Elimination Theory

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Fourth EACA International School on Computer Algebra and its Applications

What is Elimination Theory?

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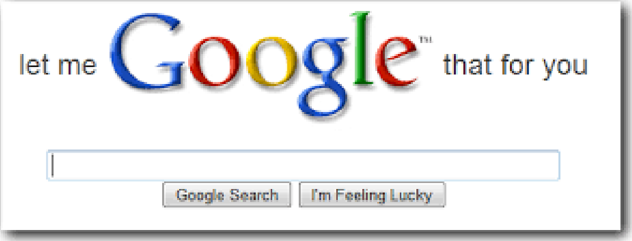

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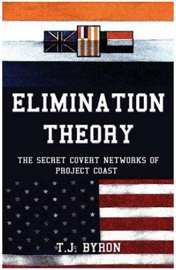

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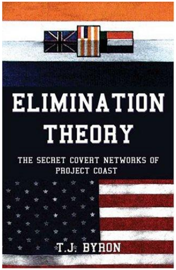
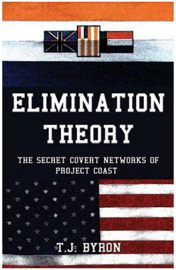


$$\operatorname{Res}(f(x),\ g(x)) = \begin{vmatrix} a_m & a_{m-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & a_m & a_{m-1} & \cdots & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_m & a_{m-1} & \cdots & a_1 & a_0 \\ b_n & b_{n-1} & \cdots & b_1 & b_0 & 0 & \cdots & 0 \\ 0 & b_n & b_{n-1} & \cdots & b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & b_n & b_{n-1} & \cdots & b_1 & b_0 \end{vmatrix}.$$

Elimination Theory



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A bit of history

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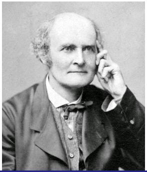
In the problem of **elimination**, one seeks the relationship that must exist between the coefficients of a function or system of functions in order that some particular circumstance (or singularity) can occur.

Cayley, 1864 Nouvelles recherches sur l'élimination et la théorie des courbes



A bit of history

A system of arbitrarily many algebraic equations for $z^0, z', z'', \ldots, z^{(n-1)}$, in which the coefficients belong to the rationality domain $(R, R', R'', \ldots.)$, define the algebraic relations between z and R, whose knowledge and representation are the purpose of the **theory of elimination**.

Kronecker, 1882 Grundzüge einer arithmetischen Theorie der algebraischen Grössen

The baby example in elimination

The baby example in elimination

Find "the condition" on

$a_{10}, a_{11}, a_{20}, a_{21}$ so that the system

$$\begin{cases} a_{10}x_0 + a_{11}x_1 = 0 \\ a_{20}x_0 + a_{21}x_1 = 0 \end{cases}$$

has a solution different from $(0,0)$

“Elimination”

$$a_{10}x_0 + a_{11}x_1, \; a_{20}x_0 + a_{21}x_1$$

$$\in \mathbb{K}[a_{10}, a_{11}, a_{20}, a_{21}, x_0, x_1]$$

"Elimination"

$$a_{10}x_0 + a_{11}x_1,\ a_{20}x_0 + a_{21}x_1$$

$$\in \mathbb{K}[a_{10}, a_{11}, a_{20}, a_{21}, x_0, x_1]$$
$$\downarrow$$
$$a_{10}a_{21} - a_{20}a_{11} \in \mathbb{K}[a_{10}, a_{11}, a_{20}, a_{21}]$$

More general

Find "the condition" for the system

$$\begin{cases} a_{10}x_0 + a_{11}x_1 + \ldots + a_{1n}x_n & = & 0 \\ a_{20}x_0 + a_{21}x_1 + \ldots + a_{2n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{(n+1)0}x_0 + a_{(n+1)1}x_1 + \ldots + a_{(n+1)n}x_n & = & 0 \end{cases}$$

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to have a solution different from

$$(0, 0, \ldots, 0)$$

Another more general

Let $d_1, d_2 \in \mathbb{N}$. Find "the condition" for the system of polynomials

$$\begin{cases} a_{10}x_0^{d_1} + a_{11}x_0^{d_1-1}x_1 + \ldots = 0 \\ a_{20}x_0^{d_2} + a_{21}x_0^{d_2-1}x_1 + \ldots = 0 \end{cases}$$

to have a solution different from $(0,0)$

More more more general...

