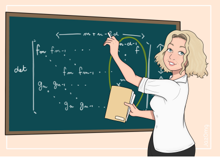
Krick Subresultants Barcelona

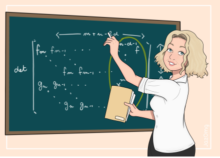
Carlos D'Andrea

Teresapalooza, December 15th 2021

Teresian Mathematical Axioms

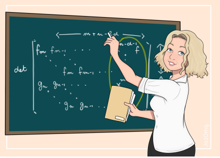


Teresian Mathematical Axioms



- You must have a motivation for the Math you are doing.

Teresian Mathematical Axioms



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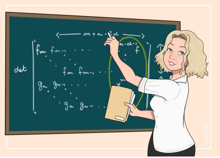

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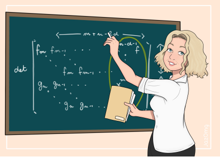

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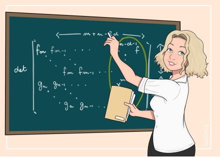

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- You must share your enthusiasm

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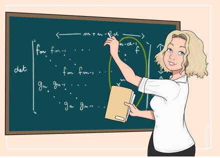

- You must have a motivation for the Math you are doing. "My advisor told me to study this" / "It is in the syllabus" are **not** valid motivations
- You must share your enthusiasm
- **Your audience must understand your explanations and share your enthusiasm**

Motivation of the talk

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$$
\begin{aligned}
f(x) &= x^4 - 3x^3 + 5x^2 + 7x + 2 \\
g(x) &= 2x^3 + 5x^2 + x + 3
\end{aligned}
$$

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$$\gcd(f, g) = 1$$

Motivation of the talk

$$\begin{aligned} f(x) &= x^4 - 3x^3 + 5x^2 + 7x + 2 \\ g(x) &= 2x^3 + 5x^2 + x + 3 \end{aligned}$$

$$\gcd(f, g) = 1 = A(x) \cdot f(x) + B(x) \cdot g(x)$$

Motivation of the talk

$$f(x) = x^4 - 3x^3 + 5x^2 + 7x + 2$$
$$g(x) = 2x^3 + 5x^2 + x + 3$$

$$\gcd(f, g) = 1 = A(x) \cdot f(x) + B(x) \cdot g(x)$$

$$A(x) = \frac{-3493 - 15019x - 5262x^2}{80341}$$
$$B(x) = \frac{29109 + 8460x - 6961x^2 + 2631x^3}{80341}$$

Questions?

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- How large the coefficients of $A(x)$, $B(x)$ are compared to those of $f(x)$, $g(x)$?

Questions?

- How large the coefficients of $A(x)$, $B(x)$ are compared to those of $f(x)$, $g(x)$?
- Can you perform the Extended Euclidean Algorithm without making divisions in $\mathbb{Z}$?


Yes We Can!

First definition of subresultants

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Subresultants are polynomials arising in the Extended Euclidean Algorithm of $f(x)$, $g(x)$

First definition of subresultants

Subresultants are polynomials arising in the Extended Euclidean Algorithm of $f(x)$, $g(x)$ without making divisions among coefficients

A Mathematica Session

```
In[5]:= PolynomialExtendedGCD[x^4 - 3 x^3 + 5 x^2 + 7 x + 2, 2 x^3 + 5 x^2 + x + 3, x]
```

Out[5]= $\left\{1, \left\{\frac{-3493 - 15019\,x - 5262\,x^2}{80341}, \frac{29109 + 8460\,x - 6961\,x^2 + 2631\,x^3}{80341}\right\}\right\}$

```
In[7]:= Subresultants[x^4 - 3 x^3 + 5 x^2 + 7 x + 2, 2 x^3 + 5 x^2 + x + 3, x]
```

Out[7]= $\{80341, -2631, 73, 2\}$

```
In[8]:= SubresultantPolynomialRemainders[x^4 - 3 x^3 + 5 x^2 + 7 x + 2, 2 x^3 + 5 x^2 + x + 3, x]
```

