On the resolution of fan algebras of principal ideals over a Noetherian ring

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Motivation

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- for all $r, s \in \mathbb{N}$?
- for $r, s \gg 0$?

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(x_1, x_2 indeterminates over R)

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The **double Rees Algebra** of I and J :

$$\begin{aligned} \mathcal{R}_{\mathrm{R}}(I, J) &:= \mathrm{R}[Ix_1, Jx_2] \\ &= \bigoplus_{r,s \in \mathbb{N}} I^r J^s x_1^r x_2^s \end{aligned}$$

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There are examples of $\mathcal{B}_{\mathrm{R}}(I, J)$ that are not even finitely generated over R!

😱

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- Principal ideals in a UFD (S. Malec, 2013)

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- "smoothness" in the length of Tors of quotient by powers of ideals
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- control on the asymptotic growth of powers of ideals
  (Ciuperca, Enescu,Spiroff, 2007)

Open Questions

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- When is $\mathcal{B}_{\mathrm{R}}(I, J)$ a finite algebra?

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- When is $\mathcal{B}_{\mathrm{R}}(I, J)$ a finite algebra?
- If so, can we compute a set of generators?

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- subadditive

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- $f(u+v) \leq f(u) + f(v)$ $\forall u, v \in |\mathcal{F}| \cap \mathbb{Z}^m$

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$$\mathcal{B}_{\mathbf{f},\mathcal{F},\mathbf{I}} := \bigoplus_{\mathbf{v}\in|\mathcal{F}|\cap\mathbb{Z}^m} I_1^{f_1(\mathbf{v})} \ldots I_n^{f_n(\mathbf{v})} \mathbf{x}^{\mathbf{v}}$$

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$$\mathcal{B}_{\mathbf{f},\mathcal{F},\mathbf{I}} = \bigoplus_{(k_1,\ldots,k_n)\in\mathbb{N}^n} I_1^{k_1} \cdots I_n^{k_n} x_1^{k_1} \cdots x_n^{k_n}$$

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The intersection algebra of $\langle p_1^{a_1} \ldots p_n^{a_n} \rangle$ and $\langle p_1^{b_1} \ldots p_n^{b_n} \rangle$

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If R is Noetherian, $\mathcal{B}_{\mathbf{f},\mathcal{F},\mathbf{I}}$ is finitely generated
and a finite set of generators can be computed!
(S. Malec, 2013)

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Theorem

$\mathcal{B}_{\mathbf{f},\mathcal{F},\mathbf{I}}$ is generated as an algebra over R by $\{r_{ijh}\,\mathbf{x}^{\mathbf{v}_{ij}};\ i = 1, \ldots, \ell,\ j = 1, \ldots, g_i,\ h = 1, \ldots, h_{ij}\}$ (S. Malec, 2013)

Moreover...

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For the intersection algebra of principal ideals in a UFD, it is a **minimal set of generators** (S. Malec, 2013)

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- minimal resolutions of $\mathcal{B}_{\mathbf{f},\mathcal{F},\mathbf{I}}$?

Our contribution

(Cortadellas-D-Enescu)

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- $I_i = \langle p_i \rangle$ principal, $i = 1, \ldots, n$

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If $|\mathcal{F}|$ is a pointed rational cone in $\mathbb{R}^2$, we can make explicit a $\mathbb{Z}^2$-graded free resolution of $\mathcal{B}_{\mathbf{f},\mathcal{F},\mathbf{p}}$ of the form

$$0 \to \mathrm{R}[\mathbf{Y}]^{M-2} \stackrel{\varphi_{M-2}}{\to} \ldots \stackrel{\varphi_3}{\to} \mathrm{R}[\mathbf{Y}]^{2\binom{M-1}{3}} \stackrel{\varphi_2}{\to} \mathrm{R}[\mathbf{Y}]^{\binom{M-1}{2}} \stackrel{\varphi_1}{\to} \mathrm{R}[\mathbf{Y}] \stackrel{\varphi_0}{\to} \mathcal{B}_{\mathbf{f},\mathcal{F},\mathbf{p}} \to 0$$

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- $\mathbf{Y} = (Y_{\mathbf{v}_{ij}})_{1\leq i\leq\ell,\, 1\leq j\leq g_i}$

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If R is *local with respect to the $\mathbb{Z}^2$-graduation, then the resolution is **minimal** iff

- **f** is **strict** with respect to $\mathcal{F}$
- none of the p_i's is a zero divisor in R
- all of the p_i's belong to the unique homogeneous maximal ideal of R

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- Minimal generators can be obtained as in Malec's case
- Minimal syzygies come from **non adjacent** vectors in the clockwise list of vectors $\mathbf{v}_{ij}$
- The rest of the resolution is computed "a la Schreyer"...(we actually show that the syzygies are minimal Gröbner bases for the induced order)

