation of a quartic can be computed as a 2×2 determinant.

If the curve has a triple point, then one row is linear and the other is cubic.

Otherwise, both rows are quadratic.

A quartic with a triple point

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$$\phi(t_0, t_1) = (t_0^4 - t_1^4 : -t_0^2 t_1^2 : t_0 t_1^3)$$
$$F(X_0, X_1, X_2) = X_2^4 - X_1^4 - X_0 X_1 X_2^2$$

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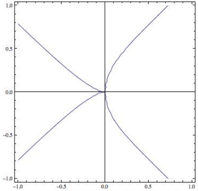

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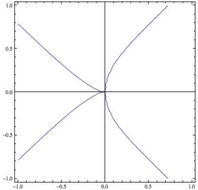

$$\begin{aligned} \mathcal{L}_{1,1}(\underline{T}, \underline{X}) &= T_0 X_2 + T_1 X_1 \\ \mathcal{L}_{1,3}(\underline{T}, \underline{X}) &= T_0 (X_1^3 + X_0 X_2^2) + T_1 X_2^3 \end{aligned}$$

$$\begin{pmatrix} X_2 & X_1 \\ X_1^3 + X_0 X_2^2 & X_2^3 \end{pmatrix}$$

A quartic without triple points

$$\phi(t_0 : t_1) = (t_0^4 : 6t_0^2t_1^2 - 4t_1^4 : 4t_0^3t_1 - 4t_0t_1^3)$$
$$F(\underline{X}) = X_2^4 + 4X_0X_1^3 + 2X_0X_1X_2^2 - 16X_0^2X_1^2 - 6X_0^2X_2^2 + 16X_0^3X_1$$

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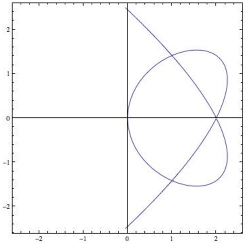

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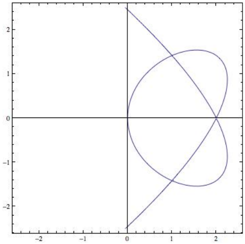

$$\begin{aligned}\mathcal{L}_{1,2}(\underline{T},\underline{X}) &= T_0(X_1X_2 - X_0X_2) + T_1(-X_2^2 - 2X_0X_1 + 4X_0^2)\\ \tilde{\mathcal{L}}_{1,2}(\underline{T},\underline{X}) &= T_0(X_1^2 + \tfrac{1}{2}X_2^2 - 2X_0X_1) + T_1(X_0X_2 - X_1X_2)\end{aligned}$$

Very concentrated singularities

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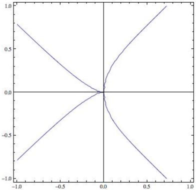
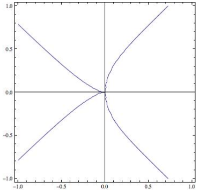


If the curve has a point of multiplicity $d-1$

Very concentrated singularities



If the curve has a point of multiplicity $d-1$
the implicit equation is always a 2×2 determinant

$$\left|\begin{array}{cc} \mathcal{L}_{1,1}(\underline{X}) & \mathcal{L}'_{1,1}(\underline{X}) \\ \mathcal{L}_{1,d-1}(\underline{X}) & \mathcal{L}'_{1,d-1}(\underline{X}) \end{array}\right|$$

In general, we do not know..

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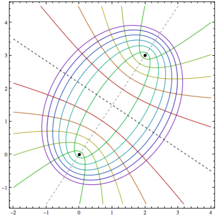
which moving lines?
which moving conics?
which moving cubics?
...



The Rees Algebra associated to the parametrization

Cox, D. **The moving curve ideal and the Rees algebra.** Theoret. Comput. Sci. 392 (2008), no. 1–3, 23–36

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$\mathcal{K}_\phi$:= {Moving curves following ϕ} = homogeneous elements in the kernel of

$$\begin{array}{ccc} \mathbb{K}[T_0, T_1, X_0, X_1, X_2] & \to & \mathbb{K}[T_0, T_1, s] \\ T_i & \mapsto & T_i \\ X_0 & \mapsto & a(\underline{T})s \\ X_1 & \mapsto & b(\underline{T})s \\ X_2 & \mapsto & c(\underline{T})s \end{array}$$

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"The ideal of moving curves following ϕ"

Method of moving curves revisited

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The implicit equation should be obtained as the determinant of a matrix with

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$$\left| \begin{array}{c} \cdots \\ \text{some minimal generators of } \mathcal{K}_\phi \\ \text{and relations among them} \\ \cdots \end{array} \right|$$

Method of moving curves revisited

The implicit equation should be obtained as the determinant of a matrix with

$$\left|\begin{array}{c} \cdots \\ \text{some minimal generators of } \mathcal{K}_\phi \\ \text{and relations among them} \\ \cdots \end{array}\right|$$