Out[8]= $\{2 + 7\,x + 5\,x^2 - 3\,x^3 + x^4, 3 + x + 5\,x^2 + 2\,x^3, 41 + 33\,x + 73\,x^2, 932 - 2631\,x, 80341\}$

```
In[9]:= Resultant[x^4 - 3 x^3 + 5 x^2 + 7 x + 2, 2 x^3 + 5 x^2 + x + 3, x]
```

Out[9]= 80341

Formulas in roots

Formulas in roots

$$
\begin{aligned}
f(x) &= (x - r_1)(x - r_2)(x - r_3)(x - r_4)\\
g(x) &= 2(x - s_1)(x - s_2)(x - s_3)
\end{aligned}
$$

Formulas in roots

$$\begin{aligned} f(x) &= (x-r_1)(x-r_2)(x-r_3)(x-r_4) \\ g(x) &= 2(x-s_1)(x-s_2)(x-s_3) \end{aligned}$$

$$SRes_0(f,g) = \mathrm{Res}(f,g) = 2^4 \prod_{i,j}(s_i - r_j)$$

Subresultants in roots

Subresultants in roots

$$\mathrm{SRes}_1(f,g) = 2^3 \sum_{j=1}^{3} (x - s_j) \frac{\prod_{i,j' \neq j} (r_i - s_{j'})}{\prod_{j' \neq j} (s_j - s_{j'})}$$

Subresultants in roots

$$\mathrm{SRes}_1(f,g) = 2^3 \sum_{j=1}^{3} (x - s_j) \frac{\prod_{i,j' \neq j}(r_i - s_{j'})}{\prod_{j' \neq j}(s_j - s_{j'})}$$

$$\mathrm{SRes}_2(f,g) = -2^2 \sum_{i,j} (x - r_i)(x - s_j) \frac{(r_i - s_j) \prod_{i' \neq i, j' \neq j}(r_{i'} - s_{j'})}{\prod_{i' \neq i}(r_i - s_{i'}) \prod_{j' \neq j}(s_j - s_{j'})}$$

Subresultants in roots

$$\mathrm{SRes}_1(f,g) = 2^3 \sum_{j=1}^{3} (x - s_j) \frac{\prod_{i,j' \neq j} (r_i - s_{j'})}{\prod_{j' \neq j} (s_j - s_{j'})}$$

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$$\mathrm{SRes}_3(f,g) =$$

Subresultants in roots

$$\mathrm{SRes}_1(f,g) = 2^3 \sum_{j=1}^{3} (x - s_j) \frac{\prod_{i,j' \neq j} (r_i - s_{j'})}{\prod_{j' \neq j} (s_j - s_{j'})}$$

$$\mathrm{SRes}_2(f,g) = -2^2 \sum_{i,j} (x - r_i)(x - s_j) \frac{(r_i - s_j) \prod_{i' \neq i, j' \neq j} (r_{i'} - s_{j'})}{\prod_{i' \neq i} (r_i - s_{i'}) \prod_{j' \neq j} (s_j - s_{j'})}$$

$$\mathrm{SRes}_3(f,g) = g(x) = 2(s - s_1)(x - s_2)(x - s_3)$$

Elementary Proof!

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Journal of Symbolic Computation 42 (2007) 290–297

Journal of Symbolic Computation

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An elementary proof of Sylvester's double sums for subresultants

Carlos D'Andrea[a], Hoon Hong[b], Teresa Krick[c], Agnes Szanto[b,*]

Elementary Proof!

