The Rees Algebra of Parametric Curves via liftings

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A3G 2018 – Bristol

UNIVERSITAT DE BARCELONA

ARCADES

Rational Plane Parametrizations

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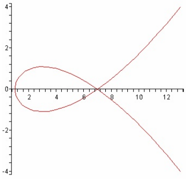
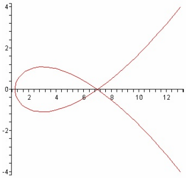


Rational Plane Parametrizations



$$\begin{array}{cccc} \phi : & \mathbb{P}^1 & \to & \mathbb{P}^2 \\ & (t_0 : t_1) & \mapsto & (a(t_0, t_1) : b(t_0, t_1) : c(t_0, t_1)) \end{array}$$

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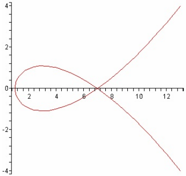

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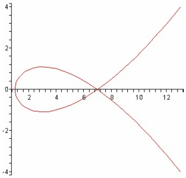

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- $\gcd(a, b, c) = 1$

From affine to projective

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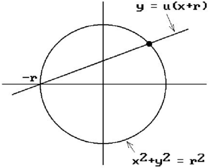
$$\boxed{\begin{array}{ccc} \mathbb{K} & \dashrightarrow & \mathbb{K}^2 \\ t & \longmapsto & \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \end{array}}$$



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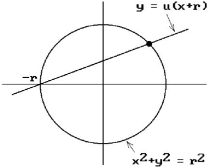

$$\begin{array}{cccc} \phi: & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ & (t_0 : t_1) & \longmapsto & (t_0^2 + t_1^2 : t_0^2 - t_1^2 : 2t_0t_1) \end{array}$$

A "factorization" Theorem

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There exist two parametrizations $\varphi_\mu(t_0 : t_1)$, $\psi_{d-\mu}(t_0 : t_1)$ such that

$$\boxed{\phi(t_0 : t_1) = \varphi_\mu(t_0 : t_1) \wedge \psi_{d-\mu}(t_0 : t_1)}$$

“Factorization” of the unit circle

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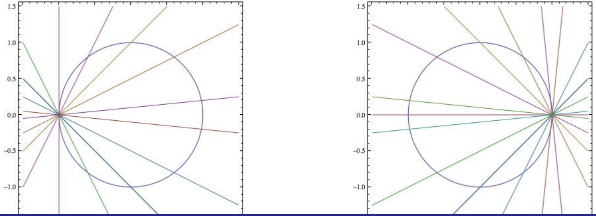
$$\boxed{\begin{array}{rcl}\varphi_1(t_0:t_1) & = & (-t_1:-t_1:t_0)\\ \psi_1(t_0:t_1) & = & (-t_0:t_0:t_1)\end{array}}$$

"Factorization" of the unit circle

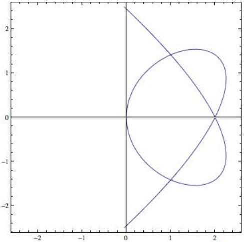
$$\boxed{\begin{array}{rcl} \varphi_1(t_0 : t_1) & = & (-t_1 : -t_1 : t_0) \\ \psi_1(t_0 : t_1) & = & (-t_0 : t_0 : t_1) \end{array}}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -t_1 & -t_1 & t_0 \\ -t_0 & t_0 & t_1 \end{vmatrix} = \left(-t_0^2 - t_1^2 : t_1^2 - t_0^2 : -2t_0t_1 \right)$$



A less trivial factorization

$$\phi(t_0 : t_1) = (t_0^4 : 6t_0^2t_1^2 - 4t_1^4 : 4t_0^3t_1 - 4t_0t_1^3)$$



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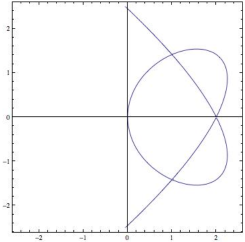
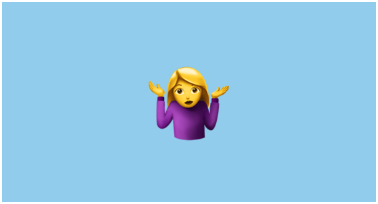


$$\begin{aligned}\varphi_2(t_0, t_1) &= (-2t_0t_1 : t_0t_1 : \tfrac{t_0^2}{2} - t_1^2)\\ \psi_2(t_0, t_1) &= (-2t_1^2 : t_0^2 : -t_0t_1)\end{aligned}$$

Why do we care about this?



