Bounds for the degrees of μ-bases of partially quadratic parametrizations

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ABSTRACT

We derive algorithms for computing μ-bases of rational parametrizations of surfaces having one of the partial degrees bounded by two. This class of parametrizations include ruled and canal surfaces. Our approach is based on a algorithmic treatment of the Smith Normal Form of a matrix of univariate polynomials, and it allows us to bound the degrees of the elements of a μ-basis of the input. A free implementation in SAGE of our method has been made by the first author.

KEYWORDS

μ-bases, syzygies, rational parametrizations, Smith Normal Form

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1 INTRODUCTION

Let $\mathbb{K}$ be an infinite field. Given a polynomial parametrization of a rational surface

$$P(s,t) = (a_1(s,t), a_2(s,t), a_3(s,t), a_4(s,t)) \in \mathbb{K}[s,t]^4, \quad (1)$$

it is well-known that the syzygy module of the sequence

$$\big(a_1(s,t), a_2(s,t), a_3(s,t), a_4(s,t)\big)$$

is free of rank 3, which is equivalent to say that there exist three parametrizations

$$X_i(s,t) = \big(x_{1i}(s,t), x_{2i}(s,t), x_{3i}(s,t), x_{4i}(s,t)\big),\ 1 \leq i \leq 3, \quad (2)$$

such that

$$X_1(s,t) \wedge X_2(s,t) \wedge X_3(s,t) = \lambda \cdot P(s,t),\ \lambda \in \mathbb{K} \setminus \{0\}. \quad (3)$$

The family $\{X_1(s,t), X_2(s,t), X_3(s,t)\}$ is called a *μ-basis* of the parametrization $P(s,t)$, and each polynomial combination of the $X_i(s,t)$'s a *moving plane* following (1). By identifying the standard basis of $\mathbb{K}[s,t]^4$ with the variables $x_1,\ x_2, x_3, x_4$ we can express (2) as

$$X_i(s,t) = x_{1i}(s,t)\, x_1 + x_{2i}(s,t)\, x_2 + x_{3i}(s,t)\, x_3 + x_{4,i}(s,t)\, x_4, \quad (4)$$

which justifies the label "moving plane" for polynomial combinations of these elements.

In general, the computation of μ-bases for rational parametrizations is a hard problem, and even a bound for the degrees of the $X_i(s,t)$'s is hard to find. In the Appendix of [Chen et al. 2005], it is shown that for $n \geq 2$, the syzygy module

$$Syz(P) = \{(A_1(s,t), \ldots, A_{n+1}(s,t)) \in \mathbb{K}[s,t]^{n+1} :$$
$$A_1(s,t)a_1(s,t) + \ldots + A_{n+1}(s,t)a_{n+1}(s,t) = 0\}$$

is free of rank n. In that paper, a *μ-basis* of $P(s,t)$ was defined as *any* basis of $Syz(P)$. A *minimal μ-basis* was defined as a basis $\{p_1(s,t), \ldots, p_n(s,t)\}$ of $Syz(P)$ such that $\sum_{i=1}^n \deg(p_i(s,t))$ is minimal among all the bases of $Syz(P)$, and the question on explicit bounds on the degree of such a minimal μ-basis was raised. Algorithms to compute μ-bases for this case can be found in [Deng et al. 2005], but no bounds on the degree of these elements can be easily derived from these algorithms.

In [Cid-Ruiz 2019] the first of such bounds is produced for surfaces in $\mathbb{K}^3$, i.e. when $n = 3$. Indeed, it is shown in [Cid-Ruiz 2019, Theorem A] that a minimal μ-basis in this situation has degree bounded by $O(d^{33})$, and several sub-cases were considered with better bounds in all of them. The last three authors of this paper in [Cortadellas et al. 2020] obtained a bound of size get $O(d^{12})$ for parametrization having a "shape basis". However, it is not clear yet whether these bounds are sharp, and there is definitely a lot of room for improvements.

In this article, we focus on parametrizations with low degree in one of the variables. We will show that the use of the Normal Smith Form of the matrix of coefficients of the parametrization with respect to the other variable can be of use to solve the problem. If one of the partial degrees is one, the input parametrizes ruled surfaces, which were studied in [Chen and Wang 2003; Chen et al.

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2001]. We show in Section 2 how to compute a μ-basis via an algorithmic treatment of the Normal Smith Form (Theorem 2.2), and produce suitable degree bounds of the output as an outcome, see Remarks 2.1 and 2.2.

The class of parametrizations with partial degree two is what we will call "partially quadratic parametrization" in this paper, and it contains canal surfaces, which were studied recently by [Yao and Jia 2019]. In that paper, an algorithm to compute a pseudo μ basis for the parametrization of canal surfaces (i.e the parameter λ in (3) is actually a polynomial in $\mathbb{K}[s]$). After that, the algorithm in [Deng et al. 2005] is used to get a proper μ-basis.

In Section 3 we show that we can compute a μ-basis of partially quadratic parametrizations with bounded degrees (cf. Theorem 3.6), and derive an algorithm from it (Algorithm 3.2). This algorithm has been implemented in SAGE by the first author, the code is publicly available in
`https://github.com/amrutha-b-nair/mu-Basis.git`. We explain the code in Section 4 and use it to compute several examples.

2 RULED SURFACES

We set $\deg(P(s,t))) := \max_{i=1,2,3,4}(\deg(a_i(s,t))$, and will pay attention to the partial degrees $m := \deg_s(P(s,t))$, and $n := \deg_t(P(s,t))$. In this section, we will deal with the case $n = 1$, i.e. the case of ruled surfaces. In contrast with the approach made in [Chen and Wang 2003; Chen et al. 2001], we will use the methods of [Hong et al. 2017] to obtain our results. We will compare our approach with previous results in Remark 2.2.

Set $a_i(s,t) = a_{i0}(s) + a_{i1}(s)t,\ i = 1,2,3,4$, and

$$A(s) := \begin{pmatrix} a_{10}(s) & a_{20}(s) & a_{30}(s) & a_{40}(s) \\ a_{11}(s) & a_{21}(s) & a_{31}(s) & a_{41}(s) \end{pmatrix} \in \mathbb{K}[s]^{2\times 4}.$$

Assume that none of the rows of $A(s)$ is identically zero, otherwise the parametrization would be degenerate and the computation of a μ-basis straightforward.

