Bounds for Degrees of Minimal μ-bases of Parametric Surfaces

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Parametric Plane Curves

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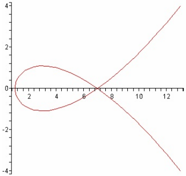

$$\begin{array}{cccc} \phi: & \mathbb{P}^1 & \rightarrow & \mathbb{P}^2 \\ & (t_0 : t_1) & \mapsto & (a(t_0, t_1) : b(t_0, t_1) : c(t_0, t_1)) \end{array}$$

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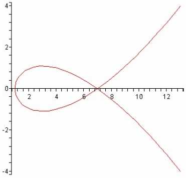

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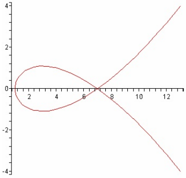

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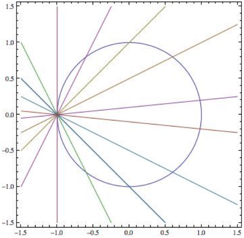
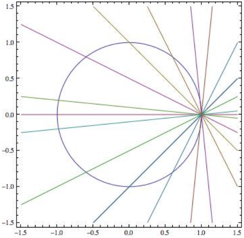
There exist $\mu \leq \frac{\delta}{2}$ and two parametrizations of degree μ and $\delta - \mu$ $\varphi_\mu(t_0 : t_1)$, $\psi_{\delta-\mu}(t_0 : t_1)$ such that

$$\boxed{\phi(t_0 : t_1) = \varphi_\mu(t_0 : t_1) \wedge \psi_{\delta-\mu}(t_0 : t_1)}$$





"Factorization" of the unit circle

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$$\boxed{\begin{array}{rcl} \varphi_1(t_0 : t_1) & = & (-t_1 : -t_1 : t_0) \\ \psi_1(t_0 : t_1) & = & (-t_0 : t_0 : t_1) \end{array}}$$

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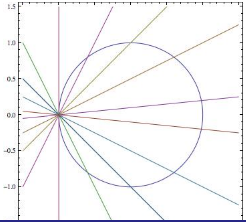
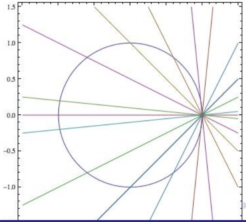
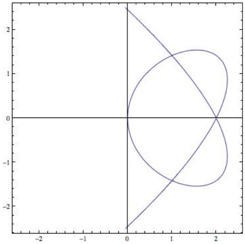
$$\begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ -t_1 & -t_1 & t_0 \\ -t_0 & t_0 & t_1 \end{vmatrix} = \left(-t_0{}^2 - t_1{}^2 : t_1{}^2 - t_0{}^2 : -2t_0t_1\right)$$





A less trivial factorization

$$\phi(t_0 : t_1) = ({t_0}^4 : 6{t_0}^2{t_1}^2 - 4{t_1}^4 : 4{t_0}^3 t_1 - 4 t_0 {t_1}^3)$$



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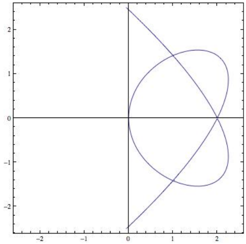

$$\begin{aligned} \varphi_2(t_0, t_1) &= (-2t_0t_1 : t_0t_1 : \tfrac{t_0{}^2}{2} - t_1{}^2) \\ \psi_2(t_0, t_1) &= (-2t_1{}^2 : t_0{}^2 : -t_0t_1) \end{aligned}$$

Algebra

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The homogeneous ideal

$$I = \langle a(T_0, T_1), b(T_0, T_1), c(T_0, T_1) \rangle \subset \mathbb{K}[\mathrm{T}]$$

Algebra

The homogeneous ideal $I = \langle a(T_0, T_1), b(T_0, T_1), c(T_0, T_1)\rangle \subset \mathbb{K}[T]$ has a **Hilbert-Burch resolution**

$$0 \to \mathbb{K}[T]^2 \xrightarrow{(\varphi_\mu, \psi_{\delta-\mu})^t} \mathbb{K}[T]^3 \xrightarrow{(a,b,c)} \mathbb{K}[T]$$

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The data $(\mu, \delta-\mu)$ is univocally determined

Surfaces

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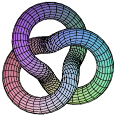

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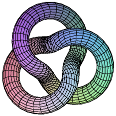

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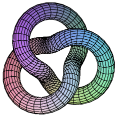

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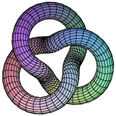

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$$0 \to \mathbb{K}[T_0, T_1, T_2]^3 \longrightarrow \mathbb{K}[T_0, T_1, T_2]^4 \xrightarrow{(a,b,c,d)} \mathbb{K}[T_0, T_1, T_2]$$

is almost never exact

However...