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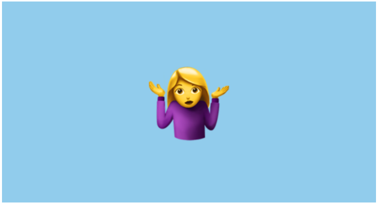

- Geometry

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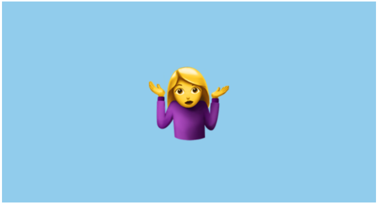

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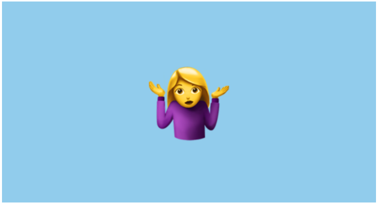

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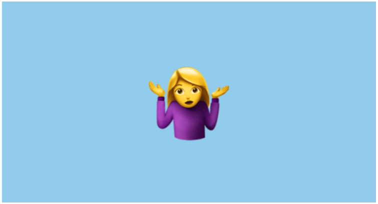

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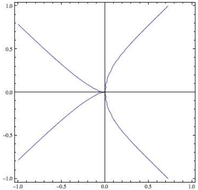
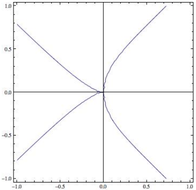
Geometry

Geometry



$$(\mu \leq d - \mu)$$

Geometry



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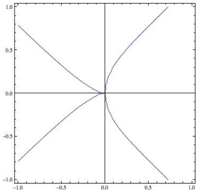

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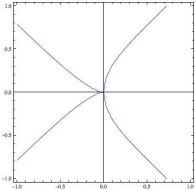

$(\mu \leq d - \mu)$

- $\mu = 1 \iff$ there is one point of multiplicity $d - 1$
- If there is a point of multiplicity $> \mu$,then there is only one and it has multiplicity $d - \mu$

Geometry II

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(Bernardi-Gimigliano-Idà 2015, Madsen 2015)

The curve has a point of multiplicity

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$$\begin{array}{ccc} & \phi(t_0:t_1) & \\ & = & \\ \varphi_\mu(t_0:t_1) & \wedge & \psi_{d-\mu}(t_0:t_1) \\ & = & \\ \left(\varphi'_{\mu_1}(t_0:t_1) \wedge \psi'_{\mu-\mu_1}(t_0:t_1)\right) & \wedge & \psi_{d-\mu}(t_0:t_1) \end{array}$$

Algebra

Algebra

The homogeneous ideal $I = (a, b, c) \subset \mathbb{K}[T_0, T_1]$ has a **Hilbert-Burch resolution** of the type

$$0 \to \mathbb{K}[\underline{T}]^2 \xrightarrow{(\varphi_\mu, \psi_{d-\mu})^{\mathbf{t}}} \mathbb{K}[\underline{T}]^3 \xrightarrow{(a,b,c)} \mathbb{K}[\underline{T}]$$

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A μ-**basis** of the parametrization is a basis of $\mathrm{Syz}(I)$ as a $\mathbb{K}[\underline{T}]$-module

Complexity: Implicitization

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$$X_2a - X_0c = X_2T_0^2 - 2X_0T_0T_1 + X_2T_1^2$$

$$X_2b - X_1c = X_2T_0^2 - 2X_1T_0T_1 - X_2T_1^2$$

Complexity: Implicitization

$$X_2 a - X_0 c \;=\; X_2 T_0^2 - 2X_0 T_0 T_1 + X_2 T_1^2$$

$$X_2 b - X_1 c \;=\; X_2 T_0^2 - 2X_1 T_0 T_1 - X_2 T_1^2$$

$$\boxed{\begin{array}{c} \mathrm{Res}_{\underline{T}}\left(X_2 \cdot a - X_0 \cdot c, X_2 \cdot b - X_1 \cdot c\right) \\ = \\ \det\begin{pmatrix} X_2 & -2X_0 & X_2 & 0 \\ 0 & X_2 & -2X_0 & X_2 \\ X_2 & -2X_1 & -X_2 & 0 \\ 0 & X_2 & -2X_1 & -X_2 \end{pmatrix} = 4X_2^2(X_1^2 + X_2^2 - X_0^2 \end{array}}$$

With μ-bases...

