Bounds for degrees of syzygies

Teresa Cortadellas, **Carlos D'Andrea**, Eulàlia Montoro

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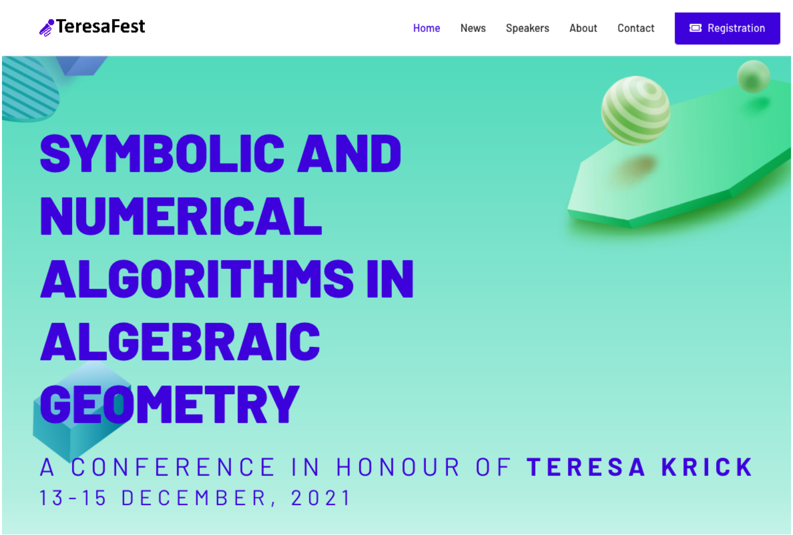

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$$\boxed{\begin{array}{c}\mathrm{Syz}(a_1, \ldots, a_m) := \\ \{(b_1, \ldots, b_m) \in \mathbb{K}[\mathbf{x}]^m \mid \sum_{i=1}^m b_i\, a_i = 0\}\end{array}}$$

The questions

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- If $\deg(a_i) \leq \delta$ can you find a basis of controlled degree?

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Hilbert-Burch Theorem: I must be either principal or have grade 2 (include the "graphic" cases $n = 1, 2$)

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Latter bounds are not optimal!

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Several effective versions in the 90's:

Fitchas-Galligo, Sturmfels,

Laubenbaucher-Woodburn

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If $\langle a_1, \ldots, a_m \rangle = \mathbb{K}[\boldsymbol{x}]$, there exist $a_{ij} \in \mathbb{K}[\boldsymbol{x}]$, $2 \leq i \leq m$, $1 \leq j \leq m$ such that

"Weak" Quillen-Suslin Theorem

If $\langle a_1, \ldots, a_m \rangle = \mathbb{K}[x]$, there exist $a_{ij} \in \mathbb{K}[x]$, $2 \leq i \leq m$, $1 \leq j \leq m$ such that

$$\det \begin{pmatrix} a_1 & a_2 & \ldots & a_m \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mm} \end{pmatrix} = 1$$

Effective Quillen-Suslin

(Fitchas-Galligo Math. Nach. 1990)

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If $\deg(a_1, \dots, a_m) \leq \delta \implies \deg(a_{ij}) \leq (m\delta)^{\mathcal{O}(n)}$

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If $C = (C_1\ C_2\ \ldots\ C_m) \in \mathbb{K}[\mathbf{x}]^{m\times m}$ is the inverse of

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$\Longrightarrow \{C_2, \ldots, C_m\}$ is a basis of $\mathrm{Syz}(a_1, \ldots, a_m)$ (and C_1 encodes a Bézout identity for these polynomials (Nullstellensatz))

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Theorem (Cortadellas-D-Montoro)

There is an algorithm which computes a basis of $\mathrm{Syz}(a_1, \ldots, a_m)$ of degree bounded by $3n^2 4^n(\delta_0^2 + \delta + 1)^{2n}$

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Theorem (Cortadellas-D-Montoro)

