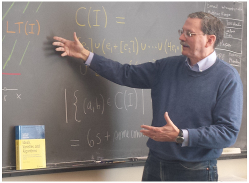
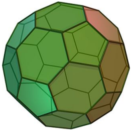
Moving Lines, Soccer, and Rees Algebras

Carlos D'Andrea

Ideals, Varieties, Applications – Amherst

UNIVERSITAT DE BARCELONA





Back in 2012...

David: Now that I am in Barcelona...

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I want to visit the 2 churches

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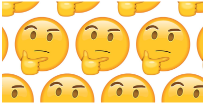
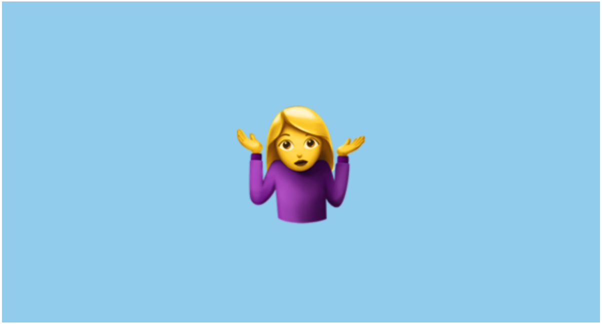
I want to visit the 2 churches



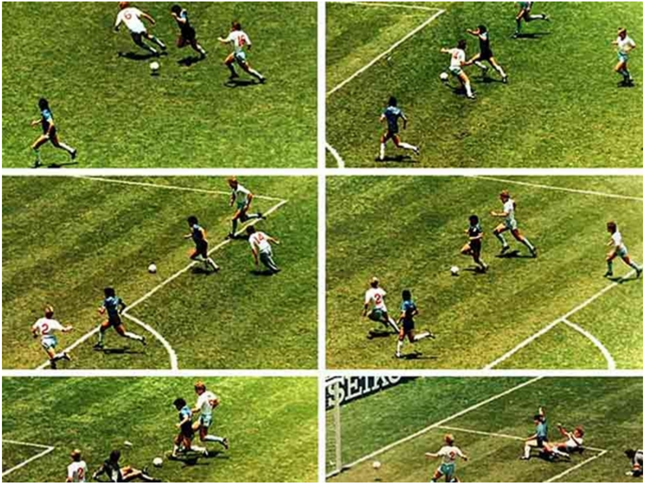
Church # 1: Sagrada Família

Church #2?



Church #2: Camp Nou!!

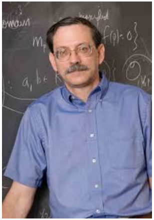
“The” score of the (20th) Century



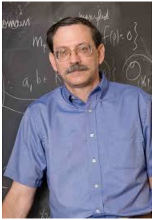
Let us enjoy it

Diego's goal

David's score of Last (and this) Century



David's score of Last (and this) Century

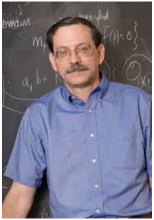


Geometric Modeling

vs

Algebraic Geometry

David's goal of Last (and this) Century

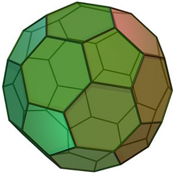


Geometric Modeling

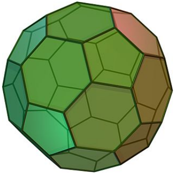
and

Algebraic Geometry

Rolling the ball...

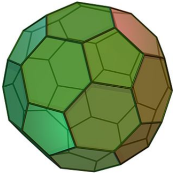


Rolling the ball...



(Sederberg & Chen 1995)

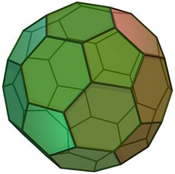
Rolling the ball...



(Sederberg & Chen 1995)

The implicit equation of a rational quartic can be computed as a 2×2 determinant.

Rolling the ball...

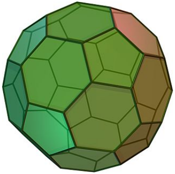


(Sederberg & Chen 1995)

The implicit equation of a rational quartic can be computed as a 2×2 determinant.

If the curve has a triple point, then one row is linear and the other is cubic.

Rolling the ball...



(Sederberg & Chen 1995)

The implicit equation of a rational quartic can be computed as a 2×2 determinant.