Let $n \in \mathbb{N}$, and $d_1, \ldots, d_{n+1} \in \mathbb{N}$, find the condition for

$$\left\{ \begin{array}{lcl} \sum_{\alpha_0+\ldots+\alpha_n=d_1} a_{1,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0} \ldots {x_n}^{\alpha_n} & = & 0 \\ \sum_{\alpha_0+\ldots+\alpha_n=d_2} a_{2,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0} \ldots {x_n}^{\alpha_n} & = & 0 \\ \vdots & \vdots & \vdots \\ \sum_{\alpha_0+\ldots+\alpha_n=d_{n+1}} a_{n+1,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0} \ldots {x_n}^{\alpha_n} & = & 0 \end{array} \right.$$

to have a solution different from $(0, 0, \ldots, 0)$

Elimination: The general problem

For $\boldsymbol{a} = (a_1, \ldots, a_N)$, k, $n \in \mathbb{N}$ let $f_1(\boldsymbol{a}, x_1, \ldots, x_n), \ldots, f_k(\boldsymbol{a}, x_1, \ldots, x_n) \in \mathbb{K}[\boldsymbol{a}, x_1, \ldots, x_n]$. Find conditions on $\boldsymbol{a}$ such that

$$\left\{ \begin{array}{rcl} f_1(\boldsymbol{a}, x_1, \ldots, x_n) & = & 0 \\ f_2(\boldsymbol{a}, x_1, \ldots, x_n) & = & 0 \\ \vdots & \vdots & \vdots \\ f_k(\boldsymbol{a}, x_1, \ldots, x_n) & = & 0 \end{array} \right.$$

has a solution

Solution?

- Depends on the ground field

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- There is not necessarily a “closed” condition

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- Depends on the ground field
- There is not necessarily a “closed” condition
- Algorithms??

Easy example

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + \ldots + a_{2n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{kn}x_1 + \ldots + a_{kn}x_n & = & 0 \end{cases}$$

with $k \geq n$

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with $k \geq n$

Conditions: all maximal minors of $(a_{ij})_{1 \leq i \leq k,\, 1 \leq j \leq n}$ equal to zero

Another "easy" example

$$k = n = 1,$$

$$a_0 + a_1 x_1 + a_2 {x_1}^2 + \ldots + a_d {x_1}^d = 0$$

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Conditions?

Geometry

$$V = \{(\boldsymbol{a}, x_1, \ldots, x_n) : f_1(\boldsymbol{a}, x_1, \ldots, x_n) = 0, \ldots f_k(\boldsymbol{a}, x_1, \ldots, x_n) = 0\}$$

Geometry

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$$\begin{array}{ccc} V & \subset & \mathbb{K}^N \times \mathbb{K}^n \\ \pi_1|_V \downarrow & & \downarrow \pi_1 \\ \pi_1(V) & \subset & \mathbb{K}^N \end{array}$$

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The set of conditions is $\pi_1(V)$ is not necessarily described by zeroes of polynomials

Projective Elimination

(homogeneous polynomials in an algebraically closed field)

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$$\pi_1(V) = \{p_1(\mathbf{a}) = 0, \ldots, p_\ell(\mathbf{a}) = 0\}$$

How do we compute these conditions?

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- Input: $f_1(a, x), \ldots f_k(a, x)$

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- Output: $p_1(a), \ldots, p_\ell(a)$

Mertens 1889

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Set $s := \max \deg(f_i)$, and $G_0, \ldots, G_n$ expressions of the form
$G_j(\boldsymbol{a}, \boldsymbol{x}) := \sum u_{\alpha,i} \boldsymbol{x}^{\alpha} f_i(\boldsymbol{a}, \boldsymbol{x})$ of degree s in $\boldsymbol{x}$ with new parameters $u_{\alpha,i}$.

