Elimination theory, sparse systems and applications

Carlos D'Andrea

Journées Nationales de Calcul Formel – Luminy, March 2020

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What is Elimination Theory?

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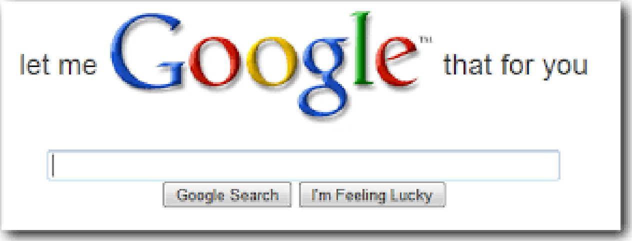
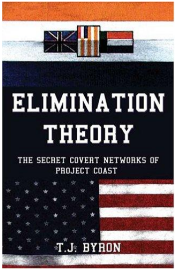
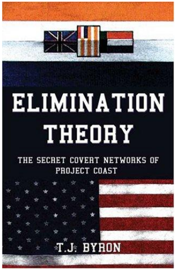


Elimination Theory

Elimination Theory

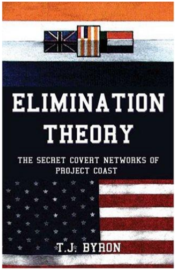


Elimination Theory



$$\mathrm{Res}(f(x),\ g(x)) = \begin{vmatrix} a_m & a_{m-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & a_m & a_{m-1} & \cdots & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_m & a_{m-1} & \cdots & a_1 & a_0 \\ b_n & b_{n-1} & \cdots & b_1 & b_0 & 0 & \cdots & 0 \\ 0 & b_n & b_{n-1} & \cdots & b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & b_n & b_{n-1} & \cdots & b_1 & b_0 \end{vmatrix}.$$

Elimination Theory



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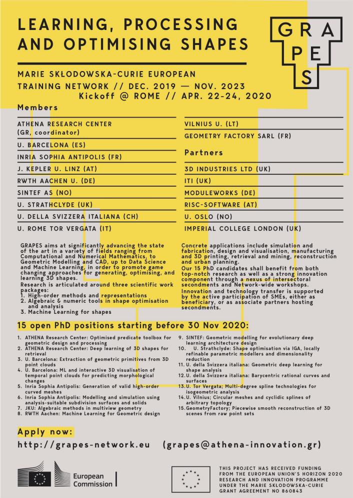
13 docs



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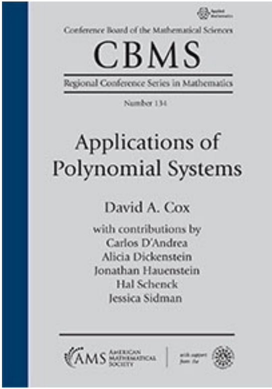
One book

One book



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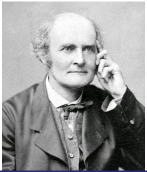
A bit of history

A bit of history

In the problem of **elimination**, one seeks the relationship that must exist between the coefficients of a function or system of functions in order that some particular circumstance (or singularity) can occur.

Cayley, 1864 Nouvelles recherches sur l'élimination et la théorie des courbes



A bit of history

A system of arbitrarily many algebraic equations for $z^0, z', z'', ..., z^{(n-1)}$, in which the coefficients belong to the rationality domain $(R, R', R'', \ldots.)$, define the algebraic relations between z and R, whose knowledge and representation are the purpose of the **theory of elimination**.

Kronecker, 1882 Grundzüge einer arithmetischen Theorie der algebraischen Grössen

The baby example in elimination

The baby example in elimination

Find "the condition" on

$a_{10},\ a_{11},\ a_{20}, a_{21}$ so that the system

$$\begin{cases} a_{10}x_0 + a_{11}x_1 & = & 0 \\ a_{20}x_0 + a_{21}x_1 & = & 0 \end{cases}$$

has a solution different from $(0,0)$

"Elimination"

$$a_{10}x_0 + a_{11}x_1,\ a_{20}x_0 + a_{21}x_1$$

$$\in \mathbb{K}[a_{10}, a_{11}, a_{20}, a_{21}, x_0, x_1]$$

“Elimination”

$$a_{10}x_0 + a_{11}x_1,\ a_{20}x_0 + a_{21}x_1$$

$$\in \mathbb{K}[a_{10}, a_{11}, a_{20}, a_{21}, x_0, x_1]$$
$$\downarrow$$
$$a_{10}a_{21} - a_{20}a_{11} \in \mathbb{K}[a_{10}, a_{11}, a_{20}, a_{21}]$$