If the curve has a triple point, then one row is linear and the other is cubic.

Otherwise, both rows are quadratic.

A quartic with a triple point

A quartic with a triple point

$$\phi(t_0, t_1) = ({t_0}^4 - {t_1}^4 : -{t_0}^2{t_1}^2 : t_0{t_1}^3)$$
$$F(X_0, X_1, X_2) = X_2^4 - X_1^4 - X_0X_1X_2^2$$

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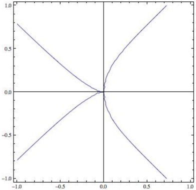

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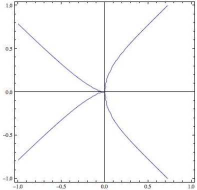

$$\begin{array}{rcl} \mathcal{L}_{1,1}(\underline{T}, \underline{X}) & = & T_0X_2 + T_1X_1 \\ \mathcal{L}_{1,3}(\underline{T}, \underline{X}) & = & T_0(X_1^3 + X_0X_2^2) + T_1\, X_2^3 \end{array}$$
$$\begin{pmatrix} X_2 & X_1 \\ X_1^3 + X_0X_2^2 & X_2^3 \end{pmatrix}$$

A quartic without triple points

$$\phi(t_0 : t_1) = ({t_0}^4 : 6{t_0}^2{t_1}^2 - 4{t_1}^4 : 4{t_0}^3 t_1 - 4 t_0 {t_1}^3)$$
$$F(\underline{X}) = X_2^4 + 4X_0X_1^3 + 2X_0X_1X_2^2 - 16X_0^2X_1^2 - 6X_0^2X_2^2 + 16X_0^3X_1$$

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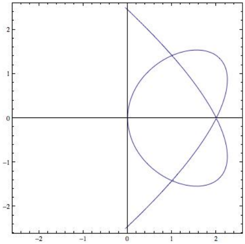

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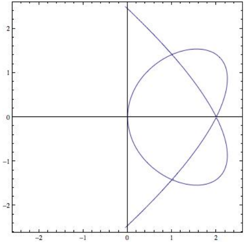
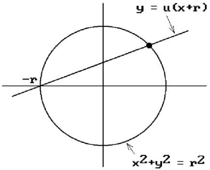


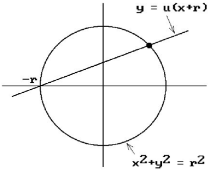
$$\mathcal{L}_{1,2}(\underline{T},\underline{X}) = T_0(X_1X_2 - X_0X_2) + T_1(-X_2^2 - 2X_0X_1 + 4X_0^2)$$
$$\tilde{\mathcal{L}}_{1,2}(\underline{T},\underline{X}) = T_0(X_1^2 + \tfrac{1}{2}X_2^2 - 2X_0X_1) + T_1(X_0X_2 - X_1X_2)$$

Parametrizing the circle

Parametrizing the circle



Parametrizing the circle



$$\phi : \quad \mathbb{P}^1 \longrightarrow \mathbb{P}^2$$

$$(t_0 : t_1) \longmapsto \left(t_0{}^2 + t_1{}^2 : t_0{}^2 - t_1{}^2 : 2t_0t_1\right)$$

$$(a(\mathbf{t}) : b(\mathbf{t}) : c(\mathbf{t}))$$

Implicitization

Implicitization

$$X_2 a(\underline{T}) - X_0 c(\underline{T}) \ = \ X_2 T_0^2 - 2X_0 T_0 T_1 + X_2 T_1^2$$

$$X_2 b(\underline{T}) - X_1 c(\underline{T}) \ = \ X_2 T_0^2 - 2X_1 T_0 T_1 - X_2 T_1^2$$

Implicitization

$$X_2 a(\underline{T}) - X_0 c(\underline{T}) \;=\; X_2 T_0^2 - 2X_0 T_0 T_1 + X_2 T_1^2$$

$$X_2 b(\underline{T}) - X_1 c(\underline{T}) \;=\; X_2 T_0^2 - 2X_1 T_0 T_1 - X_2 T_1^2$$

$$\mathrm{Res}_{\underline{T}}\left(X_2 \cdot a(\underline{T}) - X_0 \cdot c(\underline{T}), X_2 \cdot b(\underline{T}) - X_1 \cdot c(\underline{T})\right)$$

$$=$$

$$\det\begin{pmatrix} X_2 & -2X_0 & X_2 & 0 \\ 0 & X_2 & -2X_0 & X_2 \\ X_2 & -2X_1 & -X_2 & 0 \\ 0 & X_2 & -2X_1 & -X_2 \end{pmatrix} = 4X_2^2(X_1^2 + X_2^2 - X_0^2)$$

You can do it better!

