Resultants, Subresultants, and the Shape Lemma

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Starting example

In $\mathbb{C}[x_1, x_2]$ set

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$$f = x_1^2 - x_2 - 1,\ \ g = x_1^2 + x_1x_2 - 2$$

A lex Gröbner basis of the ideal $\langle f, g \rangle$ with $x_2 \prec x_1$ is $\{x_2^3 + 2x_2 - 1,\ \ x_1 - x_2^2 + 1\}$

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- One generator depending only on x_2
- One generator of degree one in x_1

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$$x_2^3 + 2x_2 - 1 = \boxed{\mathrm{Res}(f, g, x_1)}$$

$$x_1 - x_2^2 - 1 \stackrel{?}{=} \boxed{\mathrm{Sres}_1(f, g, x_1)}$$

Resultants

$$f(x_1) = f_m x_1^m + \ldots + f_1 x_1 + f_0$$
$$g(x_1) = g_n x_1^n + \ldots + g_1 x_1 + g_0$$

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$$g(x_1) = g_n x_1^n + \ldots + g_1 x_1 + g_0$$
$$\boxed{\mathrm{Res}(f, g, x_1) = \det\big(Sylv(f, g)\big)}$$

$$\begin{pmatrix} f_m & \cdots & f_0 & & & \\ & f_m & \cdots & f_0 & & \\ & & \ddots & & \ddots & \\ & & & f_m & \cdots & f_0 \\ g_n & \cdots & g_0 & & & \\ & g_n & \cdots & g_0 & & \\ & & \ddots & & \ddots & \\ & & & g_n & \cdots & g_0 \end{pmatrix}$$

Subresultants

For $0 \leq t < \min\{m, n\}$,

$$\mathrm{Sres}_t(f, g, x_1) = \det \begin{array}{|cccccc|c}
\multicolumn{6}{c}{m+n-2t} & \\ \cline{1-6}
f_m & \cdots & & \cdots & f_{t+1-(n-t-1)} & x_1^{n-t-1} f(x_1) & \\
& \ddots & & & \vdots & \vdots & n-t \\
& & f_m & \cdots & f_{t+1} & x_1^{0} f(x_1) & \\ \cline{1-6}
g_n & \cdots & & \cdots & g_{t+1-(m-t-1)} & x_1^{m-t-1} g(x_1) & \\
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\end{array}$$

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But

$$\langle f, g\rangle = \langle \mathrm{Res}(f, g, x_1), \mathrm{Sres}_1(f, g, x_1)\rangle$$

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  $I = \langle \mathrm{Res}(f, g, x_1), \mathrm{Sres}_1(f, g, x_1) \rangle$?

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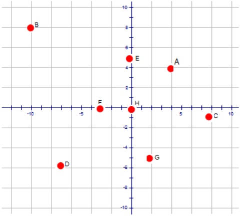
$$I = \langle p(x_2),\, x_1 - q(x_2)\rangle$$



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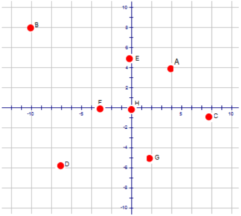

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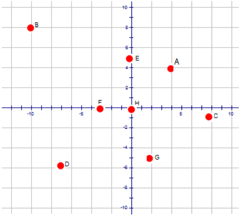

- Projection onto one of the axes should be injective
- How about multiplicities?

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If $I \cap \mathbb{C}[x_2] = \langle r(x_2) \rangle$ and the projection $V(I) \to \mathbb{C}$ onto the x_2-axis injective, TFAE:

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- $\forall \xi \in V(I)$, $T_\xi(V(I)) \overset{\simeq}{\to} T_{\xi_2}(V(r))$

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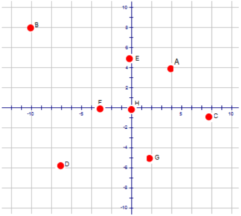

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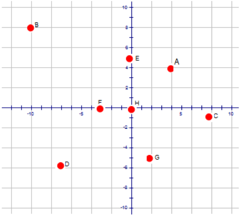

Poisson formula:

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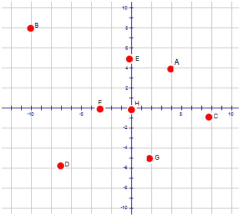

Poisson formula:

$$\mathrm{Res}(f, g, x_1) = c^* \cdot \prod_{(\xi_1,\xi_2)\in V(I)} (x_2 - \xi_2)^{m_\xi}$$

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If the projection $V(I) \to \mathbb{C}$ onto the x_2-axis is injective, any two of the following conditions imply the third:

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- $I \cap \mathbb{C}[x_2] = \langle \mathrm{Res}(f, g, x_1) \rangle$

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If the projection $V(I) \to \mathbb{C}$ onto the x_2-axis is injective, any two of the following conditions imply the third:

- I has a Shape Lemma
- $I \cap \mathbb{C}[x_2] = \langle \mathrm{Res}(f, g, x_1) \rangle$
- $\gcd(\mathrm{lc}_{x_1}(f), \mathrm{lc}_{x_1}(g)) = 1$

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- $I \cap \mathbb{C}[x_2] = \langle \mathrm{Res}(f, g) \rangle$ and $\gcd(\mathrm{Res}(f, g), \mathrm{Sres}_1(f, g)) = 1$
- $I = \langle \mathrm{Res}(f, g), \mathrm{Sres}_1(f, g)) \rangle$ and $I \cap \mathbb{C}[x_2] = \langle \mathrm{Res}(f, g) \rangle$

Works in several variables!

Cox, David; D'Andrea, Carlos

Subresultants and the Shape lemma

arXiv:2112.10306

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$$I = \langle f_1, \ldots, f_n \rangle \subset \mathbb{K}[x_1, \ldots, x_n]$$

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$\mathrm{Sres}^{x_n}_{i,d_1,\ldots,d_n}(f_1, \ldots, f_n)$, $1 \leq i \leq n-1$

Thanks!

arXiv:2112.10306

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