More general

Find "the condition" for the system

$$\begin{cases} a_{10}x_0 + a_{11}x_1 + \ldots + a_{1n}x_n & = & 0 \\ a_{20}x_0 + a_{21}x_1 + \ldots + a_{2n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{(n+1)0}x_0 + a_{(n+1)1}x_1 + \ldots + a_{(n+1)n}x_n & = & 0 \end{cases}$$

More general

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to have a solution different from

$$(0, 0, \ldots, 0)$$

Another more general

Let $d_1, d_2 \in \mathbb{N}$. Find "the condition" for the system of polynomials

$$\begin{cases} a_{10}{x_0}^{d_1} + a_{11}{x_0}^{d_1-1}x_1 + \ldots = 0 \\ a_{20}{x_0}^{d_2} + a_{21}{x_0}^{d_2-1}x_1 + \ldots = 0 \end{cases}$$

to have a solution different from $(0,0)$

More more more general...

Let $n \in \mathbb{N}$, and $d_1, \ldots, d_{n+1} \in \mathbb{N}$, find the condition for

$$\left\{\begin{array}{rcl} \sum_{\alpha_0+\ldots+\alpha_n=d_1} a_{1,\alpha_0,\ldots,\alpha_n} x_0{}^{\alpha_0} \ldots x_n{}^{\alpha_n} & = & 0 \\ \sum_{\alpha_0+\ldots+\alpha_n=d_2} a_{2,\alpha_0,\ldots,\alpha_n} x_0{}^{\alpha_0} \ldots x_n{}^{\alpha_n} & = & 0 \\ \vdots & \vdots & \vdots \\ \sum_{\alpha_0+\ldots+\alpha_n=d_{n+1}} a_{n+1,\alpha_0,\ldots,\alpha_n} x_0{}^{\alpha_0} \ldots x_n{}^{\alpha_n} & = & 0 \end{array}\right.$$

to have a solution different from $(0, 0, \ldots, 0)$

Elimination: The general problem

For $\mathbf{a} = (a_1, \ldots, a_N)$, $k,\ n \in \mathbb{N}$ let $f_1(\mathbf{a}, x_1, \ldots, x_n), \ldots, f_k(\mathbf{a}, x_1, \ldots, x_n) \in \mathbb{K}[\mathbf{a}, x_1, \ldots, x_n]$. Find conditions on $\mathbf{a}$ such that

$$\left\{ \begin{array}{rcl} f_1(\mathbf{a}, x_1, \ldots, x_n) & = & 0 \\ f_2(\mathbf{a}, x_1, \ldots, x_n) & = & 0 \\ \vdots & \vdots & \vdots \\ f_k(\mathbf{a}, x_1, \ldots, x_n) & = & 0 \end{array} \right.$$

has a solution

Solution?

- Depends on the ground field

Solution?

- Depends on the ground field
- There is not necessarily a "closed" condition

Solution?

- Depends on the ground field
- There is not necessarily a "closed" condition
- Algorithms??

Easy example

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + \ldots + a_{2n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{kn}x_1 + \ldots + a_{kn}x_n & = & 0 \end{cases}$$

with $k \geq n$

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with $k \geq n$

Conditions: all maximal minors of $\left(a_{ij}\right)_{1\leq i\leq k,\, 1\leq j\leq n}$ equal to zero

Another “easy” example

$$k = n = 1,$$

$$a_0 + a_1 x_1 + a_2 {x_1}^2 + \ldots + a_d {x_1}^d = 0$$

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Conditions?

Geometry

$$V = \{(\boldsymbol{a}, x_1, \ldots, x_n) : f_1(\boldsymbol{a}, x_1, \ldots, x_n) = 0, \ldots f_k(\boldsymbol{a}, x_1, \ldots, x_n) = 0\}$$

Geometry

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$$\begin{array}{ccc} V & \subset & \mathbb{K}^N \times \mathbb{K}^n \\ \pi_1|_V \downarrow & & \downarrow \pi_1 \\ \pi_1(V) & \subset & \mathbb{K}^N \end{array}$$

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The set of conditions is $\pi_1(V)$ is not necessarily described by zeroes of polynomials

Projective Elimination

(homogeneous polynomials in an algebraically closed field)

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$$\pi_1(V) = \{p_1(\mathbf{a}) = 0, \ldots, p_\ell(\mathbf{a}) = 0\}$$

How do we compute the conditions?