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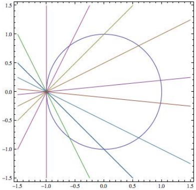

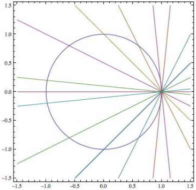

$$
\begin{aligned}
\varphi_1(\underline{T},\underline{X}) &= (-T_1 : -T_1 : T_0) \\
&= X_2 T_0 - (X_0 + X_1) T_1 \\
\psi_1(\underline{T},\underline{X}) &= (-T_0 : T_0 : T_1)
\end{aligned}
$$

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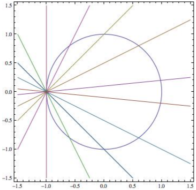

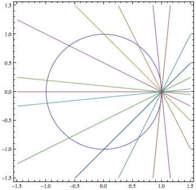

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\varphi_1(\underline{T},\underline{X}) &= (-T_1 : -T_1 : T_0) \\
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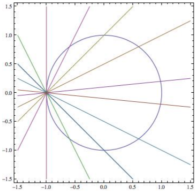
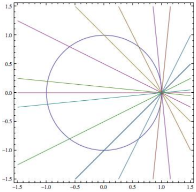
\psi_1(\underline{T},\underline{X}) &= (-T_0 : T_0 : T_1) \\
& \quad (-X_0 + X_1) T_0 + X_2 T_1
\end{aligned}
$$

You can do it better!





$$
\begin{aligned}
\varphi_1(\underline{T},\underline{X}) &= (-T_1 : -T_1 : T_0) \\
&= X_2T_0 - (X_0 + X_1)T_1 \\
\psi_1(\underline{T},\underline{X}) &= (-T_0 : T_0 : T_1) \\
& \quad (-X_0 + X_1)T_0 + X_2T_1
\end{aligned}
$$

$$
\det\begin{pmatrix} X_2 & -X_0 - X_1 \\ -X_0 + X_1 & X_2 \end{pmatrix} = X_1^2 + X_2^2 - X_0^2
$$

Moving Lines!

Moving Lines!

A moving line

$$\mathcal{L}(T_0, T_1, X_0, X_1, X_2) = v_0(\underline{T})X_0 + v_1(\underline{T})X_1 + v_2(\underline{T})X_2$$

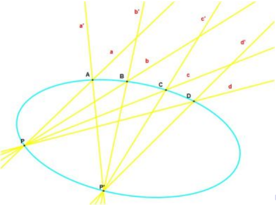
Moving Lines!

A moving line

$$\mathcal{L}(T_0, T_1, X_0, X_1, X_2) = v_0(\underline{T})X_0 + v_1(\underline{T})X_1 + v_2(\underline{T})X_2$$

follows the parametrization iff

$$\mathcal{L}(T_0, T_1, a(\underline{T}), b(\underline{T}), c(\underline{T})) = 0$$



For the unit circle...

$$\begin{aligned}
\mathcal{L}_1(\underline{T},\underline{X}) &= -2T_0^2T_1X_0 + 0X_1 + (T_0^3 + T_0T_1^2)X_2\\
\mathcal{L}_2(\underline{T},\underline{X}) &= -2T_0T_1^2X_0 + 0X_1 + (T_0^2T_1 + T_1^3)X_2\\
\mathcal{L}_3(\underline{T},\underline{X}) &= 0X_0 - 2T_0^2T_1X_1 + (T_0^3 - T_0T_1^2)X_2\\
\mathcal{L}_4(T,\underline{X}) &= 0X_0 - 2T_0T_1^2X_1 + (T_0^2T_1 - T_1^3)X_2
\end{aligned}$$

For the unit circle...