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- Linear Algebra, Sylvester and Vandermonde matrices

Sylvester double's sums

Sylvester double's sums

$$\mathrm{Sylv}^{p,q}(A,B;x) := \sum_{\substack{A'\subset A,\, B'\subset B\\ |A'|=p,\, |B'|=q}} R(x,A')\,R(x,B')\,\frac{R(A',B')\,R(A\backslash A',B\backslash B')}{R(A',A\backslash A')\,R(B',B\backslash B')},$$

where

$$R(X,Y) := \prod_{x\in X,\, y\in Y}(x-y), \quad R(x,Y) := \prod_{y\in Y}(x-y).$$

Sylvester double's sums

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For $p+q$ small, these give (scalar multiples of) $\mathrm{Sres}_{p+q}(f_A, g_B)$

Sylvester double's sums

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For $p+q$ small, these give (scalar multiples of) $\mathrm{Sres}_{p+q}(f_A, g_B)$

How about $p+q$ large?

Journal of Symbolic Computation 44 (2009) 1164–1175

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Journal of Symbolic Computation

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Sylvester's double sums: The general case☆

Carlos D'Andrea[a,1], Hoon Hong[b], Teresa Krick[c,d], Agnes Szanto[b]

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Sylvester's double sums: The general case[☆]

Carlos D'Andrea[a,1], Hoon Hong[b], Teresa Krick[c,d], Agnes Szanto[b]

simple Linear Algebra description to describe **all** Sylvester sums

What's next?

What's next?

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Sylvester's double sums: An inductive proof of the general case

Teresa Krick[a,1], Agnes Szanto[b]

What's next?

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Sylvester's double sums: An inductive proof of the general case

Teresa Krick [a,1], Agnes Szanto [b]

You can even do it simpler!

Application: rational interpolation

Application: rational interpolation

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Journal of Symbolic Computation

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Subresultants, Sylvester sums and the rational interpolation problem ☆

Carlos D'Andrea [a], Teresa Krick [b], Agnes Szanto [c]

[a] Universitat de Barcelona, Departament d'Àlgebra i Geometria, Gran Via 585, 08007 Barcelona, Spain
[b] Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires and IMAS, CONICET, Argentina
[c] Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA

Application: rational interpolation

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Subresultants, Sylvester sums and the rational interpolation problem ☆

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[a] Universitat de Barcelona, Departament d'Àlgebra i Geometria, Gran Via 585, 08007 Barcelona, Spain
[b] Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires and IMAS, CONICET, Argentina
[c] Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA

Find $r(x) \in \mathbb{K}(x)$ of bounded degree
such that $r^{(j)}(x_i) = y_{ij}$

What about multiple roots?

What about multiple roots?

The extremal case:

What about multiple roots?

The extremal case:

$$f = (x - r)^m, \; g = (x - s)^n$$

What about multiple roots?

The extremal case:

$$f = (x - r)^m, \; g = (x - s)^n$$

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Linear Algebra and its Applications

www.elsevier.com/locate/laa

Subresultants in multiple roots: An extremal case

A. Bostan [a], C. D'Andrea [b], T. Krick [c], A. Szanto [d,*], M. Valdettaro [e]

Results:

Results:

- Coefficients are easily described by expanding SRes_k as a polynomial in $(x-\alpha)^j(x-\beta)^{k-j}$

Results:

- Coefficients are easily described by expanding SRes_k as a polynomial in $(x-\alpha)^j(x-\beta)^{k-j}$
- They are hypergeometric expressions in terms of some Hankel matrices

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Subresultants of $(x-\alpha)^m$ and $(x-\beta)^n$, Jacobi polynomials and complexity

A. Bostan[a], T. Krick[b], A. Szanto[c], M. Valdettaro[d]

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- SRes_k is a Jacobi polynomial

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Subresultants of $(x-\alpha)^m$ and $(x-\beta)^n$, Jacobi polynomials and complexity

A. Bostan[a], T. Krick[b], A. Szanto[c], M. Valdettaro[d]

- SRes_k is a Jacobi polynomial
- You can compute the coefficients in the Taylor expansion of SRes_k with linear arithmetic complexity

Back to Sylvester sums...

Back to Sylvester sums...

Back to Sylvester sums...