Mertens 1889

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Then, "the Kronecker $\mathbf{u}$-Resultant"

$\mathrm{Res}_{\mathbf{x}}(G_0, \ldots, G_n) = \sum_\beta P_\beta(\mathbf{a})\mathbf{u}^\beta$ gives the conditions $\{P_\beta(\mathbf{a})\}$

van der Waerden 1926

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For arbitrary s, we use the "Macaulay matrix" $M_s(\mathbf{a})$ such that

$$M_s(\mathbf{a}) \begin{bmatrix} \vdots \\ x^{\beta} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ x^{\alpha} f_i \\ \vdots \end{bmatrix}, \ |\alpha| = s$$

van der Waerden 1926

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For $s \gg 0$, all maximal minors of $M_s(\boldsymbol{a})$ vanish iff there is a common solution

Algebra & Geometry

(van der Waerden, 1926)

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$$I\left(\pi_1(V)\right)$$
$$=$$
$$\left(\langle f_1, \ldots, f_k\rangle : \langle x_0, \ldots, x_n\rangle^{\infty}\right)$$

Algebra & Geometry

(van der Waerden, 1926)

$$\boxed{\begin{gathered} I\left(\pi_1(V)\right) \\ = \\ \left(\langle f_1, \ldots, f_k\rangle : \langle x_0, \ldots, x_n\rangle^\infty\right) \cap \mathbb{K}[a] \end{gathered}}$$

The "elimination ideal"

But what if...

But what if...

- the polynomials are not homogeneous?

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- $\mathbb{K}$ is not algebraically closed?

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Homogeneous vs non homogeneous

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Analogy with Linear Algebra...

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

Generalization of the determinant?

$$V = \{(\boldsymbol{a}, x_0, x_1, \ldots, x_n) : \, f_1(\boldsymbol{a}, x_0, x_1, \ldots, x_n) = 0, \ldots, f_{n+1}(\boldsymbol{a}, x_0, x_1, \ldots, x_n) = 0\}$$

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One hopes!

Example 1

$$\left\{\begin{array}{lcl} a_{00}x_0 + a_{01}x_1 + \ldots + a_{0n}x_n & = & 0 \\ a_{10}x_0 + a_{11}x_1 + \ldots + a_{1n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{n0}x_0 + a_{n1}x_1 + \ldots + a_{nn}x_n & = & 0 \end{array}\right.$$

Example 1

$$\begin{cases} a_{00}x_0 + a_{01}x_1 + \ldots + a_{0n}x_n &= 0 \\ a_{10}x_0 + a_{11}x_1 + \ldots + a_{1n}x_n &= 0 \\ \vdots & \vdots \quad \vdots \\ a_{n0}x_0 + a_{n1}x_1 + \ldots + a_{nn}x_n &= 0 \end{cases}$$

$$\boxed{p_1(a) = \det(a_{ij})}$$

Example 2

$$\begin{cases} f_1 = a_{10}{x_0}^{d_1} + a_{11}{x_0}^{d_1-1}x_1 + \ldots + a_{1d_1}{x_1}^{d_1} \\ f_2 = a_{20}{x_0}^{d_2} + a_{21}{x_0}^{d_2-1}x_1 + \ldots + a_{2d_2}{x_1}^{d_2} \end{cases}$$

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$$\boxed{\operatorname{Res}(f_1, f_2) = \det \begin{pmatrix} a_{10} & a_{11} & \ldots & a_{1d_1} & 0 & \ldots & 0 \\ 0 & a_{10} & \ldots & a_{1d_1-1} & a_{1d_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{10} & \ldots & \ldots & a_{1d_1} \\ a_{20} & a_{21} & \ldots & a_{2d_2} & 0 & \ldots & 0 \\ 0 & a_{20} & \ldots & a_{2d_2-1} & a_{2d_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{20} & \ldots & \ldots & a_{2d_2} \end{pmatrix}}$$