If $I = \langle b_1, b_2 \rangle$ is zero dimensional, then a basis can be found of degree bounded by

$$4(\delta^2 + \delta_0^2) + 3mn^2 4^n (3\delta^2 + \delta_0 + 1)^{2n}$$

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If $\deg(a_{1j}, a_{2j}) \leq \delta \implies \deg(C) \leq 3n^2(2(\delta+1))^{2n}$

Sketch of the proof

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$$\begin{pmatrix} a_1 & a_2 & \ldots & a_m \end{pmatrix} \cdot N = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$$

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$$
\begin{aligned}
(a_1\ a_2\ \ldots\ a_m)\cdot N &= (b_1\ b_2) \\
(b_1\ b_2)\cdot M &= (a_1\ a_2\ \ldots\ a_m)
\end{aligned}
$$

Sketch of the proof

$$\begin{aligned} \begin{pmatrix} a_1 & a_2 & \ldots & a_m \end{pmatrix} \cdot N &= \begin{pmatrix} b_1 & b_2 \end{pmatrix} \\ \begin{pmatrix} b_1 & b_2 \end{pmatrix} \cdot M &= \begin{pmatrix} a_1 & a_2 & \ldots & a_m \end{pmatrix} \end{aligned}$$

N is unimodular, and so is

$$\begin{pmatrix} M & \begin{matrix} -b_2 \\ b_1 \end{matrix} \end{pmatrix}$$

Sketch of the proof

$$\begin{aligned}(a_1\ a_2\ \ldots\ a_m)\cdot N &= (b_1\ b_2)\\ (b_1\ b_2)\cdot M &= (a_1\ a_2\ \ldots\ a_m)\end{aligned}$$

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$\Longrightarrow$] apply the Effective Q-S

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$\langle a_1, \ldots, a_m \rangle = \langle b_1, b_2 \rangle \iff$ there exists a unimodular $M \in \mathbb{K}[\mathbf{x}]^{2\times m}$ such that $(b_1 \; b_2) \cdot M = (a_1 \ldots a_m)$

Example

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- $b_1 = x_2 + 2x_1 + 1, \; b_2 = -2x_2 - x_1$
- $$\begin{cases} a_1 &= x_1 + 3x_1^2 + x_2 + 4x_1x_2 - x_2^2 \\ a_2 &= -x_1^2 + x_2 + x_2^2 \\ a_3 &= x_1 + 2x_1^2 - 2x_2^2 \\ a_4 &= 1 + x_1 - x_2 \end{cases}$$

Example (continuation)

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$$M = \begin{pmatrix} x_1 + x_2 & x_2 & x_1 & 1 \\ -x_1 + x_2 & x_2 & x_1 & 1 \end{pmatrix}$$

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$$\tilde{M} := \begin{pmatrix} x_1 + x_2 & x_2 & x_1 & 1 & -b_2 \\ -x_1 + x_2 & x_2 & x_1 & 1 & b_1 \end{pmatrix}$$ is unimodular

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Apply Effective Quillen Suslin:

$$\tilde{M} \cdot U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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We get $U =$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 2 & -2 & 2x_2 - 2x_1 & -4x_1 & 2x_1 - 2x_2 + 1 \\ 1 & -1 & -x_1 + x_2 + 1 & -2x_1 & x_1 - x_2 \\ 1 & 0 & -x_1 & -x_1 - x_2 & -x_2 \\ -1 & 1 & x_1 - x_2 & 2x_1 & x_2 - x_1 \end{pmatrix}$$

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- Our previous bound for radical ideals and $n = 2$ improved to $\mathcal{O}(\delta^8)$
- These bounds are far from being sharp

Numerical vs Symbolic?

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Let $a_{1,t}, \ldots, a_{m,t} \in \mathbb{K}[t, x]$ be such that $\langle a_{1,t}, \ldots, a_{m,t} \rangle = \langle b_{1,t},\, b_{2,t} \rangle$

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What is the connection between $\mathrm{Syz}(a_{1,t},\ldots,a_{m,t})$ and $\mathrm{Syz}(a_{1,t}|_{t=0},\ldots,a_{m,t}|_{t=0})$?

Thanks!

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