Computing Residues in Toric Varieties

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Motivation: univariate interpolation

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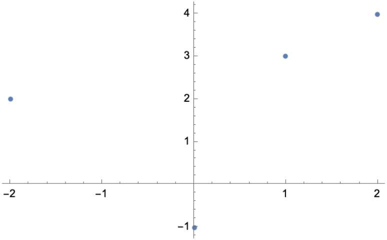
$$(x_1, y_1), \ldots, (x_N, y_N) \subset \mathbb{K}^2 \quad x_i \neq x_j \quad i \neq j$$



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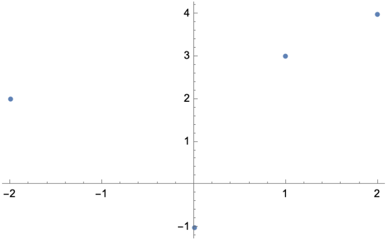
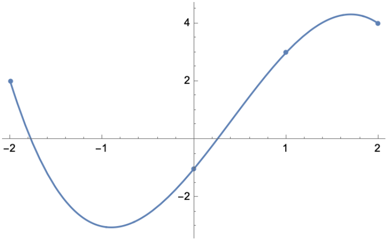


Find the polynomial of smallest degree passing through these points

Univariate interpolation



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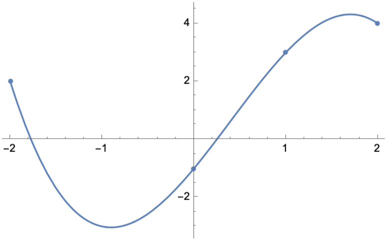

$$f(x) = \prod_{i=1}^{N} (x - x_i)$$

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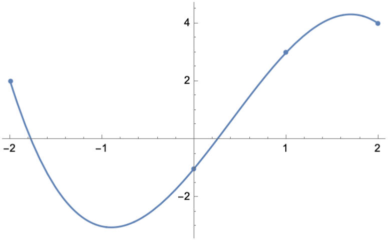

$$f(x) = \prod_{i=1}^{N}(x - x_i)$$

$$\boxed{p(x) := \sum_{i=1}^{N} y_i \,\frac{\prod_{j\neq i}(x - x_j)}{f'(x_i)}}$$

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$$= \sum_{i=1}^{N} \frac{p(x_i)}{f'(x_i)} = \mathrm{Res}_g\left(\frac{p(x)}{f(x)} dx\right)$$

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$$\sum_{i=0}^{N-2} \lambda_i x^i + \frac{\mathrm{Res_g}\left(\frac{\mathrm{p(x)}}{\mathrm{f(x)}}\mathrm{dx}\right)}{\mathrm{N}} f'(x)$$

with $\lambda_0, \ldots, \lambda_{N-2} \in \mathbb{K}$

Complex local residues

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meromorphic in $\xi \in \mathbb{C}$

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The (local) residue of $\varphi(z)\,dz$ at ξ

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The global (rational) residue

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Can be defined algebraically (over any field)

Properties

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- $\operatorname{Res}_g\left(\frac{p(z)}{f(z)}dz\right) = 0$ if $f(z) \mid p(z)$

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Euler-Jacobi Vanishing Theorem

If $\deg(p(z)) < \deg(f(z)) - 1$ then $\mathrm{Res}_g\left(\frac{p(z)}{f(z)}dz\right) = 0$

Consequences

$$p(z) = h_0 + h_1 z + \ldots + h_{d-1} z^{d-1} \bmod f(z)$$

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$$\mathrm{Res}_g\left(\frac{z^i\, p(z)}{f(z)} dz\right) = 0,\ i = 0, 1, \ldots, d-1$$

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$$\mathrm{Res}_f[\omega](\cdot) := \lim_{\delta \to 0^+} \int_{T_{f,\delta}} \omega \wedge \cdot$$

residual currents

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with equality generically

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- $\mathrm{Res}_\xi\left(\frac{p\,dz}{f_1\ldots f_n}\right) = \sum_{j=1}^{N} c_j \frac{\partial^{\mathrm{a_j}} p}{\partial z_1^{a_{j1}} \ldots \partial z_1^{a_{jn}}}(\xi)$

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- Res_g is linear in p
- $\mathrm{Res}_g\left(\frac{J_f}{f_1\ldots f_n}\right) = \#V(f)$
- If the input has coefficients in $k \subset \mathbb{K}$, $\mathrm{Res}_g\left(\frac{p\ \ dz}{f_1\ldots f_n}\right) \in k$
- $\mathrm{Res}_g\left(\frac{p\ \ dz}{f_1\ldots f_n}\right)$ is computable

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$$p = \sum_{i=1}^{n} f_i\, g_i \iff \mathrm{Res}_g\left(\frac{b_j\, p\, dz}{f_1 \ldots f_n}\right) = 0 \; j = 1, \ldots, \ell$$

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with $(b_1, \ldots, b_\ell)$ a basis of

$$\mathbb{K}[z_1, \ldots, z_n]/\langle f_1, \ldots, f_n \rangle$$

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$\Longrightarrow$ effective membership tests

Dickenstein-Sessa 1990

Euler-Jacobi vanishing Theorem

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THEOREMATA NOVA ALGEBRAICA CIRCA SYSTEMA DUARUM AEQUATIONUM INTER DUAS VARIABILES PROPOSITARUM.