Example 3

$$\begin{cases} f_1 = \sum_{\alpha_0+\ldots+\alpha_n=d_1} a_{1,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0} \ldots {x_n}^{\alpha_n} \\ f_2 = \sum_{\alpha_0+\ldots+\alpha_n=d_2} a_{2,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0} \ldots {x_n}^{\alpha_n} \\ \vdots \\ f_{n+1} = \sum_{\alpha_0+\ldots+\alpha_n=d_{n+1}} a_{n+1,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0} \ldots {x_n}^{\alpha_n} \end{cases}$$

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$$\mathrm{Res}_{d_1,\ldots,d_n}(f_1, f_2, \ldots, f_{n+1})$$

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$\mathrm{Res}_{d_1,\ldots,d_n}(f_1, f_2, \ldots, f_{n+1})$

Macaulay/dense/classical resultant

No-example

No-example

$$\left\{\begin{array}{rcl} a_{00}{x_0}^4 + a_{01}x_0^2x_1x_2 + a_{02}x_1^2{x_2}^4 & = & 0 \\ a_{10}{x_0}^4 + a_{11}x_0^2x_1x_2 + a_{12}x_1^2{x_2}^4 & = & 0 \\ a_{20}{x_0}^4 + a_{21}x_0^2x_1x_2 + a_{22}x_1^2{x_2}^4 & = & 0 \end{array}\right.$$

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$$\boxed{p_1(\mathbf{a}) \neq \det(a_{ij})}$$

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$$\boxed{p_1(\boldsymbol{a}) \neq \det(a_{ij})}$$

$$p_1(\boldsymbol{a}) = a_{00}^2a_{12}^2 - a_{00}a_{01}a_{11}a_{12} - 2a_{00}a_{02}a_{10}a_{12} + a_{00}a_{02}a_{11}^2 + a_{01}^2a_{10}a_{12} - a_{01}a_{02}a_{10}a_{11} + a_{02}^2a_{10}^2$$

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$$p_2(\mathbf{a}) = a_{00}^2a_{22}^2 - a_{00}a_{01}a_{21}a_{22} - 2a_{00}a_{02}a_{20}a_{22} + a_{00}a_{02}a_{21}^2 + a_{01}^2a_{20}a_{22} - a_{01}a_{02}a_{20}a_{21} + a_{02}^2a_{20}^2$$

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$$\begin{cases} a_{00}x_0{}^4 + a_{01}x_0^2x_1x_2 + a_{02}x_1^2x_2{}^4 & = & 0 \\ a_{10}x_0{}^4 + a_{11}x_0^2x_1x_2 + a_{12}x_1^2x_2{}^4 & = & 0 \\ a_{20}x_0{}^4 + a_{21}x_0^2x_1x_2 + a_{22}x_1^2x_2{}^4 & = & 0 \end{cases}$$

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$$p_1(\mathbf{a}) = a_{00}^2a_{12}^2 - a_{00}a_{01}a_{11}a_{12} - 2a_{00}a_{02}a_{10}a_{12} + a_{00}a_{02}a_{11}^2 + a_{01}^2a_{10}a_{12} - a_{01}a_{02}a_{10}a_{11} + a_{02}^2a_{10}^2$$

$$p_2(\mathbf{a}) = a_{00}^2a_{22}^2 - a_{00}a_{01}a_{21}a_{22} - 2a_{00}a_{02}a_{20}a_{22} + a_{00}a_{02}a_{21}^2 + a_{01}^2a_{20}a_{22} - a_{01}a_{02}a_{20}a_{21} + a_{02}^2a_{20}^2$$

$$p_3(\mathbf{a}) = a_{20}^2a_{12}^2 - a_{20}a_{21}a_{11}a_{12} - 2a_{20}a_{22}a_{10}a_{12} + a_{20}a_{22}a_{11}^2 + a_{21}^2a_{10}a_{12} - a_{21}a_{22}a_{10}a_{11} + a_{22}^2a_{10}^2$$

Cautionary Example

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“Triangulating” a system

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Two ways:

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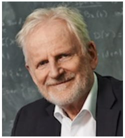
Both use some kind of **division** of polynomials

$$
\begin{array}{rl}
a_1\colon & \quad y \\
a_2\colon & \quad -1 \\
xy+1 & \\
y+1 & \sqrt{xy^2+1} \\
 & \underline{\;xy^2+y\;} \\
 & \quad -y+1 \\
 & \underline{\quad -y-1\;}
\end{array}
$$

Gröbner Bases



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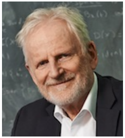

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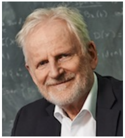

Every ideal in $\mathbb{K}[a, x]$ has a finite set of generators $I = \langle g_1, \ldots, g_N \rangle$

$\{g_1, \ldots, g_N\}$ is a **Gröbner basis iff**

$$g \in I \iff r_{g_1,\ldots g_N}(g) = 0$$

Gröbner Bases and Elimination

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Gröbner Bases and Elimination

In $\mathbb{K}[a, x]$, if $a \prec x$, then $G = GB(I) \implies G \cap \mathbb{K}[a] = GB(I \cap \mathbb{K}[a])$

And $I \cap \mathbb{K}[a]$ is the (affine) ideal of the projection

Triangular sets

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- Characteristic set's method (J.F. Ritt - 1940's)

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Membership to I is easily tested

The “elimination” dictionary

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Polynomial

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Linear	**Polynomial**
Gauss elimination	Gröbner Bases

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Cramer's rule	u-resultants
...	...

History Continues...

The rigorous foundation of the theory of algebraic varieties . . . can be formulated more simply than it has been done so far, without the help of **elimination theory**, on the sole basis of field theory and of the general theory of ideals in ring domains.

van der Waerden, 1926 First Paper on Algebraic Geometry

Fortunately....

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Work continued in the Bourbaki era:

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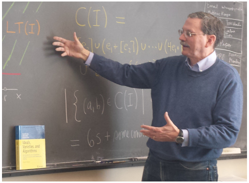
- . . .

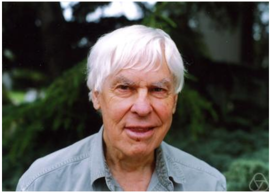
Conclusion...

Elimination Theory is Important for both algorithmic and complexity aspects of polynomial system solving

It has led and still leads the development of Algebraic Geometry and Commutative Algebra in the last 60 years



It also impacts several other areas of Mathematics (Numerical Analysis, Linear Algebra, Complexity Theory,...)



Main tools of (classic) Elimination

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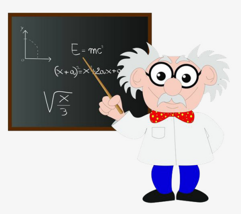
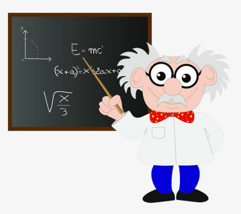
- **Resultants**

$$\operatorname{Res}(f(x),\ g(x)) = \begin{vmatrix} a_m & a_{m-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & a_m & a_{m-1} & \cdots & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_m & a_{m-1} & \cdots & a_1 & a_0 \\ b_n & b_{n-1} & \cdots & b_1 & b_0 & 0 & \cdots & 0 \\ 0 & b_n & b_{n-1} & \cdots & b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & b_n & b_{n-1} & \cdots & b_1 & b_0 \end{vmatrix}.$$

Syllabus



Syllabus



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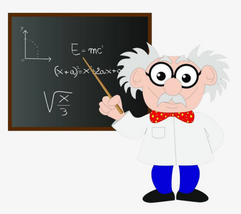
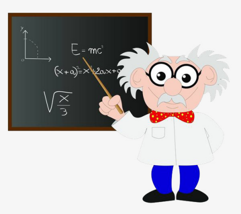
Syllabus



- Univariate Resultant (afternoon)
- Multivariate Resultants (tomorrow)

Syllabus



- Univariate Resultant (afternoon)
- Multivariate Resultants (tomorrow)
- "Other" Resultants (Thursday)