Communications in Algebra ›
Volume 45, 2017 - Issue 8

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Original Articles

Symmetric interpolation, Exchange Lemma and Sylvester sums

Teresa Krick, Agnes Szanto & Marcelo Valdettaro

Pages 3231-3250 | Received 23 Mar 2016, Accepted author version posted online: 19 Oct 2016, Published online: 09 Jan 2017

Download citation https://doi-org.sire.ub.edu/10.1080/00927872.2016.1236121 Check for updates

$$\sum_{\substack{A'\subset A\\|A'|=d}} \mathcal{R}(A\backslash A',B)\frac{\mathcal{R}(X,A')}{\mathcal{R}(A',A\backslash A')} = (-1)^{d(m-d)}\sum_{\substack{B'\subset B\\|B'|=d}}\mathcal{R}(A,B\backslash B')\frac{\mathcal{R}(X,B')}{\mathcal{R}(B',B\backslash B')}.$$

Applications: Multiple roots

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LINEAR ALGEBRA and its Applications

Closed formula for univariate subresultants in multiple roots

Carlos D'Andrea[a], Teresa Krick[b], Agnes Szanto[c,*], Marcelo Valdettaro[d]

Applications: Multiple roots

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Linear Algebra and its Applications

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Closed formula for univariate subresultants in multiple roots

Carlos D'Andrea [a], Teresa Krick [b], Agnes Szanto [c,*], Marcelo Valdettaro [d]

$$\mathrm{SylM}_d(A,B)(x) :=$$

$$(-1)^{m'(m-d)} \sum_{\substack{A'\subset\overline{A}\\ |A'|=d-m'}} \sum_{\substack{B'\subset\overline{B}\\ |B'|=m'}} \frac{\mathcal{R}(A\backslash\overline{A},\overline{B}\backslash B')\mathcal{R}(\overline{A}\backslash A',B\backslash B')\mathcal{R}(x,A')\mathcal{R}(x,B')}{\mathcal{R}(A',\overline{A}\backslash A')\mathcal{R}(B',\overline{B}\backslash B')}.$$

What about more variables?

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$$\mathrm{Res}(f, g) = 2^4 \prod_{i,j} (s_i - r_j)$$

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$$\mathrm{Res}(f,g) = 2^4 \prod_{i,j} (s_i - r_j) = \boxed{2^4 \prod_i f(s_i)}$$

What about more variables?

$$\mathrm{Res}(f,g) = 2^4 \prod_{i,j}(s_i - r_j) = \boxed{2^4 \prod_i f(s_i)}$$

$$\mathrm{Res}(f_1,\ldots,f_{n+1}) = c \prod_{f_i(\xi)=0,\,1\leq i\leq n} f_{n+1}(\xi)$$

What about more variables?

$$\mathrm{Res}(f,g) = 2^4 \prod_{i,j} (s_i - r_j) = \boxed{2^4 \prod_i f(s_i)}$$

$$\mathrm{Res}(f_1,\ldots,f_{n+1}) = c \prod_{f_i(\xi)=0,\, 1\le i\le n} f_{n+1}(\xi)$$

How is this for subresultants?

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Journal of Algebra 302 (2006) 16–36

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Multivariate subresultants in roots

Carlos D'Andrea [a,1], Teresa Krick [b,2], Agnes Szanto [c,*,3]

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Journal of Algebra 302 (2006) 16–36

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Multivariate subresultants in roots