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$\mathcal{B}_{\mathbf{f},\mathcal{F},\mathbf{I}} \subseteq \mathrm{R}[x_1, x_2]$ is the intersection algebra $\mathcal{B}_{\mathrm{R}}(I, J)$:

$$\sum_{(r,s)\in\mathbb{N}^2} \langle p_1^{\max(2r,s)} p_2^{\max(r,2s)} \rangle x_1^r x_2^s$$

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The maximal cones are

$$C_1 = \{(r,s) \in \mathbb{R}^2_{\geq 0} : 2r \leq s,\ r \leq 2s\}$$
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$$\{(0,1), (1,2)\},\ \{(1,2), (1,1), (2,1)\},\ \{(2,1), (1,0)\}$$

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$$\text{M=5},\ \mathcal{H} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$$

Minimal Generators

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$$\mathcal{B}_{\mathrm{R}}(I, J) = \mathrm{R}[p_1p_2^2x_2,\ p_1^2p_2^4x_1x_2^2,\ p_1^2p_2^2x_1x_2,\ p_1^4p_2^2x_1^2x_2,\ p_1^2p_2x_1]$$

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- $f_1(r, s) = \max(2r, s),\ f_2(r, s) = \max(r, 2s)$

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- $f_1(r,s) = \max(2r, s),\ f_2(r,s) = \max(r, 2s)$
- $\mathcal{B}_{\mathrm{R}}(I, J)$ is minimally generated by $\mathbf{p}^{f(\mathbf{v}_i)}\mathbf{x}^{\mathbf{v}_i}$, $i = 1, \ldots, 5$

Presentation

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We look for relations from the epimorphism

$$\begin{array}{ccc} \mathrm{R}[\mathbf{Y}] & \overset{\varphi_0}{\to} & \mathcal{B}_{\mathrm{R}}(I,J) \\ Y_1 & \mapsto & p_1p_2^2x_2 \\ Y_2 & \mapsto & p_1^2p_2^4x_1x_2^2 \\ Y_3 & \mapsto & p_1^2p_2^2x_1x_2 \\ Y_4 & \mapsto & p_1^4p_2^2x_1^2x_2 \\ Y_5 & \mapsto & p_1^2p_2x_1 \end{array}$$

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If $i < j - 1$, there exists $i \leq k < j$, and $\alpha, \alpha' \in \mathbb{N}$ such that $\mathbf{v}_i + \mathbf{v}_j = \alpha \mathbf{v}_k + \alpha' \mathbf{v}_{k+1}$

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$$\mathbf{s}_{ij} := Y_{\mathbf{v}_i} Y_{\mathbf{v}_j} - Y_{\mathbf{w}}^{\alpha} Y_{\mathbf{w}'}^{\alpha'} \mathbf{p}^{\gamma_{\mathbf{v}_i \mathbf{v}_j}} \in \ker(\varphi_0)$$

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with

$$\gamma_{\mathbf{v}\mathbf{v}'} = \left(f_1(\mathbf{v}) + f_1(\mathbf{v}') - f_1(\mathbf{v}+\mathbf{v}'), f_2(\mathbf{v}) + f_2(\mathbf{v}') - f_2(\mathbf{v}+\mathbf{v}')\right)$$

In our case

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$$\mathbf{s}_{13} = Y_1 Y_3 - p_1 Y_2, \quad \mathbf{s}_{14} = Y_1 Y_4 - p_1 Y_3^2$$
$$\mathbf{s}_{15} = Y_1 Y_5 - p_1 p_2 Y_3, \quad \mathbf{s}_{24} = Y_2 Y_4 - Y_3^3$$
$$\mathbf{s}_{25} = Y_2 Y_5 - p_2 Y_3^2, \quad \mathbf{s}_{35} = Y_3 Y_5 - p_2 Y_4$$

In our case

$$\mathbf{s}_{13} = Y_1Y_3 - p_1Y_2, \qquad \mathbf{s}_{14} = Y_1Y_4 - p_1Y_3^2$$
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The family $\mathbf{S} := \{\mathbf{s}_{13}, \mathbf{s}_{14}, \mathbf{s}_{15}, \mathbf{s}_{24}, \mathbf{s}_{25}, \mathbf{s}_{35}\}$ is a reduced Gröbner basis for *lex*, and a minimal set of generators of $\ker(\varphi_0)$

In our case

$$\begin{aligned} \mathbf{s}_{13} &= Y_1Y_3 - p_1Y_2, & \mathbf{s}_{14} &= Y_1Y_4 - p_1Y_3^2 \\ \mathbf{s}_{15} &= Y_1Y_5 - p_1p_2Y_3, & \mathbf{s}_{24} &= Y_2Y_4 - Y_3^3 \\ \mathbf{s}_{25} &= Y_2Y_5 - p_2Y_3^2, & \mathbf{s}_{35} &= Y_3Y_5 - p_2Y_4 \end{aligned}$$