$$
\begin{array}{rcl}
\mathcal{L}_1(\underline{T}, \underline{X}) & = & -2T_0^2T_1X_0 + 0X_1 + (T_0^3 + T_0T_1^2)X_2 \\
\mathcal{L}_2(\underline{T}, \underline{X}) & = & -2T_0T_1^2X_0 + 0X_1 + (T_0^2T_1 + T_1^3)X_2 \\
\mathcal{L}_3(\underline{T}, \underline{X}) & = & 0X_0 - 2T_0^2T_1X_1 + (T_0^3 - T_0T_1^2)X_2 \\
\mathcal{L}_4(T, \underline{X}) & = & 0X_0 - 2T_0T_1^2X_1 + (T_0^2T_1 - T_1^3)X_2
\end{array}
$$

$$
\begin{pmatrix}
X_2 & -2X_0 & X_2 & 0 \\
0 & X_2 & -2X_0 & X_2 \\
X_2 & -2X_1 & -X_2 & 0 \\
0 & X_2 & -2X_1 & -X_2
\end{pmatrix}
$$

In general

The determinant of a "matrix of moving lines" is always a multiple of the implicit equation

$$\begin{pmatrix} L_{11}(\underline{X}) & L_{12}(\underline{X}) & \dots & L_{1k}(\underline{X}) \\ L_{21}(\underline{X}) & L_{22}(\underline{X}) & \dots & L_{2k}(\underline{X}) \\ \vdots & \vdots & \vdots & \vdots \\ L_{k1}(\underline{X}) & L_{k2}(\underline{X}) & \dots & L_{kk}(\underline{X}) \end{pmatrix}$$

I dare you...

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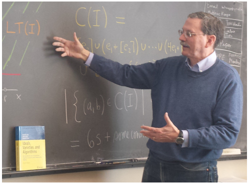
- find a non-singular square matrix of moving lines

I dare you...

- find a non-singular square matrix of moving lines
- find one where the determinant is exactly the implicit equation

David gets the ball...



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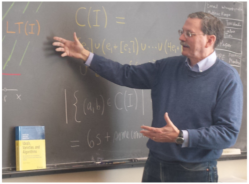

Cox, Sederberg, Chen

The moving line ideal basis of planar rational curves

CAGD 98

Moving Lines are Syzygies

Moving Lines are Syzygies

The homogeneous ideal
$I = \big(a(\underline{T}), b(\underline{T}), c(\underline{T})\big) \subset \mathbb{K}[T_0, T_1]$ has a
Hilbert-Burch resolution of the type

$$0 \to \mathbb{K}[\underline{T}]^2 \xrightarrow{(\varphi_\mu, \psi_{d-\mu})^{\mathbf{t}}} \mathbb{K}[\underline{T}]^3 \xrightarrow{(a,b,c)} \mathbb{K}[\underline{T}]$$

Moving Lines are Syzygies

The homogeneous ideal
$I = \big(a(\underline{T}),\, b(\underline{T}),\, c(\underline{T})\big) \subset \mathbb{K}[T_0, T_1]$ has a
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$$0 \to \mathbb{K}[\underline{T}]^2 \xrightarrow{(\varphi_\mu, \psi_{d-\mu})^{\mathbf{t}}} \mathbb{K}[\underline{T}]^3 \xrightarrow{(a,b,c)} \mathbb{K}[\underline{T}]$$

The moving lines are a free $\mathbb{K}[\underline{T}]$-module of rank 2
The determinant of $\mathrm{Sylv}(\varphi_\mu, \psi_{d-\mu})$ gives exactly the implicit equation

Geometrically...

Geometrically...

There are two parametrizations $\varphi_\mu(t_0 : t_1)$, $\psi_{d-\mu}(t_0 : t_1)$ such that

$$\boxed{\phi(t_0 : t_1) = \varphi_\mu(t_0 : t_1) \wedge \psi_{d-\mu}(t_0 : t_1)}$$

"Factorization" of the unit circle

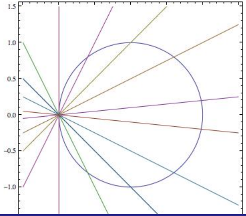
“Factorization” of the unit circle

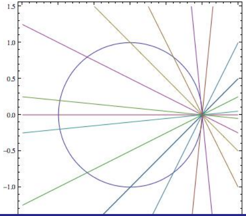
$$\begin{aligned} \varphi_1(t_0 : t_1) &= (-t_1 : -t_1 : t_0) \\ \psi_1(t_0 : t_1) &= (-t_0 : t_0 : t_1) \end{aligned}$$