Proposition 2.1. *There exists a unimodular matrix $M_1(s) \in \mathbb{K}[s]^{4\times 4}$ of degree bounded by m such that*

$$(a_{10}(s)\ a_{20}(s)\ a_{30}(s)\ a_{40}(s)) \cdot M_1(s) = \left(\gcd(a_{i0}(s))_{1\le i\le 4}\ 0\ 0\ 0\right).$$

Proof. Set $g(s) := \gcd(a_{i0}(s))_{1\le i\le 4}$. Thanks to Theorem 30 in [Hong et al. 2017], there is a matrix $M_1'(s) \in \mathbb{K}[s]^{4\times 3}$ with elements of degree less than or equal to m such that its columns are a μ-basis of $(a_{10}(s)\ a_{20}(s)\ a_{30}(s)\ a_{40}(s))$. By the Hilbert Burch Theorem (see for instance [Eisenbud 2005]), the signed maximal minors of $M_1'(s)$ are equal to $\frac{1}{g(s)}(a_{10}(s)\ a_{20}(s)\ a_{30}(s)\ a_{40}(s))$. Consider now the Bézout identity

$$b_1(s)a_{10}(s) + \ldots + b_4(s)a_{40}(s) = g(s)$$

with $\deg(b_i) \le m,\ i = 1,2,3,4$. If we set now $M_1(s)$ to have as its first column the vector $(b_1(s)\ b_2(s)\ b_3(s)\ b_4(s))$, and the last three columns of $M_1'(s)$ being the remaining three columns, the claim then follows straightforwardly. □

Set now

$$A_1(s) := A(s)\cdot M_1(s) = \begin{pmatrix} g(s) & 0 & 0 & 0 \\ a_1'(s) & a_2'(s) & a_3'(s) & a_4'(s) \end{pmatrix}.$$

We have that $\deg(A_1(s)) \le 2m$. If $(a_2'(s)\ a_3'(s)\ a_4'(s)) = (0\ 0\ 0)$, then the last three columns of $M_1(s)$ are a μ-basis of (1). Otherwise, let $M_2(s) \in \mathbb{K}[s]^{3\times 3}$ be a unimodular matrix as above such that

$$(a_2'(s)\ a_3'(s)\ a_4'(s))\cdot M_2(s) = \left(\gcd(a_j'(s))_{2\le j\le 4}\ 0\ 0\right).$$

Denote with $g'(s)$ the last gcd. Note that $\deg(M_2(s)) \le 2m$. We finally set

$$M(s) := M_1(s)\cdot \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & M_2(s) \end{pmatrix}.$$

We have that $M(s)$ is unimodular, and that $\deg(M(s)) \le 3m$. We then have that

$$A(s)\cdot M(s) = \begin{pmatrix} g(s) & 0 & 0 & 0 \\ a_1'(s) & g'(s) & 0 & 0 \end{pmatrix}. \tag{5}$$

Note that the degree of these three elements is bounded by $2m$.

Theorem 2.2. *Denote with $M^1(s)$, $M^2(s)$, $M^3(s)$, $M^4(s)$ the columns of $M(s)$ and $d(s,t) := \gcd(g(s) + a_1'(s)t, g'(s)t)$.*
If $g'(s) \neq 0$, then a μ-basis of $P(s,t)$ is

$$\{f_2(s,t)M^1(s) + f_1(s,t)M^2(s),\ M^3(s), M^4(s)\},$$

where $f_1(s,t) := \frac{g(s)+a_1'(s)t}{d(s,t)}$ and $f_2(s,t) := -\frac{g'(s)}{d(s,t)}$.
Otherwise, a μ-basis is

$$\{M^2(s),\ M^3(s),\ M^4(s)\}.$$

Proof. As $M(s)$ is unimodular, an element in

$$\mathrm{Syz}(a_1(s,t), a_2(s,t), a_3(s,t), a_4(s,t))$$

is a 4-tuple which can actually be written as $\sum_{j=1}^4 h_j(s,t)M^j(s)$ with $h_j(s,t) \in \mathbb{K}[s,t]$. From (5), we deduce then that

$$P(s,t)\cdot \Big(\sum_{j=1}^4 h_j(s,t)M^j(s)\Big) = \mathbf{0}$$

is equivalent to $h_1(s,t)(g(s) + a_1'(s)t) + h_2(s,t)g'(s)t = 0$. From here, the claim follows straightforwardly. □

Remark 2.1. The degree of $f_2(s,t)M^1(s)+f_1(s,t)M^2(s)$ is bounded by $5m$ in s, and 1 in t. The other two elements have degree bounded by $3m$ in s and do not depend on t.

Remark 2.2. In [Chen et al. 2001] it is shown that one can replace $M^3(s)$ and $M^4(s)$ with two moving planes of degrees bounded by $2m$. In [Chen and Wang 2003] there is an algorithm to find the third moving plane, the only one which is linear in t, of the lowest possible degree in s, which is also shown to be bounded by $2m$.

3 PARTIALLY QUADRATIC PARAMETRIZATIONS

Now we have $n = 2$, and we write

$$a_i(s,t) = a_{i0}(s) + a_{i1}(s)t + a_{i2}(s)t^2,\ i = 1,2,3,4. \tag{6}$$

We assume that none of these four polynomials is identically zero. Let

$$A(s) := \begin{pmatrix} a_{10}(s) & a_{20}(s) & a_{30}(s) & a_{40}(s) \\ a_{11}(s) & a_{21}(s) & a_{31}(s) & a_{41}(s) \\ a_{12}(s) & a_{22}(s) & a_{32}(s) & a_{42}(s) \end{pmatrix} \in \mathbb{K}[s]^{3\times 4}. \tag{7}$$

From Proposition 2.1 we know that there exists $M_1(s) \in \mathbb{K}[s]^{4\times 4}$ an unimodular matrix of $\deg(M_1(s)) \leq m$ such that

$$(a_{10}(s)\ a_{20}(s)\ a_{30}(s)\ a_{40}(s)) \cdot M_1(s) = \big(\gcd(a_{i0}(s))_{1\leq i\leq 4}\ 0\ 0\ 0\big),$$

set as before $g_0(s) := \gcd(a_{i0}(s))_{1\leq i\leq 4}$. Let us assume that $g_0(s) \not\equiv 0$ (if $g_0(s) \equiv 0$, dividing by t we are in the case $n = 1$).

Set now

$$A_1(s) := A(s)\cdot M_1(s) = \begin{pmatrix} g_0(s) & 0 & 0 & 0 \\ a'_{11}(s) & a'_{21}(s) & a'_{31}(s) & a'_{41}(s) \\ a'_{12}(s) & a'_{22}(s) & a'_{32}(s) & a'_{42}(s) \end{pmatrix}. \quad (8)$$

We then have that $\deg(A_1(s)) \leq 2m$.

Remark 3.1. If the last three columns $A_1(s)$ are zero, then we are done, as this would imply that the last three columns of $M_1(s)$ are a μ-basis of (6).

Proposition 3.1. *The last three columns of $M_1(s)$ are a μ-basis of the parametrization if and only if $rank(A(s)) = 1$.*

Proof. As $M_1(s)$ is unimodular, then

$$\operatorname{rank}(A(s)) = \operatorname{rank}(A(s)\cdot M_1(s)) = \operatorname{rank}(A_1(s)).$$

And from (8) we have that $\operatorname{rank}(A_1(s)) = 1$ if and only if its last three columns are 0. From here the claim follows straightforwardly. □

Otherwise, let now $M_2(s) \in \mathbb{K}[s]^{3\times 3}$ be the unimodular matrix as above such that

$$(a'_{21}(s)\ a'_{31}(s)\ a'_{41}(s)) \cdot M_2(s) = \big(\gcd(a'_{2j}(s))_{2\leq j\leq 4}\ 0\ 0\big).$$

Denote with $g'_0(s)$ the gcd to the right. Note that $\deg(M_2(s)) \leq 2m$. Set now

$$M_3(s) := M_1(s)\cdot \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & M_2(s) \end{pmatrix}. \quad (9)$$

We have now that $\deg(M_3(s)) \leq 3m$, and also

$$A(s)\cdot M_3(s) = \begin{pmatrix} g_0(s) & 0 & 0 & 0 \\ a'_{11}(s) & g'_0(s) & 0 & 0 \\ a'_{12}(s) & a^*_{22}(s) & a^*_{32}(s) & a^*_{42}(s) \end{pmatrix}, \quad (10)$$

and the degree of the elements in the right-most matrix is bounded by $4m$. If the last two columns of this matrix are identically zero, then arguing as in the proof of Theorem 2.2, we can show that (1) has a μ–basis of the type $\{f_2(s,t)M_3^1(s) + f_1(s,t)M_3^2(s),\ M_3^3(s),\ M_3^4(s)\}$ for suitable $f_1(s,t),\ f_2(s,t) \in \mathbb{K}[s,t]$. As before, we can characterize this case with the rank of $A(s)$.

