On minimal generators of the ideal of moving curves following a rational plane parametrization

Carlos D'Andrea

Computational Algebra and Geometric Modeling
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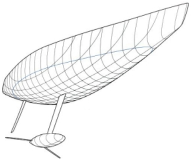

Rational Plane Parametrizations

$$\begin{array}{ccc} \mathbb{K} & \dashrightarrow & \mathbb{K}^2 \\ t & \longmapsto & \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \end{array}$$

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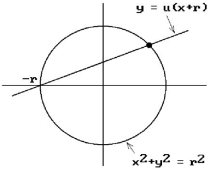
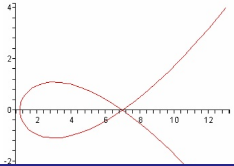
$$\begin{array}{cccc} \phi: & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ & (t_0 : t_1) & \longmapsto & (t_0^2 + t_1^2 : t_0^2 - t_1^2 : 2t_0t_1) \end{array}$$



Parametrization of Plane Curves

$$\begin{array}{cccc} \phi : & \mathbb{P}^1 & \to & \mathbb{P}^2 \\ & (t_0 : t_1) & \mapsto & (a(t_0,t_1) : b(t_0,t_1) : c(t_0,t_1)) \end{array}$$

- $a,\ b,\ c \in \mathbb{K}[T_0, T_1]$, homogeneous of the same degree $d \geq 1$
- $\gcd(a, b, c) = 1$



Rational Curves in the plane

The image of ϕ is a **rational plane curve**

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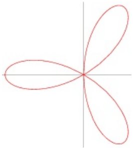

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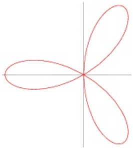

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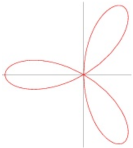

- It has degree d if ϕ is "generically" injective
- It has genus 0, which means the maximal number of multiple points $\frac{(d-1)(d-2)}{2}$
- Computing its implicit equation is relatively easy from ϕ

Sylvester's resultant

$$X_2a(\underline{T}) - X_0c(\underline{T}) = X_2T_0^2 - 2X_0T_0T_1 + X_2T_1^2$$

$$X_2b(\underline{T}) - X_1c(\underline{T}) = X_2T_0^2 - 2X_1T_0T_1 - X_2T_1^2$$

$$\mathrm{Res}_{\underline{T}}(X_2\cdot a(\underline{T}) - X_0\cdot c(\underline{T}), X_2\cdot b(\underline{T}) - X_1\cdot c(\underline{T})) = \det\begin{pmatrix} X_2 & -2X_0 & X_2 & 0 \\ 0 & X_2 & -2X_0 & X_2 \\ X_2 & -2X_1 & -X_2 & 0 \\ 0 & X_2 & -2X_1 & -X_2 \end{pmatrix}$$

From parametric to implicit

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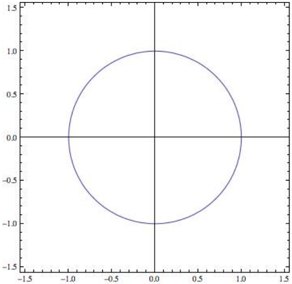
$$\mathrm{Res}_{\underline{T}}\left(X_2\cdot a(\underline{T})-X_0\cdot c(\underline{T}),X_2\cdot b(\underline{T})-X_1\cdot c(\underline{T})\right)$$
$$=$$
$$-4X_2^2(X_0^2-X_1^2-X_2^2)$$

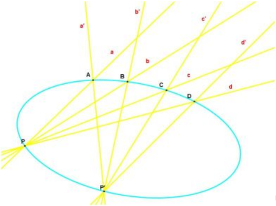


Moving lines

$$\mathcal{L}(T_0, T_1, X_0, X_1, X_2) = v_0(\underline{T})X_0 + v_1(\underline{T})X_1 + v_2(\underline{T})X_2$$

such that

$$\mathcal{L}(T_0, T_1, a(\underline{T}), b(\underline{T}), c(\underline{T})) = 0$$



In our example...