The more singular the curve, the simpler the description of $\mathcal{K}_\phi$

New Problem

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Compute a minimal system of generators of $\mathcal{K}_\phi$

New Problem

Compute a minimal system of generators of $\mathcal{K}_\phi$ for **any** ϕ

New Problem

Compute a minimal system of generators of $\mathcal{K}_\phi$ for **any** ϕ

Known for

- $\mu = 1$ (Hong-Simis-Vasconcelos, Cox-Hoffmann-Wang, Busé, Cortadellas-**D**)
- $\mu = 2$ (Busé, Cortadellas-**D**, Kustin-Polini-Ulrich)
- $(\mathcal{K}_\phi)_{(1,2)} \neq 0$ (Cortadellas- **D**)
- Monomial Parametrizations (Cortadellas-**D**)

A coarser problem

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Compute $n_0(\mathcal{K}_\phi)$, the number of minimal generators of $\mathcal{K}_\phi$

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Compute $n_0(\mathcal{K}_\phi)$, the number of minimal generators of $\mathcal{K}_\phi$

Show that if ϕ is "more singular" than ϕ' then $n_0(\mathcal{K}_\phi) \leq n_0(\mathcal{K}_{\phi'})$

Example: $\mu = 2$

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The curve has either

- one point of multiplicity $d-2$
  $n_0 = \mathcal{O}\left(\frac{d}{2}\right)$
  (Cortadellas-**D**, Kustin-Polini-Ulrich)

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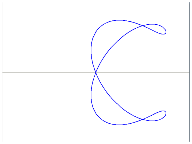
- one point of multiplicity $d - 2$
  $n_0 = \mathcal{O}\left(\frac{d}{2}\right)$
  (Cortadellas-**D**, Kustin-Polini-Ulrich)
- or only double points
  $n_0 = \mathcal{O}\left(\frac{d^2}{2}\right)$ (Busé)



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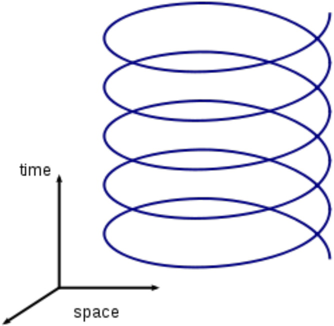
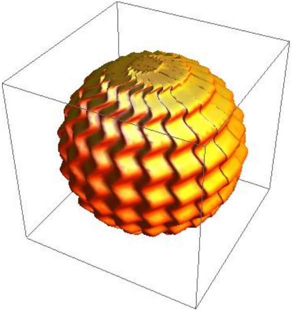
Other problems

- Describe **all** the possible values and parameters of the "function" $n_0(\mathcal{K}_\phi)$
- Does there exist a **generic** value for $n_0(\mathcal{K}_\phi)$? Is this the maximal value?
- In which "regions" is $n_0(\mathcal{K}_\phi)$ constant?

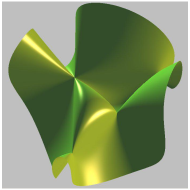
Only curves in the plane?





Rational Surfaces

$$\begin{array}{rccc} \phi_S : & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^3 \\ & \underline{t} = (t_0 : t_1 : t_2) & \longmapsto & \big(a(\underline{t}) : b(\underline{t}) : c(\underline{t}) : d(\underline{t})\big) \end{array}$$



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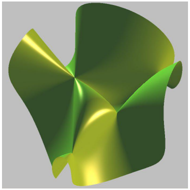

There are base points!

Implicitization via

- Resultants Macaulay, Dixon, Gelfand-Kapranov-Zelevinskii, ...

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Contrast:

- The module of moving planes is not free

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Contrast:

- The module of moving planes is not free
- There is a concept of μ-basis given by Chen-Cox-Liu

  Not easy to compute

Some results

Some results

Implicitization

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- Quadratic and cubic surfaces (Chen-Shen-Deng)

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Rees Algebras

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Rees Algebras

- "Monoid" Surfaces (Cortadellas - **D**)

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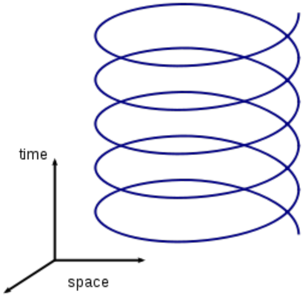
Rees Algebras

- "Monoid" Surfaces (Cortadellas - **D**)
- *de Jonquières* surfaces (Hassanzadeh- Simis)

Similar Results for

Spatial curves

$$\begin{array}{rccc} \phi_C: & \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^3 \\ & \underline{t} = (t_0 : t_1) & \longmapsto & \big(a(\underline{t}) : b(\underline{t}) : c(\underline{t}) : d(\underline{t})\big) \end{array}$$



Thanks!