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Implicit equation

$$=$$

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$$\begin{aligned}
\varphi_1(\underline{T},\underline{X}) &= (-T_1 : -T_1 : T_0)\\
&= -T_1X_0 - T_1X_1 + T_0X_2\\
&= X_2T_0 - (X_0 + X_1)T_1\\
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\end{aligned}$$

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\end{aligned}$$

$$\boxed{\det\begin{pmatrix} X_2 & -X_0-X_1\\ -X_0+X_1 & X_2\end{pmatrix} = X_1^2+X_2^2-X_0^2}$$

Syzygies are moving lines

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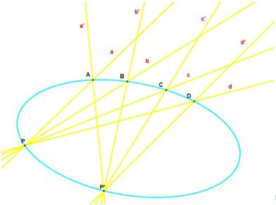
$$\mathcal{L}(T_0, T_1, X_0, X_1, X_2) = v_0(\underline{T})X_0 + v_1(\underline{T})X_1 + v_2(\underline{T})X_2$$

such that

$$\mathcal{L}(T_0, T_1, a(\underline{T}), b(\underline{T}), c(\underline{T})) = 0$$



Moving lines, conics, cubics,...

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$\mathrm{Syz}(a, b, c)$ moving lines

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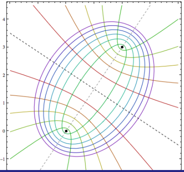
$\mathrm{Syz}(a^3, b^3, c^3, a^2b, \ldots)$	moving cubics
$\vdots$	$\vdots$



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The implicit equation of the curve can be computed as the determinant of a **small** matrix with entries

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The more **singular** the curve is, the **simpler** the description of the determinant should be

Example(Sederberg& Chen 1995)

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The implicit equation of a rational quartic can be computed as a 2×2 determinant.
If the curve has a triple point, then one row is linear and the other is cubic.
Otherwise, both rows are quadratic.

A quartic with a triple point

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$$\phi(t_0 : t_1) = (t_0^4 - t_1^4 : -t_0^2 t_1^2 : t_0 t_1^3)$$
$$F(X_0, X_1, X_2) = X_2^4 - X_1^4 - X_0 X_1 X_2^2$$

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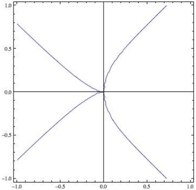

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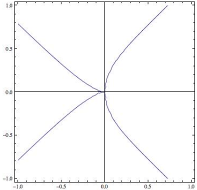

$$\begin{array}{rcl} \mathcal{L}_{1,1}(\underline{T}, \underline{X}) & = & T_0 X_2 + T_1 X_1 \\ \mathcal{L}_{1,3}(\underline{T}, \underline{X}) & = & T_0(X_1^3 + X_0 X_2^2) + T_1 X_2^3 \end{array}$$

$$\begin{pmatrix} X_2 & X_1 \\ X_1^3 + X_0 X_2^2 & X_2^3 \end{pmatrix}$$

A quartic without triple points

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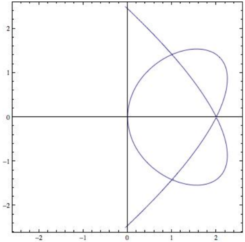

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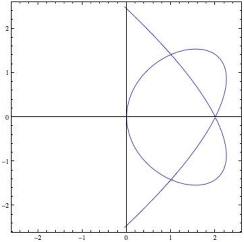

$$\begin{aligned}\mathcal{L}_{1,2}(\underline{T}, \underline{X}) &= T_0(X_1X_2 - X_0X_2) + T_1(-X_2^2 - 2X_0X_1 + 4X_0^2)\\ \tilde{\mathcal{L}}_{1,2}(\underline{T}, \underline{X}) &= T_0(X_1^2 + \tfrac{1}{2}X_2^2 - 2X_0X_1) + T_1(X_0X_2 - X_1X_2)\end{aligned}$$