$$\begin{aligned}
\mathcal{L}_1(\underline{T},\underline{X}) &= -2T_0^2T_1X_0 + 0X_1 + (T_0^3 + T_0T_1^2)X_2 \\
\mathcal{L}_2(\underline{T},\underline{X}) &= -2T_0T_1^2X_0 + 0X_1 + (T_0^2T_1 + T_1^3)X_2 \\
\mathcal{L}_3(\underline{T},\underline{X}) &= 0X_0 - 2T_0^2T_1X_1 + (T_0^3 - T_0T_1^2)X_2 \\
\mathcal{L}_4(\underline{T},\underline{X}) &= 0X_0 - 2T_0T_1^2X_1 + (T_0^2T_1 - T_1^3)X_2
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\end{aligned}$$

$$\begin{pmatrix} X_2 & -2X_0 & X_2 & 0\\ 0 & X_2 & -2X_0 & X_2\\ X_2 & -2X_1 & -X_2 & 0\\ 0 & X_2 & -2X_1 & -X_2 \end{pmatrix}$$

In general

The determinant of a "matrix of moving lines" is a multiple of the implicit equation

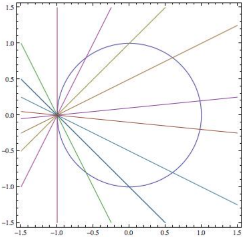
$$\begin{pmatrix} L_{11}(\underline{X}) & L_{12}(\underline{X}) & \dots & L_{1k}(\underline{X}) \\ L_{21}(\underline{X}) & L_{22}(\underline{X}) & \dots & L_{2k}(\underline{X}) \\ \vdots & \vdots & \dots & \vdots \\ L_{k1}(\underline{X}) & L_{k2}(\underline{X}) & \dots & L_{kk}(\underline{X}) \end{pmatrix}$$

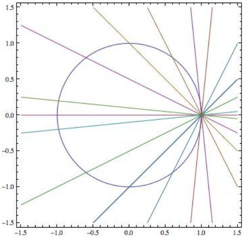
How small can the matrix be?

$$\begin{array}{rcl} \mathcal{L}_{1,1}(\underline{T},\underline{X}) & = & X_2 \quad T_0 \quad -(X_0+X_1) \quad T_1 \\ \mathcal{L}'_{1,1}(\underline{T},\underline{X}) & = & (-X_0+X_1) \quad T_0 \quad +X_2 \quad T_1 \end{array}$$

How small can the matrix be?

$$\mathcal{L}_{1,1}(\underline{T},\underline{X}) = X_2\, T_0 - (X_0 + X_1)\, T_1$$
$$\mathcal{L}'_{1,1}(\underline{T},\underline{X}) = (-X_0 + X_1)\, T_0 + X_2\, T_1$$





$$\det\begin{pmatrix} X_2 & -X_0 - X_1 \\ -X_0 + X_1 & X_2 \end{pmatrix} = X_1^2 + X_2^2 - X_0^2$$

The (free) module of moving lines

(Hilbert (1890))

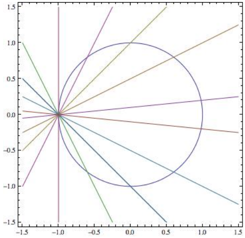
There exists $\mu \leq \frac{d}{2}$ and $p_\mu(\underline{T},\underline{X}),\ q_{d-\mu}(\underline{T},\underline{X})$ moving lines following ϕ such that any other $r_\delta(\underline{T},\underline{X})$ following ϕ is of the form

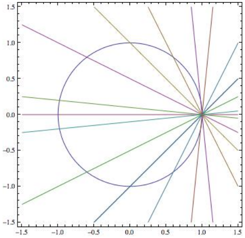
$$\mathcal{P}_{\delta-\mu}(\underline{T})p_\mu(\underline{T},\underline{X}) + \mathcal{Q}_{\delta-d+\mu}(\underline{T})q_{d-\mu}(\underline{T},\underline{X})$$

Geometric version

There exist $\mu \leq \frac{d}{2}$ and two other parametrizations $\varphi_\mu(t_0, t_1)$, $\psi_{d-\mu}(t_0, t_1)$ of degrees μ, $d - \mu$ such that

$$\boxed{\phi(t_0, t_1) = \varphi_\mu(t_0, t_1) \wedge \psi_{d-\mu}(t_0, t_1)}$$





For the unit circle...