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- Input: $f_1(a, x), \ldots f_k(a, x)$

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- Output: $p_1(a),\ldots,p_\ell(a)$

Mertens 1889

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Set $s := \max \deg(f_i)$, and $G_0, \ldots, G_n$ expressions of the form

$G_j(a, x) := \sum u_{\alpha,i} x^\alpha f_i(a, x)$ of degree s in x with new parameters $u_{\alpha,i}$.

Mertens 1889

Set $s := \max \deg(f_i)$, and $G_0, \ldots, G_n$ expressions of the form $G_j(\boldsymbol{a}, \boldsymbol{x}) := \sum u_{\alpha,i} \boldsymbol{x}^{\alpha} f_i(\boldsymbol{a}, \boldsymbol{x})$ of degree s in $\boldsymbol{x}$ with new parameters $u_{\alpha,i}$.

Then, "the Kronecker $\boldsymbol{u}$-Resultant" $\mathrm{Res}_{\boldsymbol{x}}(G_0, \ldots, G_n) = \sum_{\beta} P_{\beta}(\boldsymbol{a})\boldsymbol{u}^{\beta}$ gives the conditions $\{P_{\beta}(\boldsymbol{a})\}$

van der Waerden 1926

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For arbitrary s, we use the "Macaulay matrix" $M_s(\mathbf{a})$ such that

$$M_s(\mathbf{a}) \begin{bmatrix} \vdots \\ \mathbf{x}^{\beta} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \mathbf{x}^{\alpha} f_i \\ \vdots \end{bmatrix}, \ |\alpha| = s$$

van der Waerden 1926

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For $s \gg 0$, all maximal minors of $M_s(\mathbf{a})$ vanish iff there is a common solution

Algebra & Geometry

(van der Waerden, 1926)

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$$I\left(\pi_1(V)\right) = \left(\langle f_1, \ldots, f_k\rangle : \langle x_0, \ldots, x_n\rangle^{\infty}\right)$$

Algebra & Geometry

(van der Waerden, 1926)

$$\boxed{\begin{array}{c} I\left(\pi_1(V)\right) \\ = \\ \left(\langle f_1, \ldots, f_k \rangle : \langle x_0, \ldots, x_n \rangle^{\infty}\right) \cap \mathbb{K}[a] \end{array}}$$

The "elimination ideal"

But what if...

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- the polynomials are not homogeneous?

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- the polynomials are not homogeneous?
- $\mathbb{K}$ is not algebraically closed?

$\mathbb{K}$ is the problem?

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- Embed $\mathbb{K} \hookrightarrow \overline{\mathbb{K}}$ and work there

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- ...

Homogeneous vs non homogeneous

Homogeneous vs non homogeneous

Analogy with Linear Algebra...

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

Generalization of the determinant?

$$V = \{(a, x_0, x_1, \ldots, x_n) : f_1(a, x_0, x_1, \ldots, x_n) = 0, \ldots, f_{n+1}(a, x_0, x_1, \ldots, x_n) = 0\}$$

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One hopes!

Example 1

$$\begin{cases} a_{00}x_0 + a_{01}x_1 + \ldots + a_{0n}x_n & = & 0 \\ a_{10}x_0 + a_{11}x_1 + \ldots + a_{1n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{n0}x_0 + a_{n1}x_1 + \ldots + a_{nn}x_n & = & 0 \end{cases}$$

Example 1

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$$\boxed{p_1(a) = \det(a_{ij})}$$

Example 2

$$\begin{cases} f_1 = a_{10}{x_0}^{d_1} + a_{11}{x_0}^{d_1-1}x_1 + \ldots + a_{1d_1}{x_1}^{d_1} \\ f_2 = a_{20}{x_0}^{d_2} + a_{21}{x_0}^{d_2-1}x_1 + \ldots + a_{2d_2}{x_1}^{d_2} \end{cases}$$