"Factorization" of the unit circle

$$\begin{aligned} \varphi_1(t_0 : t_1) &= (-t_1 : -t_1 : t_0) \\ \psi_1(t_0 : t_1) &= (-t_0 : t_0 : t_1) \end{aligned}$$

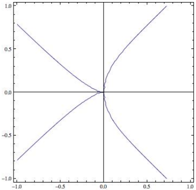
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -t_1 & -t_1 & t_0 \\ -t_0 & t_0 & t_1 \end{vmatrix} = \left(-\, {t_0}^2 - {t_1}^2 : {t_1}^2 - {t_0}^2 : -2 t_0 t_1 \right)$$





Geometry II

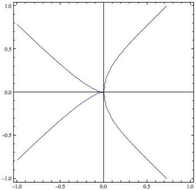
Geometry II



$$(\mu \leq d - \mu)$$

- $\mu = 1 \iff$ there is one point of multiplicity $d - 1$

Geometry II



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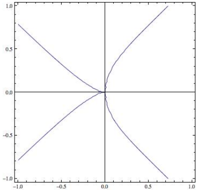
- $\mu = 1 \iff$ there is one point of multiplicity $d - 1$
- If there is a point of multiplicity $> \mu$,

Geometry II



$(\mu \leq d - \mu)$

- $\mu = 1 \iff$ there is one point of multiplicity $d - 1$
- If there is a point of multiplicity $> \mu$,then there is only one and it has multiplicity $d - \mu$

Geometry III

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(Bernardi-Gimigliano-Idà 2015, Madsen 2015)

The curve has a point of multiplicity

$$d - \mu \iff \mu' = 0$$

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$$\phi(t_0:t_1) = \varphi_\mu(t_0:t_1) \wedge \psi_{d-\mu}(t_0:t_1)$$

Geometry III

(Bernardi-Gimigliano-Idà 2015, Madsen 2015)

The curve has a point of multiplicity

$$d-\mu \iff \mu' = 0$$

$$\begin{array}{ccc} & \phi(t_0:t_1) & \\ & = & \\ \varphi_\mu(t_0:t_1) & \wedge & \psi_{d-\mu}(t_0:t_1) \\ & = & \\ \left(\varphi'_{\mu'}(t_0:t_1)\wedge\varphi'_{\mu-\mu'}(t_0:t_1)\right) & \wedge & \psi_{d-\mu}(t_0:t_1)\end{array}$$

Conics instead of lines?

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$$\mathcal{O}(\underline{T})X_0^2 + \mathcal{P}(\underline{T})X_0X_1 + \mathcal{Q}(\underline{T})X_0X_2 + \mathcal{R}(\underline{T})X_1^2 + \mathcal{S}(\underline{T})X_1X_2 + \mathcal{T}(\underline{T})X_2^2 \in \mathbb{K}[\underline{T}, \underline{X}]$$

is a **moving conic** following the parametrization if

Conics instead of lines?

$$\mathcal{O}(\underline{T})X_0^2 + \mathcal{P}(\underline{T})X_0X_1 + \mathcal{Q}(\underline{T})X_0X_2 + \mathcal{R}(\underline{T})X_1^2 + \mathcal{S}(\underline{T})X_1X_2 + \mathcal{T}(\underline{T})X_2^2 \in \mathbb{K}[\underline{T}, \underline{X}]$$

is a **moving conic** following the parametrization if

$$\mathcal{O}(\underline{T})a(\underline{T})^2 + \mathcal{P}(\underline{T})a(\underline{T})b(\underline{T}) + \mathcal{Q}(\underline{T})a(\underline{T})c(\underline{T}) + \mathcal{R}(\underline{T})b(\underline{T})^2 + \mathcal{S}(\underline{T})b(\underline{T})c(\underline{T}) + \mathcal{T}(\underline{T})c(\underline{T})^2 = 0$$

Moving lines, conics, cubics,...

Moving lines, conics, cubics,...

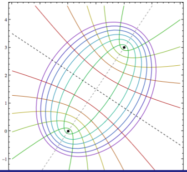
$\mathrm{Syz}(a, b, c)$ moving lines

Moving lines, conics, cubics,...