Very concentrated singularities

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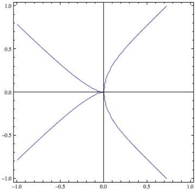

If the curve has a point of multiplicity $d - 1$

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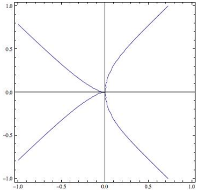

If the curve has a point of multiplicity $d-1$
the implicit equation is always a 2×2 determinant

$$\begin{vmatrix} \mathcal{L}_{1,1}(\underline{X}) & \mathcal{L}'_{1,1}(\underline{X}) \\ \mathcal{L}_{1,d-1}(\underline{X}) & \mathcal{L}'_{1,d-1}(\underline{X}) \end{vmatrix}$$

In general, we do not know..

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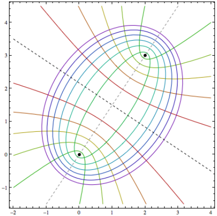
which moving lines?
which moving conics?
which moving cubics?
...



The Rees Algebra associated to the parametrization

Cox, D. **The moving curve ideal and the Rees algebra.** Theoret. Comput. Sci. 392 (2008), no. 1–3, 23–36

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$$\begin{array}{ccc} \mathbb{K}[T_0, T_1, X_0, X_1, X_2] & \rightarrow & \mathbb{K}[T_0, T_1, s] \\ T_i & \mapsto & T_i \\ X_0 & \mapsto & a(T_0, T_1)s \\ X_1 & \mapsto & b(T_0, T_1)s \\ X_2 & \mapsto & c(T_0, T_1)s \end{array}$$

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"The ideal of moving curves following ϕ"

Method of moving curves revisited

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Method of moving curves revisited

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The more singular the curve, the simpler the description of $\mathcal{K}_\phi$

New Problem

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Compute a minimal system of generators of $\mathcal{K}_\phi$

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Compute a minimal system of generators of $\mathcal{K}_\phi$ for **any** ϕ

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Show that if ϕ is "more singular" than ϕ' then $n_0(\mathcal{K}_\phi) \leq n_0(\mathcal{K}_{\phi'})$

Example: $\mu = 2$

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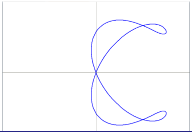

Recent Progress

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Equations of Rees algebras of ideals in two variables

arXiv:1511.04073

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Describes the bi-degrees of all minimal generators in $\underline{T}$-degree larger than or equal to μ

Our contribution

(w Teresa Cortadellas & David Cox)

- Make explicit these generators and their degrees

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- Make explicit these generators and their degrees
- Do it via "lifting" the plane parametrization

The μ-type of ϕ

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Recall that

$$\phi(t_0 : t_1) = \left(\varphi'_{\mu_1}(t_0 : t_1) \wedge \psi'_{\mu-\mu_1}(t_0 : t_1)\right) \wedge \psi_{d-\mu}(t_0 : t_1)$$

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$$=$$
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This implies

$$\phi(t_0 : t_1) = p(\underline{t}) \cdot \varphi'_{\mu_1}(t_0 : t_1) + q(\underline{t}) \cdot \varphi'_{\mu-\mu_1}(t_0 : t_1)$$

The μ-type of ϕ

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From here we can factorize

$$\mathbb{P}^1 \longrightarrow \mathcal{S}_{\mu_1,\mu_2} \hookrightarrow \mathbb{P}^{\mu+1} \dashrightarrow \mathbb{P}^2$$

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$$\mathbb{P}^1 \longrightarrow \mathcal{S}_{\mu_1,\mu_2} \hookrightarrow \mathbb{P}^{\mu+1} \dashrightarrow \mathbb{P}^2$$

with $\mathcal{S}_{\mu_1,\mu_2}$ a rational normal surface (Bernardi, Gimigliano, Idà 2017)