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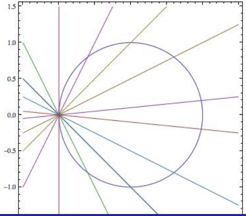
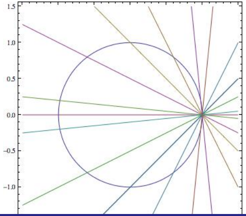
$$\begin{aligned} \varphi_1(t_0 : t_1) &= (-t_1 : -t_1 : t_0) \\ \psi_1(t_0 : t_1) &= (-t_0 : t_0 : t_1) \end{aligned}$$

For the unit circle...

$$\begin{array}{rcl}\varphi_1(t_0:t_1) & = & (-t_1:-t_1:t_0)\\ \psi_1(t_0:t_1) & = & (-t_0:t_0:t_1)\end{array}$$

$$\begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \\ -t_1 & -t_1 & t_0 \\ -t_0 & t_0 & t_1 \end{vmatrix} = \big(-t_0^2-t_1^2, t_1^2-t_0^2, -2t_0t_1\big)$$





Hilbert's Syzygy Theorem

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The homogeneous ideal $I = \big(a(\underline{T}), b(\underline{T}), c(\underline{T})\big) \subset \mathbb{K}[T_0, T_1]$ has a **Hilbert-Burch resolution** of the type

$$0 \to \mathbb{K}[\underline{T}]^2 \xrightarrow{(\varphi_\mu, \psi_{d-\mu})^{\mathbf{t}}} \mathbb{K}[\underline{T}]^3 \xrightarrow{(a,b,c)} \mathbb{K}[\underline{T}]$$

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A μ-basis of the parametrization is a basis of $\mathrm{Syz}(I)$ as a $\mathbb{K}[\underline{T}]$-module

Why do we care about μ-bases?

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Implicit equation

$$=$$

$$\mathrm{Res}_{\underline{T}}\left(p_\mu(\underline{T},\underline{X}),\ q_{d-\mu}(\underline{T},\underline{X})\right)$$

Why do we care about μ-bases?

$$\text{Implicit equation} \\ = \\ \mathrm{Res}_{\underline{T}}\left(p_\mu(\underline{T},\underline{X}),\ q_{d-\mu}(\underline{T},\underline{X})\right)$$

Busé-D (2012)

If B is a Bézout matrix, and S one of Sylester type, then

$$X_2\, S(p_\mu(\underline{T},\underline{X}),\ q_{d-\mu}(\underline{T},\underline{X})) = M \cdot B(aX_2 - cX_0, bX_2 - cX_1),$$

with $M \in \mathbb{K}^{d\times d}$ invertible

Moving conics, moving cubics,...

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$$\mathcal{O}(\underline{T})X_0^2 + \mathcal{P}(\underline{T})X_0X_1 + \mathcal{Q}(\underline{T})X_0X_2 + \mathcal{R}(\underline{T})X_1^2 + \mathcal{S}(\underline{T})X_1X_2 + \mathcal{T}(\underline{T})X_2^2 \in \mathbb{K}[\underline{T}, \underline{X}]$$

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is a **moving conic** following the parametrization if

$$\mathcal{O}(\underline{T})a(\underline{T})^2+\mathcal{P}(\underline{T})a(\underline{T})b(\underline{T})+\mathcal{Q}(\underline{T})a(\underline{T})c(\underline{T})+\mathcal{R}(\underline{T})b(\underline{T})^2+\mathcal{S}(\underline{T})b(\underline{T})c(\underline{T})+\mathcal{T}(\underline{T})c(\underline{T})^2=0$$

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the more **singular** the curve, the **simpler** the description of the determinant

Example (Sederberg & Chen 1995)

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The implicit equation of a quartic can be computed as a 2×2 determinant.
If the curve has a triple point, then one row is linear and the other is cubic.
Otherwise, both rows are quadratic.

A quartic with a triple point

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$$\phi(t_0, t_1) = (t_0^4 - t_1^4 : -t_0^2 t_1^2 : t_0 t_1^3)$$
$$F(X_0, X_1, X_2) = X_2^4 - X_1^4 - X_0 X_1 X_2^2$$

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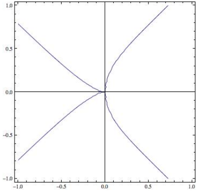

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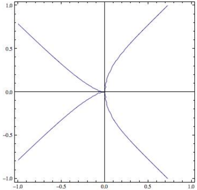

$$\begin{array}{rcl} \mathcal{L}_{1,1}(\underline{T},\underline{X}) & = & T_0 X_2 + T_1 X_1 \\ \mathcal{L}_{1,3}(\underline{T},\underline{X}) & = & T_0 (X_1^3 + X_0 X_2^2) + T_1 X_2^3 \end{array}$$

$$\begin{pmatrix} X_2 & X_1 \\ X_1^3 + X_0 X_2^2 & X_2^3 \end{pmatrix}$$

A quartic without triple points

$$\phi(t_0 : t_1) = (t_0^4 : 6t_0^2t_1^2 - 4t_1^4 : 4t_0^3t_1 - 4t_0t_1^3)$$
$$F(\underline{X}) = X_2^4 + 4X_0X_1^3 + 2X_0X_1X_2^2 - 16X_0^2X_1^2 - 6X_0^2X_2^2 + 16X_0^3X_1$$