Proposition 3.2. *The parametrization (6) has a μ–basis of the type $\{f_2(s,t)M_3^1(s) + f_1(s,t)M_3^2(s),\ M_3^3(s),\ M_3^4(s)\}$ if and only if $rank(A(s)) = 2$.*

If this is not the case, let $M_4(s) \in \mathbb{K}[s]^{2\times 2}$ the unimodular matrix such that

$$(a^*_{32}(s)\ a^*_{42}(s)) \cdot M_4(s) = (g''_0(s)\ 0),$$

with $g''_0(s) = \gcd(a^*_{32}(s),\ a^*_{42}(s))$. Clearly $\deg(M_4(s)) \leq 4m$, and if we set

$$M(s) := M_3(s)\cdot \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_4(s) \end{pmatrix}, \quad (11)$$

we have finally that $\deg(M(s)) \leq 7m$, and

$$A(s)\cdot M(s) = \begin{pmatrix} g_0(s) & 0 & 0 & 0 \\ a'_{11}(s) & g'_0(s) & 0 & 0 \\ a'_{12}(s) & a^*_{22}(s) & g''_0(s) & 0 \end{pmatrix}. \quad (12)$$

Now we need to operate on the rows of this matrix, but with extra care because in our context it implies invertible changes of variables. Recall from [Villard 1995] that a *good conditioning* in the matrix $A(s)$ is a change in its first row of the type $R_1 + \alpha_2 R_2 + \alpha_3 R_3 \mapsto R_1$, with $\alpha_2,\ \alpha_3 \in \mathbb{K}$, in such a way that the gcd of the elements in the first row is equal to the gcd of all the elements in $A(s)$. A good conditioning for $A(s)$ can be found if $\mathbb{K}$ has enough elements (which is our case because we are assuming that $\mathbb{K}$ is infinite). It can be represented as an invertible matrix $U \in \mathbb{K}^{3\times 3}$ of the type

$$U = \begin{pmatrix} 1 & \alpha_2 & \alpha_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

such that $U \cdot A(s)$ has the desired properties. This does not change the degree of $A(s)$, so actually we will assume w.l.o.g. that at the beginning of our algorithm $A(s)$ is good conditioned by multiplying to the left by such an invertible U.

In addition, by applying another a good conditioning to

$$\begin{pmatrix} a'_{21}(s) & a'_{31}(s) & a'_{41}(s) \\ a'_{22}(s) & a'_{32}(s) & a'_{42}(s) \end{pmatrix}$$

before computing the matrix $M_2(s)$ above, which implies another multiplication of $U \cdot A(s)$ by another invertible matrix

$$U^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha' \\ 0 & 0 & 1 \end{pmatrix}$$

to the left. Note that we have

$$U^*\cdot U = \begin{pmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha' \\ 0 & 0 & 1 \end{pmatrix}, \quad (13)$$

so we can actually put our matrix $A(s)$ in a very good condition after multiplying it to the left to a matrix like (13) with $\alpha_1, \alpha_2, \alpha' \in \mathbb{K}$ generic. After this operation, from (12) we arrive to an expression of the form

$$U^*\cdot U\cdot A(s)\cdot M(s) = \begin{pmatrix} g(s) & 0 & 0 & 0 \\ g(s)a''_{11}(s) & g(s)g'(s) & 0 & 0 \\ g(s)a''_{12}(s) & g(s)g'(s)a''_{22}(s) & g(s)g'(s)g''(s) & 0 \end{pmatrix}.$$

In terms of the representation of the parametrization (1), we have that the columns of $U^*\cdot U\cdot A(s)$ now represent the coefficients of $a_i(s,t),\ i = 1,2,3,4$, with respect to a new basis $\{\phi_1(t),\ \phi_2(t),\ \phi_3(t)\}$ of $\mathbb{K}[t]_{\leq 2}$ given by $(1\ t\ t^2)\cdot(U^*\cdot U)^{-1}$.

Due to the nature of the problem, we can actually assume that $g(s) = 1$, as the parametrization $P(s,t)$ and the same one divided by $g(s)$ have the same μ-basis.

In addition, if $g'(s) = 0$, then we will have straightforwardly that the last three columns of $M(s)$ are a μ-basis of the input, so suppose that this is not the case. We will see now that we can also assume that $g'(s) = 1$.

Lemma 3.3. *Let $h(s,t)\in\mathbb{K}[s,t]$. The parametrization $P(s,t)$ as in (1) has a μ-basis $\{X_1(s,t),\, X_2(s,t),\, X_3(s,t)\}$ with $X_i(s,t)$ as in (4), $i=1,2,3$, if and only if*

$$P_h(s,t) := \big(a_1(s,t),\ h(s,t)a_2(s,t),\ h(s,t)a_3(s,t),\ h(s,t)a_4(s,t)\big)$$

has $\{\tilde{X}_1(s,t),\, \tilde{X}_2(s,t),\, \tilde{X}_3(s,t)\}$ as μ-basis, where

$$\tilde{X}_i(s,t) = h(s,t)\,x_{1i}(s,t)\,x_1 + x_{2i}(s,t)\,x_2 + x_{3i}(s,t)\,x_3 + x_{4,i}(s,t)\,x_4,$$

Proof. Use (3) and the result follows straightforwardly. □

With all of the above, we can actually assume w.l.o.g. that (12) is of the form

$$U^*\cdot U\cdot A(s)\cdot M(s) = \begin{pmatrix} 1 & 0 & 0 & 0\\ a'_{11}(s) & 1 & 0 & 0\\ a'_{12}(s) & a^*_{22}(s) & g''(s) & 0\end{pmatrix}.$$

If $g''(s)=0$, then as in Theorem 2.2, we will have that, by setting $d(s,t) := \gcd(\phi_1(t)+a'_{11}(s)\phi_2(t)+a'_{12}(s)\phi_3(t),\, \phi_2(t)+a^*_{22}(s)\phi_3(t))$, $f_1(s,t) = \frac{\phi_1(t)+a'_{11}(s)\phi_2(t)+a'_{12}(s)\phi_3(t)}{d(s,t)}$ and $f_2(s,t) = \frac{\phi_2(t)+a^*_{22}(s)\phi_3(t)}{d(s,t)}$ the set

$$\{G'(s)(f_2(s,t)M^1(s) - f_1(s,t)M^2(s)),\ M^3(s),\ M^4(s)\}$$

is a μ-basis of (6), with

$$G'(s) = \begin{pmatrix} g'(s) & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \qquad (14)$$