Example 2

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$$\boxed{\mathrm{Res}(f_1, f_2) = \det \begin{pmatrix} a_{10} & a_{11} & \ldots & a_{1d_1} & 0 & \ldots & 0 \\ 0 & a_{10} & \ldots & a_{1d_1-1} & a_{1d_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{10} & \ldots & \ldots & a_{1d_1} \\ a_{20} & a_{21} & \ldots & a_{2d_2} & 0 & \ldots & 0 \\ 0 & a_{20} & \ldots & a_{2d_2-1} & a_{2d_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{20} & \ldots & \ldots & a_{2d_2} \end{pmatrix}}$$

Example 3

$$\begin{cases} f_1 = \sum_{\alpha_0+\ldots+\alpha_n=d_1} a_{1,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0} \ldots {x_n}^{\alpha_n} \\ f_2 = \sum_{\alpha_0+\ldots+\alpha_n=d_2} a_{2,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0} \ldots {x_n}^{\alpha_n} \\ \vdots \\ f_{n+1} = \sum_{\alpha_0+\ldots+\alpha_n=d_{n+1}} a_{n+1,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0} \ldots {x_n}^{\alpha_n} \end{cases}$$

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$$\mathrm{Res}_{d_1,\ldots,d_n}(f_1, f_2, \ldots, f_{n+1})$$

Example 3

$$\begin{cases} f_1 = \sum_{\alpha_0+\ldots+\alpha_n=d_1} a_{1,\alpha_0,\ldots,\alpha_n} x_0{}^{\alpha_0} \ldots x_n{}^{\alpha_n} \\ f_2 = \sum_{\alpha_0+\ldots+\alpha_n=d_2} a_{2,\alpha_0,\ldots,\alpha_n} x_0{}^{\alpha_0} \ldots x_n{}^{\alpha_n} \\ \vdots \\ f_{n+1} = \sum_{\alpha_0+\ldots+\alpha_n=d_{n+1}} a_{n+1,\alpha_0,\ldots,\alpha_n} x_0{}^{\alpha_0} \ldots x_n{}^{\alpha_n} \end{cases}$$

$\mathrm{Res}_{d_1,\ldots,d_n}(f_1, f_2, \ldots, f_{n+1})$
Macaulay/dense/classical resultant

No-example

No-example

$$\begin{cases} a_{00}{x_0}^4 + a_{01}x_0^2x_1x_2 + a_{02}x_1^2{x_2}^2 & = & 0 \\ a_{10}{x_0}^4 + a_{11}x_0^2x_1x_2 + a_{12}x_1^2{x_2}^2 & = & 0 \\ a_{20}{x_0}^4 + a_{21}x_0^2x_1x_2 + a_{22}x_1^2{x_2}^2 & = & 0 \end{cases}$$

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$$\boxed{p_1(a) \neq \det(a_{ij})}$$

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$$p_1(\mathbf{a}) = a_{00}^2a_{12}^2 - a_{00}a_{01}a_{11}a_{12} - 2a_{00}a_{02}a_{10}a_{12} + a_{00}a_{02}a_{11}^2 + a_{01}^2a_{10}a_{12} - a_{01}a_{02}a_{10}a_{11} + a_{02}^2a_{10}^2$$

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$$p_3(\mathbf{a}) = a_{20}^2a_{12}^2 - a_{20}a_{21}a_{11}a_{12} - 2a_{20}a_{22}a_{10}a_{12} + a_{20}a_{22}a_{11}^2 + a_{21}^2a_{10}a_{12} - a_{21}a_{22}a_{10}a_{11} + a_{22}^2a_{10}^2$$

Cautionary Example

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“Triangulating” a system

“Triangulating” a system

Two ways:

“Triangulating” a system

Two ways:

- Gröbner Bases (w.r.t. “lex”)

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- Triangular Decomposition

"Triangulating" a system

Two ways:

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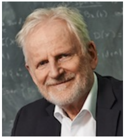
- Triangular Decomposition

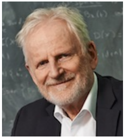
Both use some kind of **division** of polynomials

$$
\begin{array}{rr}
a_1: & y \\
a_2: & -1 \\
xy+1 & \\
y+1 & \sqrt{xy^2+1} \\
 & xy^2+y \\
\hline
 & -y+1 \\
 & -y-1 \\
\hline
\end{array}
$$