(Chen-Cox-Liu JSC 2005)

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is exact

$\{\phi, \psi, \chi\}$ is also called a μ-**basis** of the affine parametrization

Applications of μ-bases

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$$\deg(\psi, \phi, \chi) \leq ??$$

First Bounds

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First Bounds

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$$\deg(\psi, \phi, \chi) = \mathcal{O}(\delta^{33})$$

- $\mathcal{O}(\delta^{22})$ if $V_{\mathbb{P}^2}(a, b, c, d) = \emptyset$
- $\mathcal{O}(\delta^{12})$ if $\langle a, b, c, d \rangle$ is a general Artinian almost complete intersection

General Belief

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These bounds are not optimal

Our contribution

(Cortadellas -**D**- Montoro 2020)

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Apply the **Effective Quillen-Suslin Theorem** to improve these bounds

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Several effective versions in the 90's:
Fitchas-Galligo, Sturmfels,
Laubenbaucher-Woodburn

“Weak” Quillen-Suslin Theorem

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Effective Quillen-Suslin

(Fitchas-Galligo Math. Nach. 1990)

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If $\deg(f_1, \dots, f_m) \leq \delta \implies \deg(f_{ij}) \leq (m\delta)^{\mathcal{O}(n)}$

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If $B = (B_1\, B_2\, \ldots\, B_m) \in \mathbb{K}[\mathrm{T}]^{m\times m}$ is the inverse of

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$\Longrightarrow$ $\{B_2, \ldots, B_m\}$ is a μ-basis of $(f_1, \ldots, f_m)$ (and B_1 encodes a Bézout identity for these polynomials (Nullstellensatz))

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Previous bound: $\mathcal{O}(\delta^{22})$

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(Cortadellas -**D**- Montoro 2020)
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- Effective Quillen-Suslin in two steps

A running example

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$$\begin{cases} \tilde{a}(t_1, t_2) & = & 11 - 4t_1 + 3{t_1}^2 + 4t_2 \\ \tilde{b}(t_1, t_2) & = & 5 - 4t_1 + 2{t_1}^2 + 4t_2 - 2t_1t_2 + {t_2}^2 \\ \tilde{c}(t_1, t_2) & = & 1 + 3{t_1}^2 - {t_1}^3 + {t_1}^2t_2 \\ \tilde{d}(t_1, t_2) & = & 7 - 3t_1 + {t_1}^2 + 3t_2 \end{cases}$$

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We have

$$\begin{pmatrix} 4 & 3 \\ 2 - t_1 + t_2 & 1 \\ {t_1}^2 & 1 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 - t_1 + t_2 \\ 1 + {t_1}^2 \end{pmatrix} = \begin{pmatrix} \tilde{a} \\ \tilde{b} \\ \tilde{c} \\ \tilde{c} \end{pmatrix}$$

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- always depends only of the variable t_1
- the gcd of these elements is 1

Simplify

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Apply Effective Quillen-Suslin in $\mathbb{K}[t_1]$ to get

$$\begin{pmatrix} 2 - t_1 + t_2 & 1 \\ -2 + 3t_1 - 3t_2 & 0 \\ 4 - 3{t_1}^2 & 0 \\ -5 & 0 \end{pmatrix}$$

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The new (equivalent) parametrization $(\tilde{\tilde{a}}, \tilde{\tilde{b}}, \tilde{\tilde{c}}, \tilde{\tilde{d}})$has the last three polynomials being multiple of

$$2 - t_1 + t_2$$

Factor out

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"Add" the common factor at the end

In our case

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$$(\tilde{\tilde{a}}, \tilde{\tilde{b}}_0, \tilde{\tilde{c}}_0, \tilde{\tilde{d}}_0)$$
$$= (5 - 4t_1 + 2{t_1}^2 + 4t_2 - 2t_1t_2 + {t_2}^2, -2 + 3t_1 - 3t_2, 4 - 3{t_1}^2, -5)$$

A Quillen-Suslin matrix B such that $(\tilde{\tilde{a}}, \tilde{\tilde{b}}_0, \tilde{\tilde{c}}_0, \tilde{\tilde{d}}_0) \cdot B = (1\,0\,0\,0)$ is

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$$\begin{pmatrix} 0 & 5 - \frac{150t_2}{37} + \frac{45t_1t_2}{37} - \frac{540{t_2}^2}{1369} + \frac{405{t_1}^2{t_2}^2}{1369} & \frac{135t_2}{37} & 0 \\ 0 & m_1 & 5 + \frac{150t_2}{37} - \frac{45t_1t_2}{37} + \frac{45{t_2}^2}{37} & 0 \\ 0 & -\frac{60t_2}{37} + \frac{30t_1t_2}{37} - \frac{180{t_2}^2}{1369} + \frac{135{t_1}^2{t_2}^2}{1369} & \frac{45t_2}{37} & 5 \\ -\frac{1}{5} & 5 - 4t_1 + 2{t_1}^2 & -2 + 3t_1 & 4 - 3{t_1}^2 \end{pmatrix}$$