$\mathrm{Syz}(a, b, c)$	moving lines
$\mathrm{Syz}(a^2, b^2, c^2, ab, ac, bc)$	moving conics

Moving lines, conics, cubics,...

$\mathrm{Syz}(a, b, c)$	moving lines
$\mathrm{Syz}(a^2, b^2, c^2, ab, ac, bc)$	moving conics
$\mathrm{Syz}(a^3, b^3, c^3, a^2b, \ldots)$	moving cubics
$\vdots$	$\vdots$



The method of moving curves

The method of moving curves

The implicit equation can be computed as the determinant of a **small** matrix with entries

The method of moving curves

The implicit equation can be computed as the determinant of a **small** matrix with entries

$$\left| \begin{array}{c} \text{some moving lines} \\ \text{some moving conics} \\ \text{some moving cubics} \\ \ldots \end{array} \right|$$

The method of moving curves

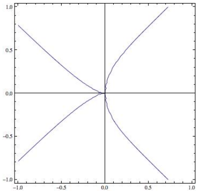
The implicit equation can be computed as the determinant of a **small** matrix with entries

$$\begin{vmatrix} \text{some moving lines} \\ \text{some moving conics} \\ \text{some moving cubics} \\ \ldots \end{vmatrix}$$

The more **singular** the curve, the **simpler** the description of the determinant

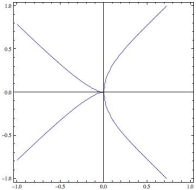
Extreme case

Extreme case



If the curve has a point of multiplicity $d - 1$

Extreme case



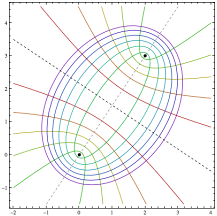
If the curve has a point of multiplicity $d-1$
the implicit equation is always a 2×2 determinant

$$\left|\begin{array}{cc} \mathcal{L}_{1,1}(\underline{X}) & \mathcal{L}'_{1,1}(\underline{X}) \\ \mathcal{L}_{1,d-1}(\underline{X}) & \mathcal{L}'_{1,d-1}(\underline{X}) \end{array}\right|$$

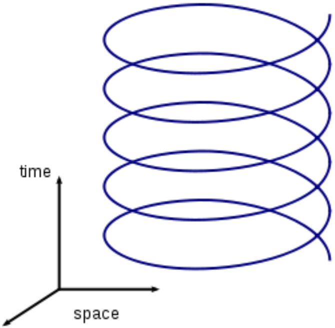
In general, we do not know..

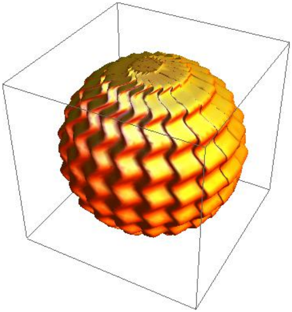
In general, we do not know..

which moving lines?
which moving conics?
which moving cubics?
. . .



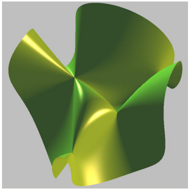
Spatial Curves/Surfaces





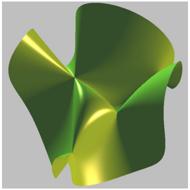
Rational Surfaces

$$\begin{array}{rccc} \phi_S : & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^3 \\ & \underline{t} = (t_0 : t_1 : t_2) & \longmapsto & \big(a(\underline{t}) : b(\underline{t}) : c(\underline{t}) : d(\underline{t})\big) \end{array}$$



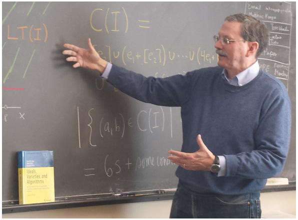
Rational Surfaces

$$\phi_S : \quad \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$$
$$\underline{t} = (t_0 : t_1 : t_2) \longmapsto \big(a(\underline{t}) : b(\underline{t}) : c(\underline{t}) : d(\underline{t})\big)$$

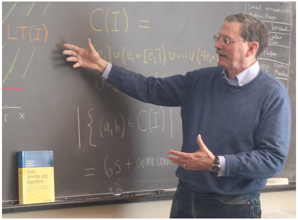


There are base points!

And David goes...



And David goes...