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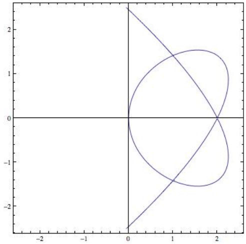

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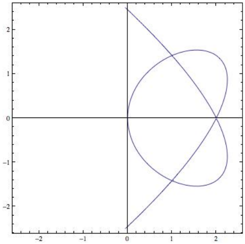

$$\begin{aligned}\mathcal{L}_{1,2}(\underline{T}, \underline{X}) &= T_0(X_1X_2 - X_0X_2) + T_1(-X_2^2 - 2X_0X_1 + 4X_0^2) \\ \tilde{\mathcal{L}}_{1,2}(\underline{T}, \underline{X}) &= T_0(X_1^2 + \tfrac{1}{2}X_2^2 - 2X_0X_1) + T_1(X_0X_2 - X_1X_2)\end{aligned}$$

Very concentrated singularities

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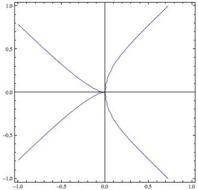

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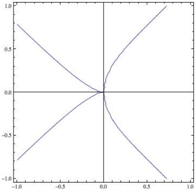

If the curve has a point of multiplicity $d-1$
the implicit equation is always a 2×2 determinant

$$\left| \begin{array}{cc} \mathcal{L}_{1,1}(\underline{X}) & \mathcal{L}'_{1,1}(\underline{X}) \\ \mathcal{L}_{1,d-1}(\underline{X}) & \mathcal{L}'_{1,d-1}(\underline{X}) \end{array} \right|$$

In general, we do not know..

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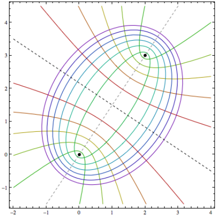
which moving lines?
which moving conics?
which moving cubics?
. . .



The Rees Algebra associated to the parametrization

Cox, D. **The moving curve ideal and the Rees algebra.** Theoret. Comput. Sci. 392 (2008), no. 1–3, 23–36

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$\mathcal{K}_\phi :=$ {Moving curves following ϕ} = homogeneous elements in the kernel of

$$\begin{array}{ccc} \mathbb{K}[T_0, T_1, X_0, X_1, X_2] & \to & \mathbb{K}[T_0, T_1, s] \\ T_i & \mapsto & T_i \\ X_0 & \mapsto & a(\underline{T})s \\ X_1 & \mapsto & b(\underline{T})s \\ X_2 & \mapsto & c(\underline{T})s \end{array}$$

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"The ideal of moving curves following ϕ"

Method of moving curves revisited

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Method of moving curves revisited

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The more singular the curve, the simpler the description of $\mathcal{K}_\phi$

New Problem

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Compute a minimal system of generators of $\mathcal{K}_\phi$

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Compute a minimal system of generators of $\mathcal{K}_\phi$ for **any** ϕ

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Known for

- $\mu = 1$ (Hong-Simis-Vasconcelos, Cox-Hoffmann-Wang, Busé, Cortadellas-**D**)
- $\mu = 2$ (Busé, Cortadellas-**D**, Kustin-Polini-Ulrich)
- $\left(\mathcal{K}_\phi\right)_{(1,2)} \neq 0$ (Cortadellas- **D**)
- Monomial Parametrizations (Cortadellas-**D**)

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Compute $n_0(\mathcal{K}_\phi)$, the number of minimal generators of $\mathcal{K}_\phi$

A coarser problem

Compute $n_0(\mathcal{K}_\phi)$, the number of minimal generators of $\mathcal{K}_\phi$

Show that if ϕ is "more singular" than ϕ' then $n_0(\mathcal{K}_\phi) \leq n_0(\mathcal{K}_{\phi'})$

Example: $\mu = 2$

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The curve has either

- one point of multiplicity $d-2$
  $n_0 = \mathcal{O}\left(\frac{d}{2}\right)$
  (Cortadellas-**D**, Kustin-Polini-Ulrich)

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  $n_0 = \mathcal{O}\left(\frac{d}{2}\right)$
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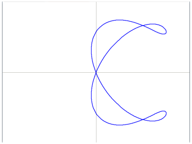
- or only double points
  $n_0 = \mathcal{O}\left(\frac{d^2}{2}\right)$ (Busé)



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- In which "regions" is $n_0(\mathcal{K}_\phi)$ constant?