The family $\mathbf{S} := \{\mathbf{s}_{13}, \mathbf{s}_{14}, \mathbf{s}_{15}, \mathbf{s}_{24}, \mathbf{s}_{25}, \mathbf{s}_{35}\}$ is a reduced Gröbner basis for *lex*, and a minimal set of generators of $\ker(\varphi_0)$

$$\#\mathbf{S} = 6 = \binom{M-1}{2}$$

The Resolution

With similar combinatorial arguments we obtain explicitly the minimal free resolution

$$0 \to \mathrm{R}[\mathbf{Y}]^3 \stackrel{\varphi_3}{\to} \mathrm{R}[\mathbf{Y}]^8 \stackrel{\varphi_2}{\to} \mathrm{R}[\mathbf{Y}]^6 \stackrel{\varphi_1}{\to} \mathrm{R}[\mathbf{Y}] \stackrel{\varphi_0}{\to} \mathcal{B} \to 0$$

The maps are explicit

$$\varphi_1 = \begin{pmatrix} Y_1 Y_3 - p_1 Y_2 & Y_1 Y_4 - p_1 Y_3^2 & Y_1 Y_5 - p_1 p_2 Y_3 & Y_2 Y_4 - Y_3^3 & Y_2 Y_5 - p_2 Y_3^2 & Y_3 Y_5 - p_2 Y_4 \end{pmatrix}$$

$$\varphi_2 = \begin{pmatrix} -Y_3^2 & -p_2 Y_3 & Y_4 & Y_5 & 0 & 0 & 0 & 0 \\ Y_2 & 0 & -Y_3 & 0 & -p_2 & Y_5 & 0 & 0 \\ 0 & Y_2 & 0 & -Y_3 & Y_3 & -Y_4 & 0 & 0 \\ -Y_1 & 0 & p_1 & 0 & 0 & 0 & -p_2 & Y_5 \\ 0 & -Y_1 & 0 & p_1 & 0 & 0 & Y_3 & -Y_4 \\ 0 & 0 & 0 & 0 & -Y_1 & p_1 Y_3 & -Y_2 & Y_3^2 \end{pmatrix}$$

$$\varphi_3 = \begin{pmatrix} -p_2 & Y_5 & 0 \\ Y_3 & -Y_4 & 0 \\ 0 & -p_2 Y_3 & Y_5 \\ 0 & Y_3^2 & -Y_4 \\ -Y_2 & Y_3^2 & 0 \\ 0 & -Y_2 & Y_3 \\ 0 & Y_1 & -p_1 \\ Y_1 & -p_1 Y_3 & 0 \end{pmatrix}$$

References

Cortadellas, T., D'Andrea, C., Enescu, F. *On the resolution of fan algebras of principal ideals over a Noetherian ring*
arXiv:1612.02939

References

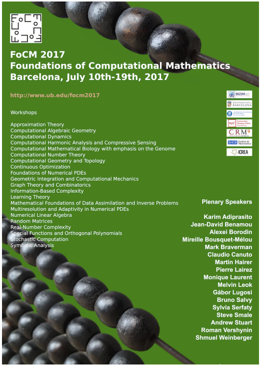
- Cortadellas, T., D'Andrea, C., Enescu, F. *On the resolution of fan algebras of principal ideals over a Noetherian ring* **arXiv:1612.02939**
- Fields, J.B. *Lengths of Tors determined by killing powers of ideals in a local ring.* Journal of Algebra, vol. 247, no. 1, 104–133, 2002

References

- Cortadellas, T., D'Andrea, C., Enescu, F. *On the resolution of fan algebras of principal ideals over a Noetherian ring* **arXiv:1612.02939**
- Fields, J.B. *Lengths of Tors determined by killing powers of ideals in a local ring.* Journal of Algebra, vol. 247, no. 1, 104–133, 2002
- Malec, S. *On the Intersection Algebra of Principal Ideals.* Comm. Algebra 43 (2015), no. 2, 623–635

References

- Cortadellas, T., D'Andrea, C., Enescu, F. *On the resolution of fan algebras of principal ideals over a Noetherian ring* **arXiv:1612.02939**
- Fields, J.B. *Lengths of Tors determined by killing powers of ideals in a local ring.* Journal of Algebra, vol. 247, no. 1, 104–133, 2002
- Malec, S. *On the Intersection Algebra of Principal Ideals.* Comm. Algebra 43 (2015), no. 2, 623–635
- Malec, S. *Intersection Algebras and pointed rational cones.* Ph.D. Thesis, 2013.



See you in a month!