Cox, Goldman, Zhang

On the validity of implicitization by moving quadrics of rational surfaces with no base points

JSC 2000

Main result

The "method of moving quadrics" works without base points, for proper parametrizations with the "right number" of moving planes of degree $d-1$

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The "method of moving quadrics" works without base points, for proper parametrizations with the "right number" of moving planes of degree $d-1$

Proof uses Commutative Algebra (Cohen-Macaulay rings) and Algebraic Geometry (sheaf cohomology)

Got the community's attention

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- Laurent Busé

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- Laurent Busé
- Marc Chardin

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- Marc Chardin
- D

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- D
- William Hoffman

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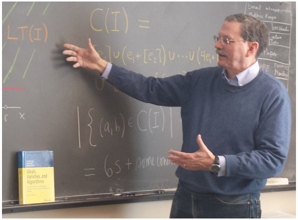
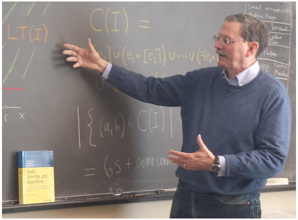
And David goes...



And David goes...



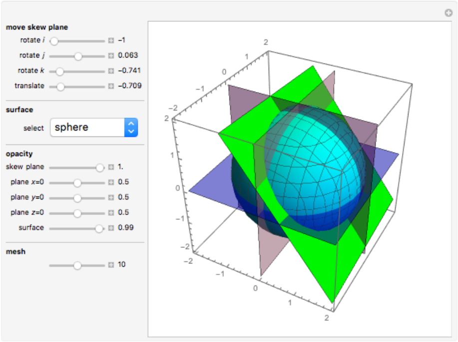
Chen, Cox, Liu

The μ-basis and implicitization of a rational parametric surface

JSC 2005

Main result

Every parametrization of a rational surface has a μ-basis.



- Nicolás Botbol

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- Yairon Cid Ruiz

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- Alicia Dickenstein

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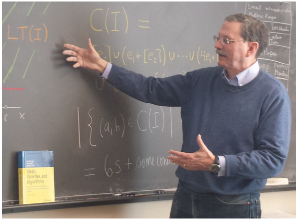

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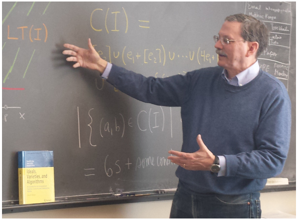

Cox, Kustin, Polini, Ulrich

A study of singularities on rational curves via syzygies

MAMS 2013

Main results

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- Study of the singularities of a rational curve via the Hilbert-Burch matrix

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- Classification of curves with points of "highest" multiplicity

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- ...

- Alessandra Bernardi

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- Eduardo Casas-Alvero

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- Louiza Fouli

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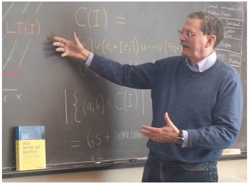
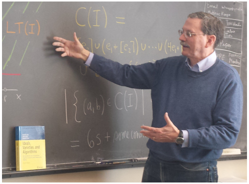
And David scores



And David scores



Cox

The moving curve ideal and the Rees algebra

TCS 2008

What did we learn there?

$\mathcal{K}_\phi :=$ {Moving curves following ϕ} $=$ homogeneous elements in the kernel of

$$
\begin{array}{ccc}
\mathbb{K}[T_0, T_1, X_0, X_1, X_2] & \to & \mathbb{K}[T_0, T_1, s] \\
T_i & \mapsto & T_i \\
X_0 & \mapsto & a(T_0, T_1)s \\
X_1 & \mapsto & b(T_0, T_1)s \\
X_2 & \mapsto & c(T_0, T_1)s
\end{array}
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$$

"The ideal of moving curves following ϕ"

Method of moving curves revisited

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The implicit equation should be obtained as the determinant of a matrix with

Method of moving curves revisited

The implicit equation should be obtained as the determinant of a matrix with

$$\left| \begin{array}{c} \cdots \\ \text{some minimal generators of } \mathcal{K}_\phi \\ \text{and relations among them} \\ \cdots \end{array} \right|$$

Method of moving curves revisited

The implicit equation should be obtained as the determinant of a matrix with

$$\left|\begin{array}{c}\cdots \\ \text{some minimal generators of } \mathcal{K}_\phi \\ \text{and relations among them} \\ \cdots\end{array}\right|$$

The more singular the curve, the simpler the description of $\mathcal{K}_\phi$

Goal!