Gröbner Bases

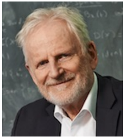


Gröbner Bases



Every ideal in $\mathbb{K}[a, x]$ has a finite set of generators $I = \langle g_1, \ldots, g_N \rangle$

Gröbner Bases



Every ideal in $\mathbb{K}[a, x]$ has a finite set of generators $I = \langle g_1, \ldots, g_N \rangle$

$\{g_1, \ldots, g_N\}$ is a **Gröbner basis iff**

$$g \in I \iff r_{g_1, \ldots g_N}(g) = 0$$

Gröbner Bases and Elimination

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In $\mathbb{K}[a, x]$, if $a \prec x$, then

Gröbner Bases and Elimination

In $\mathbb{K}[a, x]$, if $a \prec x$, then $G = GB(I) \implies G \cap \mathbb{K}[a] = GB(I \cap \mathbb{K}[a])$

Gröbner Bases and Elimination

In $\mathbb{K}[a, x]$, if $a \prec x$, then $G = GB(I) \implies G \cap \mathbb{K}[a] = GB(I \cap \mathbb{K}[a])$

And $I \cap \mathbb{K}[a]$ is the (affine) ideal of the projection

Triangular sets

Triangular sets

- Characteristic set's method (J.F. Ritt - 1940's)

Triangular sets

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- Wu's method (Wenjun Wu - 1970's)

Triangular sets

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Membership to I is easily tested

The “dictionary” of elimination

Linear **Polynomial**

The “dictionary” of elimination

Linear	**Polynomial**
Gauss elimination	Gröbner Bases

The “dictionary” of elimination

Linear	**Polynomial**
Gauss elimination	Gröbner Bases
Triangulation	Triangular sets

The "dictionary" of elimination

Linear	**Polynomial**
Gauss elimination	Gröbner Bases
Triangulation	Triangular sets
Determinants	Resultants

The "dictionary" of elimination

Linear	**Polynomial**
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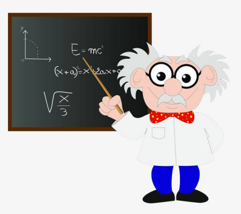
Triangulation	Triangular sets
Determinants	Resultants
Cramer's rule	u-resultants
...	...

And we will focus on Resultants!



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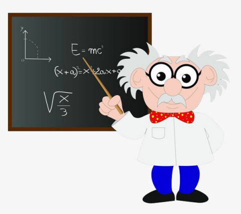

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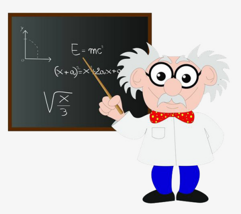
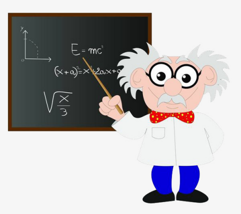


- Univariate Resultant
- Multivariate Resultants

And we will focus on Resultants!



- Univariate Resultant
- Multivariate Resultants
- Sparse Resultants

Univariate Resultants

$$\begin{cases} f_1 = a_{10}{x_0}^{d_1} + a_{11}{x_0}^{d_1-1}x_1 + \ldots + a_{1d_1}{x_1}^{d_1} \\ f_2 = a_{20}{x_0}^{d_2} + a_{21}{x_0}^{d_2-1}x_1 + \ldots + a_{2d_2}{x_1}^{d_2} \end{cases}$$

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What is the resultant of f_1 and f_2?

Algebraic Definitions

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- the "condition" to test if $\gcd(f_1, f_2) = 1$ (Sylvester)

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$\mathrm{Res}(f_1, f_2)$ is:

- the "condition" to test if $\gcd(f_1, f_2) = 1$ (Sylvester)
- One of the (two) generators of the elimination ideal

$$\langle f_1(x_0, 1),\ f_2(x_0, 1)\rangle \cap \mathbb{Z}[a_{10}, \ldots, a_{1d_1},\ a_{20}, \ldots, a_{2d_2}]$$

Geometry

$\mathbb{K}$ any field

Geometry

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$$V = \{(\alpha_{10}, \ldots, \alpha_{2d_2}; p_0 : p_1) : f_1(\boldsymbol{\alpha}, p_0, p_1) = f_2(\boldsymbol{\alpha}, p_0, p_1) = 0\}$$