1.

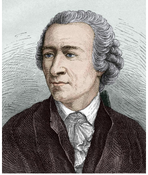
E theorematis, quae in elementis algebraicis traduntur, vix extat aliud magis utile in aequationibus maxime diversis, quam notum illud:

„Designante X functionem ipsius x rationalem integram, fieri

$$\Sigma\left(\frac{U}{\frac{dX}{dx}}\right)=0,$$

„si quidem extendatur summa ad omnes radices x aequationis $X=0$,

Jacobi 1835



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Multidimensional Euler-Jacobi

If $\#V(f) = d_1 \ldots d_n$ and

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Multidimensional Euler-Jacobi

If $\#V(f) = d_1 \ldots d_n$ and

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then

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Multivariate Interpolation

If $\# V(f) = d_1 \ldots d_n$
any $p \in \mathbb{K}[z_1, \ldots, z_n]$ can be
expressed mod $\langle f_1, \ldots, f_n \rangle$
as

$$p_0 + \frac{\mathrm{Res}_g\left(\frac{h\,dz}{f_1 \ldots f_n}\right)}{d_1 \ldots d_n} J_f$$

with $\deg(p_0) < d_1 + \ldots + d_n - n$

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$\exists p \in \mathbb{K}[z_1, \ldots, z_n]$ such that

$p(\xi_j) = y_j$ with

$$\deg(p) < d_1 + \ldots + d_n - n \iff$$

$$\sum_{j=1}^{d} \frac{y_j}{J_f(\xi_j)} = \mathrm{Res}_g\left(\frac{p\,dz}{f_1 \ldots f_n}\right) = 0$$

A "negative" consequence

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A “negative” consequence

If $\# V(f) = d_1 \ldots d_n$
you will never find a $\mathbb{K}$-basis of
$\mathbb{K}[z_1, \ldots, z_n]/\langle f_1, \ldots, f_n \rangle$ made of
polynomials of degree
$< d_1 + \ldots + d_n - n$

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Compactifications and Stokes Theorem

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$

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Algebraic "compactifications"

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$$f_1^h, \ldots, f_n^h \in \mathbb{K}[z_0, z_1, \ldots, z_n]$$

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Homogeneous residues

Homogeneous residues

The map

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If $\deg(p) < d_1+\ldots+d_n-n$

$z_0 \,|\, p^h \implies$ Euler-Jacobi

Limitations

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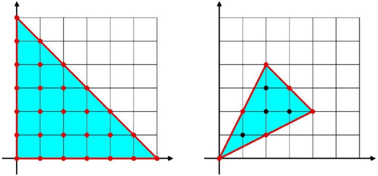
$$\#V(f) = d_1 \ldots d_n$$

is a very **strong** condition



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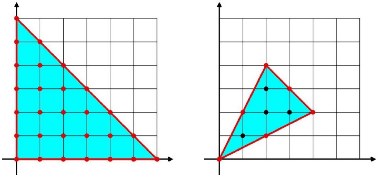

"Sparse" philosophy: $f = \sum_{a \in \mathcal{A}} c_a t^a$

$\mathcal{A} \subset \mathbb{Z}^n$ finite

Newton polytopes

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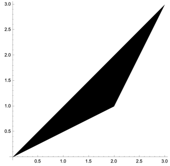

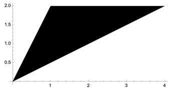

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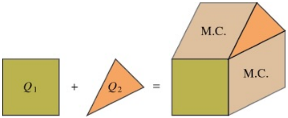
$$\# V(\boldsymbol{f}) \leq MV(N(f_1), \ldots, N(f_n))$$

with generic equality



the mixed volume

Residues in the torus

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local and global residues are defined as usual:

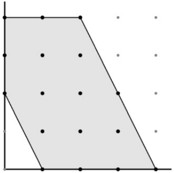
$$\mathrm{Res}_g\left(\frac{h\,dz}{f_1 \ldots f_n}\right) := \sum_{\xi \in V(f)} \mathrm{Res}_\xi\left(\frac{h\,dz}{f_1 \ldots f_n}\right)$$

Computing residues

Computing residues

Easy in the generic case

$$\#V(f) = \mathrm{MV}(N(f_1), \ldots, N(f_n))$$



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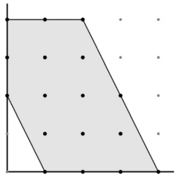