Recent Progress

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Jeff Madsen

Equations of Rees algebras of ideals in two variables

arXiv:1511.04073

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Describes the bi-degrees of all minimal generators in $\underline{T}$-degree larger than or equal to μ

Ongoing Project

(w/Teresa Cortadellas and David Cox)

- Make explicit these generators and their degrees

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- Complete the description to a set of **all** minimal generators

The μ-type of ϕ

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It also has a μ-basis!

The μ-type of ϕ

The μ-basis of

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is

$$A_h = (A_0(\underline{T}), A_1(\underline{T}), A_2(\underline{T}))$$
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with

$$\deg(A_h) = h \leq \deg(B_\ell) = \ell$$
$$h + \ell = \mu$$

$(\boldsymbol{h}, \ell)$ is the μ type of ϕ

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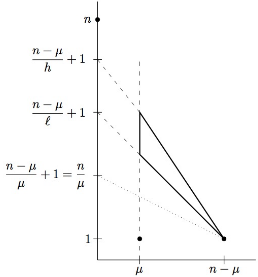
Bernardi, Gimigliano, Idà (arXiv:1507.02227)

$h = 0 \iff$ there is an axial moving line

Theorem

(Jeff Madsen `arXiv:1511.04073`)

The minimal generators of $(\mathcal{K}_\phi)_{\geq \mu, *}$ are inside the triangle



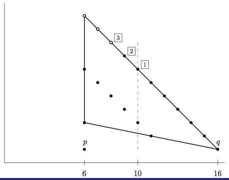
Moreover...

(Jeff Madsen arXiv:1511.04073)

If $h < \ell$ there is one minimal generator at bidegree

$$\boxed{(i,j) = (n - \mu - \alpha h - \beta\ell, \alpha + \beta + 1)}$$

$$(\alpha, \beta \geq 0,\ n - \mu - \alpha h - \beta\ell \geq \mu)$$

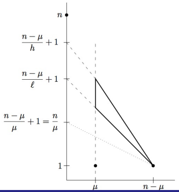


(Jeff Madsen arXiv:1511.04073)

If $h = \ell$ there are exactly j minimal at

$$\boxed{(i,j) = (n-\mu-\alpha h, \alpha+1)}$$

$$(\alpha \geq 0,\ n-\mu-\alpha h \geq \mu)$$



Our Contribution

(Cortadellas-Cox-D 2016)

Construction of explicit generators

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Recall: The μ-basis of

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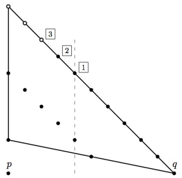
$$D_A(G) := \begin{vmatrix} G_0 & G_1 & G_2 \\ x_0 & x_1 & x_2 \\ A_0 & A_1 & A_2 \end{vmatrix} \in (\mathcal{K}_\phi)_{i-\ell,j+1}$$

Analogously

(Cortadellas-Cox-D 2016)

$$D_B(G) := \begin{vmatrix} G_0 & G_1 & G_2 \\ x_0 & x_1 & x_2 \\ B_0 & B_1 & B_2 \end{vmatrix} \in (\mathcal{K}_\phi)_{i-h,j+1}$$



Creating minimal generators

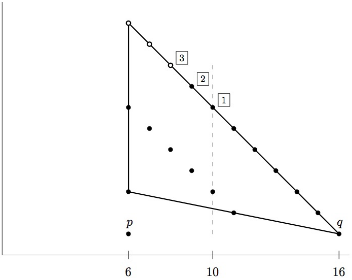


FIGURE 2. Degrees when $n = 22, \mu = 6, h = 1, \ell = 5$

Starting from $q_{d-\mu}$

Creating minimal generators

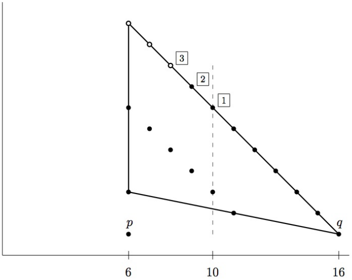


FIGURE 2. Degrees when $n = 22, \mu = 6, h = 1, \ell = 5$

Starting from $q_{d-\mu}$ we apply either D_A or D_B to get **almost** all the generators...