So we have this diagram

$$
\begin{tikzcd}
\mathbb{P}^1 \arrow[drr, "1"] \arrow[dr, "2"] \arrow[ddr, "\phi"'] & & \\
& \mathcal{S}_{\mu_1,\mu_2} \arrow[r, hook] \arrow[d] & \mathbb{P}^{\mu+1} \arrow[dl, dashed] \\
& \mathbb{P}^2 &
\end{tikzcd}
$$

Our contribution

(Cortadellas-Cox-**D** `arXiv:1810.02077`)

"Lift" this to a map of Rees Algebras

$$\begin{tikzcd}
 & \mathbb{K}[\underline{T}, \underline{X}, \underline{Y}] \arrow[d, "\Phi'"] \arrow[ddr, "\Phi"] & \\
\mathbb{K}[\underline{T}, \underline{Z}] \arrow[ur, "\Gamma"] \arrow[r, "\Omega"] \arrow[drr, "\eta"'] & \mathbb{K}[\underline{T}, X, Y] \arrow[dr, "\chi"] & \\
 & & \mathbb{K}[\underline{T}, s]
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$$

where Φ comes from arrow 1, χ from arrow 2, and η from ϕ

Consequences

(Cortadellas-Cox-**D** `arXiv:1810.02077`)

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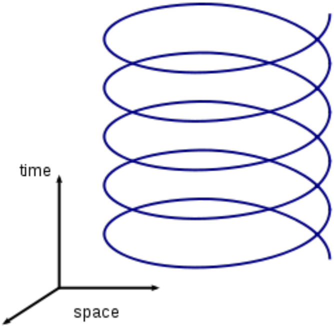
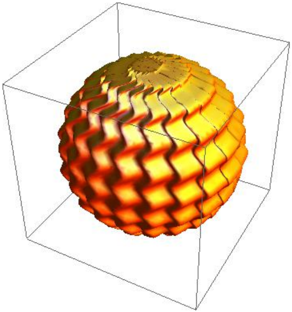
- Construction of explicit generators of the Rees ideal of the lifted curve
- Construction of some explicit generators of the Rees ideal of ϕ

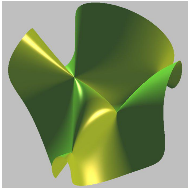
Only curves in the plane?





Rational Surfaces

$$\begin{array}{rccc} \phi_S : & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^3 \\ & \underline{t} = (t_0 : t_1 : t_2) & \longmapsto & \big(a(\underline{t}) : b(\underline{t}) : c(\underline{t}) : d(\underline{t})\big) \end{array}$$



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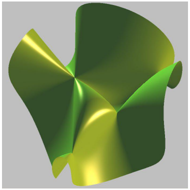

There are base points!

Implicitization via

- Resultants Macaulay, Dixon, Gelfand-Kapranov-Zelevinskii, ...

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Moving planes, moving quadrics,...

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(Sederberg-Chen, Cox-Goldman-Zhang, Busé-Cox, **D**, **D**-Khetan)

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Contrast:

- The module of moving planes is not free

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(Sederberg-Chen, Cox-Goldman-Zhang, Busé-Cox, **D**, **D**-Khetan)

Contrast:

- The module of moving planes is not free
- There is a concept of μ-basis given by Chen-Cox-Liu
  Not easy to compute (recent bounds on the degree by Cid Ruiz)

Some results

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Implicitization

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- Quadratic and cubic surfaces (Chen-Shen-Deng)

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Rees Algebras

Some results

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Rees Algebras

- "Monoid" Surfaces (Cortadellas - **D**)

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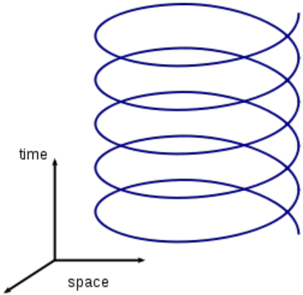
Rees Algebras

- "Monoid" Surfaces (Cortadellas - **D**)
- *de Jonquières* surfaces (Hassanzadeh- Simis)

Similar Results for

Spatial curves

$$\begin{array}{rccc} \phi_C: & \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^3 \\ & \underline{t} = (t_0 : t_1) & \longmapsto & \big(a(\underline{t}) : b(\underline{t}) : c(\underline{t}) : d(\underline{t})\big) \end{array}$$



Thanks!