Geometry

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$$V=\{(\alpha_{10},\ldots,\alpha_{2d_2};p_0:p_1):f_1(\boldsymbol{\alpha},p_0,p_1)=f_2(\boldsymbol{\alpha},p_0,p_1)=0\}$$

$$\boxed{\begin{array}{ccc} V & \subset & \mathbb{K}^{d_1+d_2+2}\times\mathbb{P}^1_{\mathbb{K}} \\ \pi_1|_V\downarrow & & \downarrow\pi_1 \\ \pi_1(V) & \subset & \mathbb{K}^{d_1+d_2+2} \end{array}}$$

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$\mathrm{Res}(f_1, f_2) \in \mathbb{K}[a_{10}, \ldots, a_{2d_2}]$ should be the equation of $\pi_1(V)$

Does Algebra meet Geometry?

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- gcd

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- gcd
- elimination ideal

Does Algebra meet Geometry?

- gcd
- elimination ideal
- equation of a hypersurface

?

gcd

gcd

For $\tilde{f}_1,\ \tilde{f}_2 \in \mathbb{K}[x_0, x_1]$ homogeneous of degrees $d_1,\ d_2$ to have a nontrivial common factor

gcd

For $\tilde{f}_1,\ \tilde{f}_2 \in \mathbb{K}[x_0, x_1]$ homogeneous of degrees $d_1,\ d_2$ to have a nontrivial common factor the map

$$\begin{array}{ccc} \mathbb{K}[x_0, x_1]_{d_2-1} \oplus \mathbb{K}[x_0, x_1]_{d_1-1} & \stackrel{\phi}{\to} & \mathbb{K}[x_0, x_1]_{d_1+d_2-1} \\ (g_1, g_2) & \mapsto & g_1\tilde{f}_1 + g_2\tilde{f}_2 \end{array}$$

fails to be an isomorphism

Matrix of ϕ

Matrix of ϕ

$$\begin{pmatrix}
a_{10} & a_{11} & \dots & a_{1d_1} & 0 & \dots & 0 \\
0 & a_{10} & \dots & a_{1d_1-1} & a_{1d_1} & \dots & 0 \\
\vdots & \vdots & \ddots & \dots & \dots & \ddots & \vdots \\
0 & 0 & \dots & a_{10} & \dots & \dots & a_{1d_1} \\
a_{20} & a_{21} & \dots & \dots & a_{2d_2} & \dots & 0 \\
0 & a_{20} & \dots & \dots & a_{2d_2-1} & \dots & 0 \\
\vdots & \vdots & \ddots & \dots & \dots & \ddots & \vdots \\
0 & 0 & \dots & a_{20} & \dots & \dots & a_{2d_2}
\end{pmatrix}$$

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- weighted homogeneous of degree $d_1 d_2$

$$\boxed{\mathrm{Res}(f_1(\lambda_0 x_0, \lambda_1 x_1), f_2(\lambda_0 x_0, \lambda_1 x_1)) = (\lambda_0 \lambda_1)^{d_0 d_1} \,\mathrm{Res}(f_1, f_2)}$$

More Properties

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(exercise)

Geometry

Geometry

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- Is $\pi_1(V)$ a hypersurface?

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- Is $\pi_1(V)$ a hypersurface?
- Is it irreducible?
- is $\pi_1|_V$ birational?

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$\pi_1(V)$ is an irreducible hypersurface!

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Algebra meets Geometry!

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- $\Longrightarrow$ $\boxed{{a_{2d_2}}^{d_1} \prod_{f_2(\xi,1)=0} f_1(\xi,1) = \mathrm{Res}(f_1, f_2)}$

Corollary: Multiplicativity

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Other "hybrid" formulae (GKZ)

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What if $p \mid m$??

Theorem

Gomez-Perez, Gutierrez, Ibeas, Sevilla

(2009)

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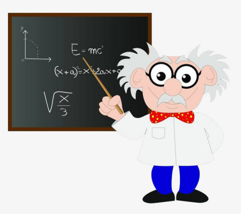
(Exercise)

See you tomorrow!



- Multivariate Resultants
- Sparse Resultants

http://www.ub.edu/arcades/notes.html