A Quillen-Suslin matrix B such that $(\tilde{\tilde{a}}, \tilde{\tilde{b}}_0, \tilde{\tilde{c}}_0, \tilde{\tilde{d}}_0) \cdot B = (1\,0\,0\,0)$ is

$$\begin{pmatrix} 0 & 5-\frac{150t_2}{37}+\frac{45t_1t_2}{37}-\frac{540{t_2}^2}{1369}+\frac{405{t_1}^2{t_2}^2}{1369} & \frac{135t_2}{37} & 0 \\ 0 & m_1 & 5+\frac{150t_2}{37}-\frac{45t_1t_2}{37}+\frac{45{t_2}^2}{37} & 0 \\ 0 & -\frac{60t_2}{37}+\frac{30t_1t_2}{37}-\frac{180{t_2}^2}{1369}+\frac{135{t_1}^2{t_2}^2}{1369} & \frac{45t_2}{37} & 5 \\ -\frac{1}{5} & 5-4t_1+2{t_1}^2 & -2+3t_1 & 4-3{t_1}^2 \end{pmatrix}$$

with

$$m_1 = -\frac{125t_2}{37}+\frac{100t_1t_2}{37}-\frac{2450{t_2}^2}{1369}+\frac{735t_1{t_2}^2}{1369}+\frac{450{t_1}^2{t_2}^2}{1369}-\frac{135{t_1}^3{t_2}^2}{1369}-\frac{180{t_2}^3}{1369}+\frac{135{t_1}^2{t_2}^3}{1369}$$

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- Remove the first column of B
- Multiply the first row by the “common factor” $2 - t_1 + t_2$

$$\begin{pmatrix} \left(5 - \frac{150t_2}{37} + \frac{45t_1t_2}{37} - \frac{540{t_2}^2}{1369} + \frac{405{t_1}^2{t_2}^2}{1369}\right)(2 - t_1 + t_2) & \frac{135t_2}{37}(2 - t_1 + t_2) & 0 \\ m_1 & 5 + \frac{150t_2}{37} - \frac{45t_1t_2}{37} + \frac{45{t_2}^2}{37} & 0 \\ -\frac{60t_2}{37} + \frac{30t_1t_2}{37} - \frac{180{t_2}^2}{1369} + \frac{135{t_1}^2{t_2}^2}{1369} & \frac{45t_2}{37} & 5 \\ 5 - 4t_1 + 2{t_1}^2 & -2 + 3t_1 & 4 - 3{t_1}^2 \end{pmatrix}$$

Change back the coordinates

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We get ψ, ϕ, χ with

$$\begin{aligned}
\psi_1 &= 5 - 4t_1 + 2{t_1}^2 - 5t_2 + \frac{130t_1t_2}{37} - \frac{2630{t_2}^2}{1369} + \frac{735t_1{t_2}^2}{1369} \\
&\quad + \frac{585{t_1}^2{t_2}^2}{1369} - \frac{135{t_1}^3{t_2}^2}{1369} - \frac{180{t_2}^3}{1369} + \frac{135{t_1}^2{t_2}^3}{1369} \\
\psi_2 &= 10 - 5t_1 + \frac{260t_2}{37} - \frac{60t_1t_2}{37} - \frac{45{t_1}^2t_2}{37} + \frac{720{t_2}^2}{1369} \\
&\quad - \frac{540{t_1}^2{t_2}^2}{13} \\
\psi_3 &= \frac{180t_2}{37} - \frac{90t_1t_2}{37} + \frac{540{t_2}^2}{1369} - \frac{405{t_1}^2{t_2}^2}{1369} \\
\psi_4 &= -15 + 12t_1 - 6{t_1}^2
\end{aligned}$$

$$
\begin{aligned}
\phi_1 &= 3 + 3t_1 + \frac{195t_2}{37} - \frac{45t_1t_2}{37} + \frac{45t_2{}^2}{37}\\
\phi_2 &= -15 - \frac{180t_2}{37}\\
\phi_3 &= -\frac{135t_2}{37}\\
\phi_4 &= 6 - 9t_1
\end{aligned}
$$

$$
\begin{aligned}
\chi_1 &= 9 - 3t_1{}^2\\
\chi_2 &= 0\\
\chi_3 &= -15\\
\chi_4 &= -12 + 9t_1{}^2
\end{aligned}
$$

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- Cannot extrapolate from here to "all the cases"
- Bounds still not known to be sharp

Thanks!

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