Carlos D'Andrea[a,1], Teresa Krick[b,2], Agnes Szanto[c,*,3]

$$\Delta_{\mathcal{S}} = \pm \left(\prod_{j=t-d_{n+1}+1}^{t} \Delta_{\mathcal{T}_j} \right) \frac{|\mathcal{O}_{\mathcal{S}}|}{\mathcal{V}_{\mathcal{T}}},$$

$$\mathcal{O}_{\mathcal{S}} = \begin{bmatrix} \xi_1^{\gamma_1} & \cdots & \xi_{\mathbf{d}}^{\gamma_1} \\ \vdots & & \vdots \\ \xi_1^{\gamma_k} & \cdots & \xi_{\mathbf{d}}^{\gamma_k} \\ \hline \xi_1^{\alpha_1} & \cdots & \xi_{\mathbf{d}}^{\alpha_1} \\ \vdots & & \vdots \\ \xi_1^{\alpha_s} & \cdots & \xi_{\mathbf{d}}^{\alpha_s} \\ \hline \xi_1^{\beta_1} f_{n+1}(\xi_1) & \cdots & \xi_{\mathbf{d}}^{\beta_1} f_{n+1}(\xi_{\mathbf{d}}) \\ \vdots & & \vdots \\ \xi_1^{\beta_r} f_{n+1}(\xi_1) & \cdots & \xi_{\mathbf{d}}^{\beta_r} f_{n+1}(\xi_{\mathbf{d}}) \end{bmatrix} \in \overline{K}^{\mathbf{d}\times\mathbf{d}}.$$

Multiple roots?

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Subresultants in multiple roots[☆]

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Subresultants in multiple roots[☆]

Carlos D'Andrea[a,*], Teresa Krick[b], Agnes Szanto[c]

$$\Delta_S(f_1,\dots,f_{n+1}) = \pm \left(\prod_{j=t-D_{n+1}+1}^{t} \widetilde{\Delta}_{T_j}\right) \frac{\det\left(\mathcal{O}_S(\mathbf{A})\right)}{\det\left(V_T(\mathbf{A})\right)}.$$

To conclude...

To conclude...

Two gifts for Teresa!

To conclude...

Two gifts for Teresa!

Gift #1

Gift #1

SUBRESULTANTS AND THE SHAPE LEMMA

DAVID A. COX AND CARLOS D'ANDREA

This paper is dedicated to Teresa Krick on the occasion of the TeresaFest 2021 conference in her honor

ABSTRACT. In nice cases, a zero-dimensional complete intersection ideal over a field of characteristic zero has a Shape Lemma. There are also cases where the ideal is generated by the resultant and first subresultant polynomials of the generators. This paper explores the relation between these representations and studies when the resultant generates the elimination ideal. We also prove a Poisson formula for resultants arising from the hidden variable method.

The fabulous main result

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$$f(x,y),\ g(x,y) \in \mathbb{K}[x,y]$$

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$$f(x,y),\ g(x,y) \in \mathbb{K}[x,y]$$

$$\boxed{\langle f, g\rangle \stackrel{?}{=} \langle \mathrm{SRes}_0(f,g,y),\ \mathrm{SRes}_1(f,g,y)\rangle}$$

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$$\Longleftrightarrow\ \gcd(\mathrm{SRes}_0, \mathrm{SRes}_1) = 1$$

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$$\iff \ \gcd(\mathrm{SRes}_0, \mathrm{SRes}_1) = 1$$

(the ideal has a Shape Lemma)

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(the ideal has a Shape Lemma)

$$\langle \mathrm{SRes}_0\rangle = \langle f,g\rangle \cap \mathbb{K}[x]$$

$$\iff \ \gcd(LC_y(f), LC_y(g)) = 1$$

The fabulous main result

$$f(x,y),\ g(x,y) \in \mathbb{K}[x,y]$$

$$\boxed{\langle f,g\rangle \stackrel{?}{=} \langle \mathrm{SRes}_0(f,g,y),\ \mathrm{SRes}_1(f,g,y)\rangle}$$

$$\Longleftrightarrow\ \gcd(\mathrm{SRes}_0, \mathrm{SRes}_1) = 1$$

(the ideal has a Shape Lemma)

$$\langle \mathrm{SRes}_0\rangle = \langle f,g\rangle \cap \mathbb{K}[x]$$

$$\Longleftrightarrow\ \gcd(LC_y(f), LC_y(g)) = 1$$

(no roots at infinity)

Gift #2

Gift #2

Happy Birthday Teresa!