Another approach

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$$\boxed{\phi = \alpha_{d-h}(\underline{T}) A_h + \beta_{d-\ell}(\underline{T}) B_\ell}$$

The “lifting” of ϕ

(Bernardi, Gimigliano, Idà `arXiv:1507.02227`)

$$\begin{array}{ccc} \mathbb{P}^1 & \to & \mathbb{P}^{\mu+1} \\ \underline{t} & \mapsto & \left(\alpha_{d-h}(\underline{t})t_0^h : \ldots : \beta_{d-\ell}(\underline{t})t_1^\ell\right) \end{array}$$

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The projection $\mathbb{P}^{\mu+1} \to \mathbb{P}^2$ is linear

Algebraically...

(Cortadellas-Cox-D 2016)

$$(a(\underline{T}), b(\underline{T}), c(\underline{T})) \subset$$
$$\left(\alpha_{d-h}(\underline{T})\, T_0^i T_1^{h-i},\ \beta_{d-\ell}(\underline{T})\, T_0^j T_1^{\ell-j}\right)_{0\leq i\leq h,\, 0\leq j\leq \ell}$$

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$$\Longrightarrow$$

$$\mathcal{K}_\phi = \mathrm{Rees}(\phi) \text{``} \subset\text{''}\ \mathrm{Rees}(\text{lifted curve})$$

Main Result

(Cortadellas-Cox-D 2016)

Explicit generators of

Rees(lifted curve)

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Rees(lifted curve)

- Some generators from the normal scroll
- Some coming from "polarizing"
  $\alpha_{d-h}(\underline{T}),\ \beta_{d-\ell}(\underline{T})$
- Some coming from the "monomial ideal"
  $(i,j,k) \in \mathbb{N}^3 :\ i + hj + \ell k \geq d - \mu$

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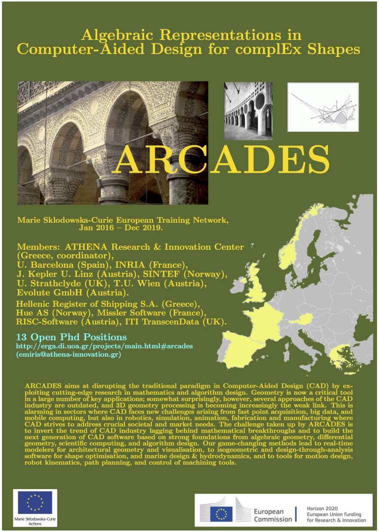
- And also for smaller values of i!

Thanks!



Algebraic Representations in
Computer-Aided Design for complEx Shapes
ARCADES
Marie Sklodowska-Curie European Training Network,
Jan 2016 – Dec 2019.
Members: ATHENA Research & Innovation Center
(Greece, coordinator),
U. Barcelona (Spain), INRIA (France),
J. Kepler U. Linz (Austria), SINTEF (Norway),
U. Strathclyde (UK), T.U. Wien (Austria),
Evolute GmbH (Austria).
Hellenic Register of Shipping S.A. (Greece),
Hue AS (Norway), Missler Software (France),
RISC-Software (Austria), ITI TranscenData (UK).
13 Open Phd Positions
http://erga.di.uoa.gr/projects/main.html#arcades
(emiris@athena-innovation.gr)
ARCADES aims at disrupting the traditional paradigm in Computer-Aided Design (CAD) by exploiting cutting-edge research in mathematics and algorithm design. Geometry is now a critical tool in a large number of key applications; somewhat surprisingly, however, several approaches of the CAD industry are outdated, and 3D geometry processing is becoming increasingly the weak link. This is alarming in sectors where CAD faces new challenges arising from fast point acquisition, big data, and mobile computing, but also in robotics, simulation, animation, fabrication and manufacturing where CAD strives to address crucial societal and market needs. The challenge taken up by ARCADES is to invert the trend of CAD industry lagging behind mathematical breakthroughs and to build the next generation of CAD software based on strong foundations from algebraic geometry, differential geometry, scientific computing, and algorithm design. Our game-changing methods lead to real-time modelers for architectural geometry and visualisation, to isogeometric and design-through-analysis software for shape optimisation, and marine design & hydrodynamics, and to tools for motion design, robot kinematics, path planning, and control of machining tools.
Marie Skłodowska-Curie Actions
European Commission
Horizon 2020
European Union funding
for Research & Innovation