If $g''(s)\neq 0$, set

$$M(s) := M(s)\cdot\begin{pmatrix} 1 & 0 & 0 & 0\\ -a'_{11}(s) & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

which will make increase the bound on the degree of $M(s)$ to $9m$, then

$$U^*\cdot U\cdot A(s)\cdot M(s) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ a''_{12}(s) & a^*_{22}(s) & g''(s) & 0\end{pmatrix}, \qquad (15)$$

with

$$\deg(a''_{12}(s),\ a^*_{22}(s),\ g''(s)) \leq 10m. \qquad (16)$$

Let

$$\alpha(t) := \begin{pmatrix} \alpha_{11}(t) & \alpha_{12}(t) & \alpha_{13}(t)\\ \alpha_{21}(t) & \alpha_{22}(t) & \alpha_{23}(t)\end{pmatrix} = \begin{pmatrix}\alpha_1(t)\\ \alpha_2(t)\end{pmatrix}\in\mathbb{K}[t]^{2\times 3}$$

be a μ-basis of $(\phi_1(t),\ \phi_2(t),\ \phi_3(t))$. As these three polynomials are linearly independent, we have that $\deg(\alpha(t))=1$. As before, denote with $M^1(s),\ M^2(s),\ M^3(s),\ M^4(s)$ the columns of $M(s)$. We proceed as in the proof of Theorem 2.2, we have that $\sum_{i=1}^4 h_i(s,t)M^i(s)\in \mathrm{Syz}(P(s,t))$ if and only if $A(s)\cdot M(s)\begin{pmatrix}h_1(s,t)\\ h_2(s,t)\\ h_3(s,t)\\ h_4(s,t)\end{pmatrix}$ is a $\mathbb{K}[s,t]$-linear combination of $\alpha_1(t),\ \alpha_2(t)$. From (15), we deduce that $M^4(s)$ is an element of $\mathrm{Syz}(P(s,t))$, and in addition we have that

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ a''_{12}(s) & a^*_{22}(s) & g''(s)\end{pmatrix}\cdot\begin{pmatrix}h_1(s,t)\\ h_2(s,t)\\ h_3(s,t)\end{pmatrix} = \big(\alpha_1(t)\ \alpha_2(t)\big)\cdot\begin{pmatrix}v_1(s,t)\\ v_2(s,t)\end{pmatrix},$$

with $v_1(s,t), v_2(s,t)\in\mathbb{K}[s,t]$. We multiply this equality by the inverse of the matrix in the left-hand-side to get

$$\begin{pmatrix}h_1(s,t)\\ h_2(s,t)\\ h_3(s,t)\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ -\frac{a''_{12}(s)}{g''(s)} & -\frac{a^*_{22}(s)}{g''(s)} & \frac{1}{g''(s)}\end{pmatrix}\cdot\big(\alpha_1(t)\ \alpha_2(t)\big)\cdot\begin{pmatrix}v_1(s,t)\\ v_2(s,t)\end{pmatrix} = \qquad (17)$$

$$= \begin{pmatrix} \alpha_{11}(t) & \alpha_{21}(t)\\ \alpha_{12}(t) & \alpha_{22}(t)\\ \frac{\tilde{\alpha}_{13}(s,t)}{g''(s)} & \frac{\tilde{\alpha}_{23}(s,t)}{g''(s)}\end{pmatrix}\cdot\begin{pmatrix}v_1(s,t)\\ v_2(s,t)\end{pmatrix}.$$

with $\tilde{\alpha}_{13}(s,t) = -a''_{12}(s)\alpha_{11}(t) - a^*_{22}(s)\alpha_{12}(t) + \alpha_{13}(t)$ and $\tilde{\alpha}_{23}(s,t) = -a''_{12}(s)\alpha_{21}(t) - a^*_{22}(s)\alpha_{22}(t) + \alpha_{23}(t)$. This is equivalent to requiring that

$$\tilde{\alpha}_{13}(s,t)v_1(s,t) + \tilde{\alpha}_{23}(s,t)v_2(s,t)$$

should be a multiple of $g''(s)$, i.e. there exists $v_3(s,t)\in\mathbb{K}[s,t]$ such that

$$\tilde{\alpha}_{13}(s,t)v_1(s,t) + \tilde{\alpha}_{23}(s,t)v_2(s,t) + g''(s)v_3(s,t) = 0. \qquad (18)$$

Note that the triplet

$$(\tilde{\alpha}_{13}(s,t), \tilde{\alpha}_{23}(s,t), g''(s)) \qquad (19)$$

does not really parametrize a ruled surface in $\mathbb{K}^3$ (it has three coordinates and not four), but we can apply the methods of the previous section to it to deduce that there exists a μ-basis of (19) of the form $\{X_1(s,t),\ X_2(s,t)\}$ with

$$X_i(s,t) = (x_{i1}(s,t), x_{i2}(s,t), x_{i3}(s,t)),\ i=1,2.$$

of degree bounded by $5\cdot 10m = 50m$ (thanks to Remark 2.1).
The first two components of each of the members of the μ-basis are what we are looking for, i.e. by setting for $i=1,2$

$$\begin{pmatrix}L_{i1}(s,t)\\ L_{i2}(s,t)\\ L_{i3}(s,t)\end{pmatrix} := \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ -\frac{a''_{12}(s)}{g''(s)} & -\frac{a^*_{22}(s)}{g''(s)} & \frac{1}{g''(s)}\end{pmatrix}\cdot\big(\alpha_1(t)\ \alpha_2(t)\big)\cdot\begin{pmatrix}x_{i1}(s,t)\\ x_{i2}(s,t)\end{pmatrix}, \qquad (20)$$

we have that this is a vector of polynomials in $\mathbb{K}[s,t]$, and that

$$L_i(s,t) := \sum_{j=1}^{3} L_{ij}(s,t)M^j(s)\in \mathrm{Syz}(P(s,t)),\quad i=1,2. \qquad (21)$$

Moreover, from (18), we have that

$$(v_1(s,t), v_2(s,t), v_3(s,t)) = f_1(s,t)\cdot X_1(s,t) + f_2(s,t)\cdot X_2(s,t)$$

for suitable $f_1(s,t),\ f_2(s,t)\in\mathbb{K}[s,t]$. This, combined with (17) and (20) implies straightforwardly that

$$\begin{pmatrix}h_1(s,t)\\ h_2(s,t)\\ h_3(s,t)\end{pmatrix} = f_1(s,t)L_1(s,t) + f_2(s,t)L_2(s,t).$$

So we have proven the following.

Theorem 3.4. *Given $P(s,t)$ as in (6). If $rank(A(s))=3$, a μ-basis of this parametrization is given by $\{G'(s)L_1(s,t),\ G'(s)L_2(s,t),\ M^4(s)\}$, where $G'(s)$ is as in (14).*

Note that the degree in t of $L_i(s,t)$ is equal to one. To bound the degree in s of these polynomials, from (20) and (16) we deduce that the s-degree of $L_i(s,t),\ i=1,2$ is bounded by $50m+10m=60m$. So, each $L_{ij}(s,t)M^j(s)$ has degree bounded by $60m+9m=69m$. We still need to add $g'(s)$ to the first row of the μ-basis (following Lemma 3.3), which makes the degree bound increase to $69m+4m=73m$. Note that the degree of $M^4(s)$ -the third syzygy- is bounded by $9m$ but nevertheless this degree can be improved as a consequence of the following Lemma.