New Problem(s)

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Compute a minimal system of generators of $\mathcal{K}_\phi$

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Compute a minimal system of generators of $\mathcal{K}_\phi$ for **any** ϕ

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Known for

- $\mu = 1$ (Hong-Simis-Vasconcelos, Cox-Hoffmann-Wang, Busé, Cortadellas-**D**)

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- $\left(\mathcal{K}_\phi\right)_{(1,2)} \neq 0$ (Cortadellas- **D**)

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- $\mu = 2$ (Busé, Cortadellas-**D**, Kustin-Polini-Ulrich)
- $\left(\mathcal{K}_\phi\right)_{(1,2)} \neq 0$ (Cortadellas- **D**)
- Monomial Parametrizations (Cortadellas-**D**)

A coarser problem

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Compute $n_0(\mathcal{K}_\phi)$, the number of minimal generators of $\mathcal{K}_\phi$

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Compute $n_0(\mathcal{K}_\phi)$, the number of minimal generators of $\mathcal{K}_\phi$

Show that if ϕ is "more singular" than ϕ' then $n_0(\mathcal{K}_\phi) \leq n_0(\mathcal{K}_{\phi'})$

Example: $\mu = 2$

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The curve has either

- one point of multiplicity $d - 2$
  $n_0 = \mathcal{O}\left(\frac{d}{2}\right)$
  (Cortadellas-**D**, Kustin-Polini-Ulrich)

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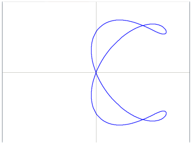
- one point of multiplicity $d-2$
  $n_0 = \mathcal{O}\left(\frac{d}{2}\right)$
  (Cortadellas-**D**, Kustin-Polini-Ulrich)
- or only double points
  $n_0 = \mathcal{O}\left(\frac{d^2}{2}\right)$ (Busé)



Last Progress

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- Madsen, `arXiv:1511.04073`
- Cortadellas–Cox–**D**, JPAA to appear

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Describe the bi-degrees of all minimal generators in $\underline{T}$-degree larger than or equal to μ

We "got" this region

We “got” this region

$$\phi(t_0 : t_1)$$
$$=$$
$$\left(\varphi'_{\mu_1}(t_0 : t_1) \wedge \psi'_{\mu_2}(t_0 : t_1)\right) \wedge \psi_{d-\mu}(t_0 : t_1)$$

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$$=$$
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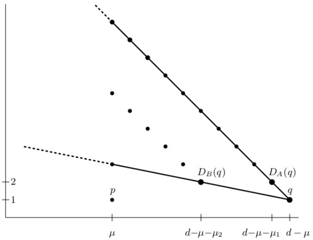

Further results

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Implicitization

Further results

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- Quadratic and cubic surfaces (Chen-Shen-Deng)

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Further results

Implicitization

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- Steiner surfaces (Wang-Chen)
- Revolution surfaces (Shi-Goldman)
- ...

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Rees Algebras

Further results

Implicitization

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Rees Algebras

- "Monoid" Surfaces (Cortadellas - **D**)

Further results

Implicitization

- Quadratic and cubic surfaces (Chen-Shen-Deng)
- Steiner surfaces (Wang-Chen)
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- ...

Rees Algebras

- "Monoid" Surfaces (Cortadellas - **D**)
- de Jonquières surfaces (Hassanzadeh- Simis)
- **D**-modules approach (Cid Ruiz)

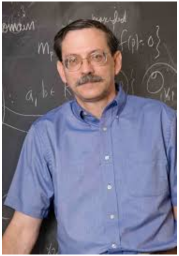
This was quite a journey...

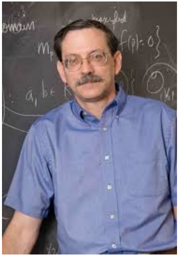
This was quite a journey...

(Do not clap yet!)

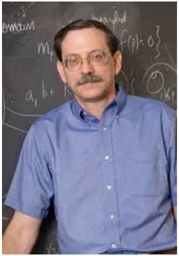
A gift for David

A gift for David





Happy Retirement, David!



Happy Retirement, David!

\end{talk}