Lemma 3.5. *Let*

$$[a_i,a_j,a_k]=\begin{vmatrix} a_{i0} & a_{j0} & a_{k0}\\ a_{i1} & a_{j1} & a_{k1}\\ a_{i1} & a_{j2} & a_{k2}\end{vmatrix}$$

with $i,j,k\in\{1,2,3,4\}$ *and*

$$\tilde{g}(s)=gcd([a_1,a_2,a_3],[a_1,a_2,a_4],[a_1,a_3,a_4],[a_2,a_3,a_4]).$$

If $\tilde{g}(s)\not\equiv 0$ *then, up to a nonzero constant in* $\mathbb{K}$,

$$M^4(s):=\frac{1}{\tilde{g}(s)}([a_2,a_3,a_4],-[a_1,a_3,a_4],[a_1,a_2,a_4],-[a_1,a_2,a_3]) \tag{22}$$

Proof. It's obvious that for $i=0,1,2$ one has

$$0=\begin{vmatrix} a_{1i} & a_{2i} & a_{3i} & a_{4i}\\ a_{10} & a_{20} & a_{30} & a_{40}\\ a_{11} & a_{21} & a_{31} & a_{41}\\ a_{12} & a_{22} & a_{32} & a_{42}\end{vmatrix}=$$

$=a_{1i}[a_2,a_3,a_4]-a_{2i}[a_1,a_3,a_4]+a_{3i}[a_1,a_2,a_4]-a_{4i}[a_1,a_2,a_3]$. As $\tilde{g}(s)\neq 0$, the $\mathbb{K}[s]$-syzygy module of (6) (i.e. the set of syzygies which only depend on s) is free of rank one. Both $M^4(s)$ and the syzygy to the right of (22) are $\mathbb{K}[s]$-syzygies without a common polynomial factor. Hence, they must coincide up to a nonzero constant in $\mathbb{K}$. □

All together we have the following

Theorem 3.6. *Given* $P(s,t)$ *as in* (6), *with* $\deg_s(P(s,t))=m$. *A* μ*-basis of this parametrization can be found of respective* t*-degrees* $0,1,1$, *and corresponding* s*-degrees bounded by* $3m$, $73m$, $73m$.

In light of these results, we can present the following algorithm:

Algorithm 3.2.
Input: A parametrization $(a_1,\ a_2,\ a_3,\ a_4)$ as in (6).
Output: A μ-basis of this parametrization.

(1) Compute the matrix $A(s)$ from (7).
(2) Multiply to the left of $A(s)$ a 3×3 matrix U_0 of the shape (13) for generic values of $\alpha_1,\ \alpha_2,\ \alpha'$.
(3) Compute the matrix $M_1(s)$ from (8).
(4) If $A_1(s)$ has the last three columns equal to zero, return the last three columns of $M_1(s)$ and stop the algorithm.
(5) Set $(\phi_1(t)\ \phi_2(t)\ \phi_3(t))=(1\ t\ t^2)U_0^{-1}$.
(6) Compute the matrix $M_3(s)$ from (9).
(7) If the last two columns of $A(s)\cdot M_3(s)$ are zero, return $\{f_2(s,t)M_3^1(s)+f_1(s,t)M_3^2(s),\ M_3^3(s),\ M_3^4(s)\}$, with -using the notation of (10)-

$$f_1=\frac{g_0(s)\phi_1(t)+a'_{11}(s)\phi_2(t)+a'_{12}(s)\phi_3(t)}{d(s,t)},\ f_2=\frac{g'_0(s)\phi_2(t)+a^*_{22}(s)\phi_3(t)}{d(s,t)},$$

with

$$d(s,t)=\gcd(g_0(s)\phi_1(t)+a'_{11}(s)\phi_2(t)+a'_{12}(s)\phi_3(t),g'_0(s)\phi_2(t)+a^*_{22}(s)\phi_3(t)),$$

and stop the algorithm.
(8) Compute the matrix $M(s)$ from (15).
(9) Compute $\{\alpha_1(t),\alpha_2(t)\}$ a μ–basis of $\{\phi_1(t),\phi_2(t),\phi_3(t)\}$.
(10) Compute $\{X_1(s,t),X_2(s,t)\}$ a μ–basis of (19).
(11) Compute $L_1(s,t)$ and $L_2(s,t)$ from (21).
(12) Return $\{G'(s)L_1(s,t),\ G'(s)L_2(s,t),\ M^4(s)\}$.

Remark Note that this approach also covers the case $n=1$, as one can just set the last row of $A(s)$ to be all of them equal to zero.

Example 3.7. Let

$$P_1(s,t)=(s^2+t^2,2st,2t^2,s^2-t^2)$$

If we take $(\phi_1(t),\phi_2(t),\phi_3(t))=(t^2,t,1)$ we obtain that

$$A(s)=\begin{pmatrix}1 & 0 & 2 & -1\\ 0 & 2s & 0 & 0\\ s^2 & 0 & 0 & s^2\end{pmatrix}$$

is a well-conditioned matrix. Moreover we have that

$$M_1(s)=\begin{pmatrix}1&0&0&1\\0&1&0&0\\0&0&1&0\\0&0&2&1\end{pmatrix},$$

$$M_2(s)=Id,\ M_3(s)=M_1(s),\ M_4(s)=\begin{pmatrix}1&-1\\0&1\end{pmatrix},$$

and therefore

$$M(s)=\begin{pmatrix}1&0&0&1\\0&1&0&0\\0&0&1&-1\\0&0&2&-1\end{pmatrix},$$

and

$$A(s)\cdot M(s)=\begin{pmatrix}1&0&0&0\\0&2s&0&0\\s^2&0&2s^2&0\end{pmatrix}.$$

Notice that $g(s)=1$ and $g'(s)=2s$. Moreover $\alpha_1(t)=(1,-t,0)$ and $\alpha_2(t)=(0,1,-t)$ is a μ–basis of $(\phi_1(t),\phi_2(t),\phi_3(t))$. As $g'(s)=1$, then (17) is

$$\begin{pmatrix}h_1(s,t)\\h_2(s,t)\\h_3(s,t)\end{pmatrix}=\begin{pmatrix}1&0\\-t&1\\-s&-\frac{t}{s}\end{pmatrix}\cdot\begin{pmatrix}v_1(s,t)\\v_2(s,t)\end{pmatrix}.$$

And a μ–basis of $(-s^2,-t,s)$ is given by $X_1(s,t)=(1,0,s)$ and $X_2(s,t)=(0,s,t)$, so

$$\begin{pmatrix}L_{11}(s,t)\\L_{12}(s,t)\\L_{13}(s,t)\end{pmatrix}=\begin{pmatrix}1&0\\-t&1\\-s&-\frac{t}{s}\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\-t\\-s\end{pmatrix}$$

and

$$\begin{pmatrix}L_{21}(s,t)\\L_{22}(s,t)\\L_{23}(s,t)\end{pmatrix}=\begin{pmatrix}1&0\\-t&1\\-s&-\frac{t}{s}\end{pmatrix}\cdot\begin{pmatrix}0\\s\end{pmatrix}=\begin{pmatrix}0\\s\\-t\end{pmatrix}$$

Finally, we compute

$$L_1(s,t) = L_{11}M^1 + L_{12}M^2 + L_{13}M^3 = 1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} - t\begin{pmatrix}0\\1\\0\\0\end{pmatrix} - s\begin{pmatrix}0\\0\\1\\2\end{pmatrix} =$$

$$= \begin{pmatrix}1\\-t\\-s\\-2s\end{pmatrix},$$

and

$$L_2(s,t) = L_{21}M^1 + L_{22}M^2 + L_{23}M^3 = 0\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + s\begin{pmatrix}0\\1\\0\\0\end{pmatrix} - t\begin{pmatrix}0\\0\\1\\2\end{pmatrix} =$$

$$= \begin{pmatrix}0\\s\\-t\\-2t\end{pmatrix}.$$

By applying Lemma 3.3, we conclude that a $\mu-$basis of $P_1(s,t)$ is

$$\left\{\begin{pmatrix}2s\\-t\\-s\\-2s\end{pmatrix}, \begin{pmatrix}0\\s\\-t\\-2t\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\-1\end{pmatrix}\right\}$$

4 SAGE CODE AND EXAMPLES

In this section, we report on the SAGE code designed by the first author which implements Algorithm 3.2, and can be accessed freely through `https://github.com/amrutha-b-nair/mu-Basis.git`.

Below we list some explanations of the main commands of code.

(1) The algorithm checks for the t-degree of the given parametrization, and outputs the $\mu-$basis if the degree is at most two.
(2) The function **vector_to_matrix** takes the parametrization as input and gives the matrix $A(s)$ as output.
(3) The function **good_conditioning** will return a matrix of good conditioning, say U and a modified matrix, say A_1 such that $U \cdot A = A_1$
(4) The function **GaussJordan** can be used to compute the $\mu-$basis for univariate case following Proposition 2.1.
(5) The function **reduce** will take the parametriization P as input and give the matrices $U, A(s), M(s)$ and $N(s)$ so that

$$U \cdot A(s) \cdot M(s) = N(s)$$

where, N(s) will have the form

$$N(s) = \begin{pmatrix}1 & 0 & 0 & 0\\ a'_{11}(s) & g'(s) & 0 & 0\\ a''_{12}(s) & g'(s)a^*_{22}(s) & g'(s)g''(s) & 0\end{pmatrix}$$

(6) If at least two rows of $N(s)$ are zero, the $\mu-$basis is computed using the function **mu_det_zero** following Algorithm 3.2, step 7.
(7) If three columns of $N(s)$ are non-zero, it computes a μ-basis following Lemma 3.3 and Algorithm 3.2 Steps 9 to 12 with the function **mu_basis_two**, which uses the commands described above.

The following examples have been computed with the aid of this code:

Example 4.1. Let $P(s,t) = (t^2-1, st^2+t^2, t^2+1, 1)$. We have then

$$A(s) = \begin{pmatrix}-1 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\\ 1 & s+1 & 1 & 0\end{pmatrix}.$$

Note that $rank(A(s)) \neq 2$. If $(\phi_1(t), \phi_2(t), \phi_3(t)) = (1, t, t^2-t)$, then

$$A(s) = \begin{pmatrix}-1 & 1 & 0 & 1\\ 1 & s+1 & 1 & 0\\ 1 & s+1 & 1 & 0\end{pmatrix}$$

is well-conditioned. In this case

$$M(s) = \begin{pmatrix}0 & 1 & \frac{1}{2} & s+1\\ 0 & 0 & 0 & -2\\ 0 & 0 & \frac{-1}{2} & s+1\\ 1 & 1 & 1 & 0\end{pmatrix},$$

and

$$A(s)M(s) = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}.$$

We have $g''(s) = 0$, then a μ-basis of $P(s,t)$ is

$$\{t^2M^1 - M^2, M^3, M^4\} = \left\{\begin{pmatrix}-1\\0\\0\\t^2-1\end{pmatrix}, \begin{pmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\\1\end{pmatrix}, \begin{pmatrix}s+1\\-2\\s+1\\0\end{pmatrix}\right\}.$$

Example 4.2. Let

$$P(s,t) = (s^3+st^2+7st-1, 2st+s^2, t, s^2+t^2+1)$$

Taking $\phi(t) = (1, t, t^2)$, we obtain

$$A(s) = \begin{pmatrix}s^3-1 & s^2 & 0 & s^2+1\\ 7*s & 2*s & 1 & 0\\ s & 0 & 0 & 1\end{pmatrix}$$

which is a well-conditioned matrix. We compute

$$M(s) = \begin{pmatrix}-1 & 0 & -s+1 & -\frac{1}{7}s^2\\ s & 0 & -2 & -\frac{1}{7}(s+1)\\ -2s^2+7s & 1 & 7s^2-3s & s^3+\frac{2}{7}(s^2+s)\\ 0 & 0 & s^2-s+1 & \frac{1}{7}s^3\end{pmatrix},$$

and have that

$$A(s) \cdot M(s) = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -s & 0 & 1 & 0\end{pmatrix}.$$

In this case $g'(s) = 1$ and a $\mu-$basis of $(\phi_1(t), \phi_2(t), \phi_3(t))$ is $\alpha_1(t) = (t, -1, 0)$ and $\alpha_2(t) = (0, t, -1)$. So, (17) is

$$\begin{pmatrix}h_1(s,t)\\h_2(s,t)\\h_3(s,t)\end{pmatrix} = \begin{pmatrix}t & 0\\ -1 & t\\ st & -1\end{pmatrix} \cdot \begin{pmatrix}v_1(s,t)\\v_2(s,t)\end{pmatrix},$$

and a μ–basis of $(st,-1,1)$ is $X_1(s,t)=(1,0,-st)$ and $X_2(s,t)=(0,1,1)$. Hence,

$$\begin{pmatrix}L_{11}(s,t)\\L_{12}(s,t)\\L_{13}(s,t)\end{pmatrix}=\begin{pmatrix}t&0\\-1&t\\st&-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}t\\-1\\st\end{pmatrix},$$

and

$$\begin{pmatrix}L_{21}(s,t)\\L_{22}(s,t)\\L_{23}(s,t)\end{pmatrix}=\begin{pmatrix}t&0\\-1&t\\st&-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\t\\-1\end{pmatrix}.$$

Finally

$$L_1(s,t)=L_{11}M^1+L_{12}M^2+L_{13}M^3=$$

$$=t\begin{pmatrix}-1\\s\\-2s^2+7s\\0\end{pmatrix}-1\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+st\begin{pmatrix}-s+1\\-2\\7s^2-3s\\s^2-s+1\end{pmatrix}=\begin{pmatrix}-(s^2-s+1)t\\-st\\(7s^3-5s^2+7s)t-1\\(s^3-s^2+s)t\end{pmatrix},$$

and

$$L_2(s,t)=L_{21}M^1+L_{22}M^2+L_{23}M^3=$$

$$=0\begin{pmatrix}-1\\s\\-2s^2+7s\\0\end{pmatrix}+t\begin{pmatrix}0\\0\\1\\0\end{pmatrix}-1\begin{pmatrix}-s+1\\-2\\7s^2-3s\\s^2-s+1\end{pmatrix}=\begin{pmatrix}s-1\\-st\\-7s^2+3s+t\\-s^2+s-1\end{pmatrix}.$$

So, a μ– basis of the input parametrization is $\{L_1(s,t),L_2(s,t),M^4\}$,

$$\left\{\begin{pmatrix}-(s^2-s+1)t\\-st\\(7s^3-5s^2+7s)t-1\\(s^3-s^2+s)t\end{pmatrix},\begin{pmatrix}s-1\\-st\\-7s^2+3s+t\\-s^2+s-1\end{pmatrix},\begin{pmatrix}-\frac{1}{7}s^2\\-\frac{1}{7}(s+1)\\s^3+\frac{2}{7}(s^2+s)\\\frac{1}{7}s^3\end{pmatrix}\right\}.$$

Example 4.3. Set $P(s,t)=(s^9t+t^2+1,2t^2+t,s^2+7,t)$.
Taking $(\phi_1(t),\phi_2(t),\phi_3(t))=(1,t,t^2)$, we obtain

$$A(s)=\begin{pmatrix}1&0&s^2+7&0\\s^9&1&0&1\\1&2&0&0\end{pmatrix}$$

which is a well-conditioned matrix. We have then

$$M(s)=\begin{pmatrix}1&0&0&-s^2-7\\0&0&\frac{1}{2}&\frac{1}{2}(s^2+7)\\0&0&0&1\\-s^9&1&-\frac{1}{2}&s^{11}+7*s^9-\frac{1}{2}(s^2+7)\end{pmatrix},$$

and

$$A(s)\cdot M(s)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\end{pmatrix}.$$

In this case we have $g'(s)=1$ and a μ–basis of $(\phi_1(t),\phi_2(t),\phi_3(t))$ is $\alpha_1(t)=(t,-1,0)$ and $\alpha_2(t)=(0,t,-1)$. Identity (17) turns into

$$\begin{pmatrix}h_1(s,t)\\h_2(s,t)\\h_3(s,t)\end{pmatrix}=\begin{pmatrix}t&0\\-1&t\\-t&-1\end{pmatrix}\cdot\begin{pmatrix}v_1(s,t)\\v_2(s,t)\end{pmatrix},$$

and a μ–basis of $(-t,-1,1)$ is $X_1(s,t)=(-1,0,t)$ and $X_2(s,t)=(0,1,1)$. Hence,

$$\begin{pmatrix}L_{11}(s,t)\\L_{12}(s,t)\\L_{13}(s,t)\end{pmatrix}=\begin{pmatrix}t&0\\-1&t\\-t&-1\end{pmatrix}\cdot\begin{pmatrix}-1\\0\end{pmatrix}=\begin{pmatrix}-t\\1\\t\end{pmatrix},$$

and

$$\begin{pmatrix}L_{21}(s,t)\\L_{22}(s,t)\\L_{23}(s,t)\end{pmatrix}=\begin{pmatrix}t&0\\-1&t\\-t&-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\t\\-1\end{pmatrix}.$$

We then compute

$$L_1(s,t)=L_{11}M^1+L_{12}M^2+L_{13}M^3=$$

$$=-t\begin{pmatrix}1\\0\\0\\s^9\end{pmatrix}+1\begin{pmatrix}0\\0\\0\\1\end{pmatrix}+t\begin{pmatrix}0\\\frac{1}{2}\\0\\-\frac{1}{2}\end{pmatrix}=\begin{pmatrix}-t\\\frac{1}{2}t\\0\\\frac{1}{2}(2s^9-1)t+1\end{pmatrix}$$

$$L_2(s,t)=L_{21}M^1+L_{22}M^2+L_{23}M^3=$$

$$=0\begin{pmatrix}1\\0\\0\\s^9\end{pmatrix}+t\begin{pmatrix}0\\0\\0\\1\end{pmatrix}-1\begin{pmatrix}0\\\frac{1}{2}\\0\\-\frac{1}{2}\end{pmatrix}=\begin{pmatrix}0\\\frac{1}{2}\\0\\t+\frac{1}{2}\end{pmatrix}.$$

A μ– basis of the input parametrization is then $\{L_1(s,t),L_2(s,t),M^4\}$

$$=\left\{\begin{pmatrix}-t\\\frac{1}{2}t\\0\\\frac{1}{2}(2s^9-1)t+1\end{pmatrix},\begin{pmatrix}0\\\frac{1}{2}\\0\\t+\frac{1}{2}\end{pmatrix},\begin{pmatrix}-s^2-7\\\frac{1}{2}(s^2+7)\\1\\s^{11}+7*s^9-\frac{1}{2}(s^2+7)\end{pmatrix}\right\}.$$

Example 4.4. This parametrization is Example 3.1 in [Yao and Jia 2019]:
$P(s,t)=(4s^3+st^2+4s^2-12st+t^2+s+1,4s^4+s^2t^2+s^2+6t,6t^2,4s^2+t^2+1)$
We apply algorithm 3.2 to this input to get

$$A(s)=\begin{pmatrix}4s^3+4s^2+s+1&4s^4+s^2&0&4s^2+1\\-12s&6&0&0\\s+1&s^2&6&1\end{pmatrix}$$

which is not a good conditioned matrix. So we take

$$U=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$$

to get

$$U\cdot A(s)=\begin{pmatrix}4s^3+4s^2-11s+1&4s^4+s^2+6&0&4s^2+1\\-11s+1&s^2+6&6&1\\s+1&s^2&6&1\end{pmatrix}$$

which is now good conditioned, and

$$(\phi_1(t),\phi_2(t),\phi_3(t))=(1,t,t^2)\cdot U^{-1}=(1,t-1,t^2-t+1).$$

Then, $M(s)=[M^1,M^2,M^3,M^4]$, with

$$M^1=\begin{pmatrix}\frac{1}{3453}(160s^3-44s^2-63s-69)\\\frac{1}{3453}(-160s^2-116s+587)\\\frac{1}{3453}(320s^4-88s^3-46s^2)-\frac{1}{6}\\0\end{pmatrix},\ M^2=\begin{pmatrix}0\\0\\\frac{1}{6}\\0\end{pmatrix},$$

$$M^3=\begin{pmatrix}\frac{1}{12}s\\-\frac{1}{24}\\\frac{1}{6}s^2\\-\frac{1}{24}(s^2+2s-6)\end{pmatrix},\ M^4=\begin{pmatrix}-\frac{1}{2}\\-s\\0\\s^3+\frac{1}{2}(s+1)\end{pmatrix}.$$

We have then

$$U \cdot A(s) \cdot M(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_{31}(s) & 1 & s^2 + \frac{1}{4} & 0 \end{pmatrix}$$

with $a_{31}(s) = \frac{1}{1151}(640s^4 - 176s^3 + 68s^2 - 44s - 1174)$.

In this case $g'(s) = 1$ and $g''(s) = s^2 + \frac{1}{4}$. A μ–basis of $(\phi_1(t), \phi_2(t), \phi_3(t))$ is $\alpha_1(t) = (t-1, -1, 0)$ and $\alpha_2(t) = (1, t, -1)$. Then, Identity (17) is

$$\begin{pmatrix} h_1(s,t) \\ h_2(s,t) \\ h_3(s,t) \end{pmatrix} = \begin{pmatrix} t-1 & 1 \\ -1 & t \\ \frac{\alpha_{31}(s,t)}{g''(s)} & \frac{\alpha_{32}(s,t)}{g''(s)} \end{pmatrix} \cdot \begin{pmatrix} v_1(s,t) \\ v_2(s,t) \end{pmatrix},$$

with $\alpha_{31}(s,t) = \frac{4}{1151}(640s^4 - 176s^3 + 68s^2 - 2(320s^4 - 88s^3 + 34s^2 - 22s - 587)t - 44s - 23)$, $\alpha_{32}(s,t) = \frac{4}{1151}(640s^4 - 176s^3 + 68s^2 - 44s + 1151t - 23)$ and $g''(s) = (4s^2 + 1)$.

A μ–basis of $(\alpha_{31}(s,t), \alpha_{32}(s,t), g''(s))$ is

$$X_1(s,t) = (x_{11}(s,t), x_{12}(s,t), x_{13}(s,t))$$
$$= (0, -s^2 - \frac{1}{4}, -\frac{1}{1151}(160s^2 - 44s - 23)(4s^2+1) - t)$$

and

$$X_2(s,t) = (x_{21}(s,t), x_{22}(s,t), x_{23}(s,t))$$
$$= (-\frac{1324801}{409600}, \frac{1151}{640}s^4 - \frac{12661}{25600}s^3 + \frac{19567}{102400}s^2 - \frac{12661}{102400}s - \frac{675637}{204800},$$
$$s^6 - \frac{11}{20}s^5 + \frac{61}{1600}s^4 - \frac{187}{3200}s^3 - \frac{827}{25600}s^2 + \frac{253}{12800}s + \frac{529}{102400}).$$

Hence,

$$\begin{pmatrix} L_{11}(s,t) \\ L_{12}(s,t) \\ L_{13}(s,t) \end{pmatrix} = \begin{pmatrix} t-1 & 1 \\ -1 & t \\ \frac{\alpha_{31}(s,t)}{g''(s)} & \frac{\alpha_{32}(s,t)}{g''(s)} \end{pmatrix} \cdot \begin{pmatrix} x_{11}(s,t) \\ x_{12}(s,t) \end{pmatrix}$$
$$= \begin{pmatrix} -s^2 - \frac{1}{4} \\ -\frac{1}{4}(4s^2+1)t \\ \frac{1}{1151}(640s^4 - 176s^3 + 68s^2 - 44s + t - 23) \end{pmatrix},$$

and

$$\begin{pmatrix} L_{21}(s,t) \\ L_{22}(s,t) \\ L_{23}(s,t) \end{pmatrix} = \begin{pmatrix} t-1 & 1 \\ -1 & t \\ \frac{\alpha_{31}(s,t)}{g''(s)} & \frac{\alpha_{32}(s,t)}{g''(s)} \end{pmatrix} \cdot \begin{pmatrix} x_{21}(s,t) \\ x_{22}(s,t) \end{pmatrix} =$$
$$= \begin{pmatrix} \frac{1151}{640}s^4 - \frac{12661}{25600}s^3 + \frac{19567}{102400}s^2 - \frac{12661}{102400}s - \frac{1324801}{409600}t - \frac{26473}{409600} \\ \frac{1151}{204800}(320s^4 - 88s^3 + 34s^2 - 22s - 587)t + \frac{1324801}{409600} \\ -s^6 + \frac{11}{20}s^5 - \frac{61}{1600}s^4 + \frac{187}{3200}s^3 + \frac{827}{25600}s^2 - \frac{253}{12800}s - \frac{529}{102400} \end{pmatrix}.$$

Finally, we have

$$L_1(s,t) = L_{11}M^1 + L_{12}M^2 + L_{13}M^3 = \begin{pmatrix} p_{11}(s,t) \\ p_{12}(s,t) \\ p_{13}(s,t) \\ p_{14}(s,t) \end{pmatrix},$$

with $p_{11}(s,t) = \frac{1}{3453}(40s^3 + 69s^2 + 10s) + \frac{1}{12}st + \frac{23}{4604}$, $p_{12}(s,t) = \frac{1}{6906}(160s^4 + 276s^3 - 1111s^2 + 69s) - \frac{1}{24}(t+1)$, $p_{13}(s,t) = \frac{1}{24}(4s^2 - t + 1)$ and $p_{14}(s,t) = -\frac{80}{3453}s^6 - \frac{46}{1151}s^5 + \frac{1031}{6906}s^4 - \frac{287}{6906}s^3 + \frac{173}{9208}s^2 - \frac{1}{24}(s^2 + 2s - 6)t - \frac{109}{13812}s - \frac{23}{4604}$, and

$$L_2(s,t) = L_{21}M^1 + L_{22}M^2 + L_{23}M^3 = \begin{pmatrix} p_{21}(s,t) \\ p_{22}(s,t) \\ p_{23}(s,t) \\ p_{24}(s,t) \end{pmatrix},$$

with $p_{21}(s,t) = -\frac{1}{48}s^5 - \frac{29}{960}s^4 + \frac{589}{76800}s^3 - \frac{733}{307200}s^2 - \frac{1151}{1228800}(160s^3 - 44s^2 - 63s - 69)t + \frac{989}{307200}s + \frac{529}{409600}$, $p_{22}(s,t) = -\frac{1}{24}s^6 - \frac{29}{480}s^5 + \frac{4033}{12800}s^4 - \frac{6697}{76800}s^3 + \frac{7841}{204800}s^2 + \frac{1151}{1228800}(160s^2 + 116s - 587)t - \frac{5537}{307200}s - \frac{26473}{2457600}$, $p_{23}(s,t) = -\frac{1151}{3840}s^4 + \frac{12661}{153600}s^3 - \frac{19567}{614400}s^2 + \frac{1151}{2457600}(160s^2 - 44s - 23)t + \frac{12661}{614400}s + \frac{675637}{1228800}$ $p_{24}(s,t) = \frac{1}{24}s^8 + \frac{29}{480}s^7 - \frac{11299}{38400}s^6 + \frac{3539}{25600}s^5 - \frac{129}{8192}s^4 + \frac{1957}{153600}s^3 + \frac{977}{98304}s^2 - \frac{5543}{1228800}s - \frac{529}{409600}$.

A μ– basis of the input is then $\{L_1(s,t), L_2(s,t), M^4\}$ as above. Moreover, we can retrieve the parametization back from the μ–multiplied by a constant, as in (3): ,

$$L_1(s,t) \wedge L_2(s,t) \wedge M^4 = \left(\begin{vmatrix} p_{12}(s,t) & p_{13}(s,t) & p_{14}(s,t) \\ p_{22}(s,t) & p_{23}(s,t) & p_{24}(s,t) \\ m_{42} & m_{43} & m_{44} \end{vmatrix}, \right.$$
$$-\begin{vmatrix} p_{11}(s,t) & p_{13}(s,t) & p_{14}(s,t) \\ p_{21}(s,t) & p_{23}(s,t) & p_{24}(s,t) \\ m_{41} & m_{43} & m_{44} \end{vmatrix}, \begin{vmatrix} p_{11}(s,t) & p_{12}(s,t) & p_{14}(s,t) \\ p_{21}(s,t) & p_{22}(s,t) & p_{24}(s,t) \\ m_{41} & m_{42} & m_{44} \end{vmatrix},$$
$$\left. -\begin{vmatrix} p_{11}(s,t) & p_{12}(s,t) & p_{13}(s,t) \\ p_{21}(s,t) & p_{22}(s,t) & p_{23}(s,t) \\ m_{41} & m_{42} & m_{43} \end{vmatrix} \right) = \frac{1324801}{117964800}P(s,t).$$

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