Cattani-Dickenstein-Sturmfels 1992

Cattani-Dickenstein 1994

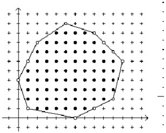
Euler-Jacobi in the torus

Euler-Jacobi in the torus

$$\text{If } \#V(f) = \mathrm{MV}(N(f_1), \ldots, N(f_n))$$

and

$$N(t_1 \ldots t_n\, p) \subset (N(f_1) + \ldots + N(f_n))^{\circ}$$



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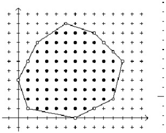

$$\Longrightarrow \boxed{\mathrm{Res}_g\left(\frac{p\, dz}{f_1 \ldots f_n}\right) = 0}$$

Sparse Interpolation

If $\# V(f) = \mathrm{MV}(N(f_1), \ldots, N(f_n))$

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$$= p_0 + \frac{\mathrm{Res}_g\left(\frac{p\,dz}{f_1 \ldots f_n}\right)}{\#V(f)} J_f$$

with

$$N(t_1 \ldots t_n\, p_0) \subset (N(f_1) + \ldots + N(f_n))^\circ$$

Sparse Interpolation II

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$$V(f) = \{\xi_1, \ldots, \xi_m\} \subset (\mathbb{K}^\times)^n$$

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$$N(t_1 \ldots t_n p) \subset$$

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$$N(t_1\ldots t_n\,p)\subset$$

$$(N(f_1)+\ldots+N(f_n))^\circ\iff$$

$$\sum_{j=1}^{\mathrm{m}}\frac{y_j}{J_f(\xi_j)}=\mathrm{Res}_g\left(\frac{p\,dz}{f_1\ldots f_n}\right)=0$$

Soprunov 07

Sparse compactifications

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$$\mathbb{C}^n \hookrightarrow \mathbb{P}^n$$

$$(\mathbb{C}^\times)^n \hookrightarrow ??$$

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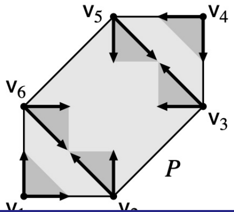
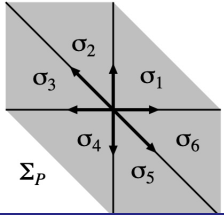
Poltyope $P \leftrightarrow$ Fan $\Sigma_P \leftrightarrow$ Toric Variety X_P

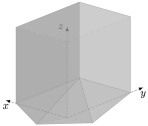




“Homogeneous” polynomials

“Homogeneous” polynomials

$P := N(f_1) + N(f_2) + \dots N(f_n)$ The **Cox ring** is $\mathbb{K}[x_1, x_2, \dots, x_N]$ (one variable per facet)



and a “homogenization” rule $f_i \mapsto f_i^h \in \mathbb{K}[x_1, \dots, x_N]$

BKK bound and Toric residues

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$$\# V(f_1, \ldots, f_n) = \mathrm{m} \iff V_{X_P}(x^\alpha, f_1^h, \ldots, f_n^h) = \emptyset$$

BKK bound and Toric residues

$$\#V(f_1,\ldots,f_n)=\mathrm{m} \iff$$
$$V_{X_P}(x^\alpha,f_1^h,\ldots,f_n^h)=\emptyset$$

There is a trace map

$$\left(\mathbb{K}[x_1,\ldots,x_N]/\langle x^\alpha,f_1^h,\ldots,f_n^h\rangle\right)_\rho\to\mathbb{K}$$

giving the global residue

Cox 96

Cattani, Cox, Dickenstein 97

Toric jacobians?

$$J_{x^\alpha, f_1^h, \ldots, f_n^h} \in \mathbb{K}[x_1, \ldots, x_N]$$

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Cox 96

Cattani, Cox, Dickenstein 97

"Combinatorial" jacobian

Cattani, Cox, Dickenstein 97

D, Khetan 2003

Khetan, Soprunov 05

Codimension one?

$$\dim\left(\mathbb{K}[x_1,\ldots,x_N]/\langle x^\alpha, f_1^h,\ldots,f_n^h\rangle\right)_\rho = 1$$

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Combinatorial characterization given in

Cox, Dickenstein 05

Current work

w/ A. Dickenstein & I. Soprunov

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Is it true that

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Is it true that

Euler-Jacobi $\Longleftrightarrow$ $\#V(f) = \mathrm{m}$?

Current work

w/ A. Dickenstein & I. Soprunov

Is it true that

Euler-Jacobi $\iff \#V(f) = \mathrm{m}$?

$$J_{f_0^h,\dots,f_n^h} \in \langle f_0^h, \dots, f_n^h \rangle \iff$$

$$V(f_h^0, \dots, f_n^h) \neq \emptyset?$$

Euler-Jacobi

Euler-Jacobi

Is it true that if $\#V(f) < \text{m?}$ then $\exists p:\ N(t_1 \ldots t_n\, p) \subset (N(f_1) + \ldots + N(f_n))^{\circ}$ such that $\mathrm{Res}_g\left(\frac{p}{f_1 \ldots f_n}\right) \neq 0$?

Answer: look at the codimension

$$\dim \left(\mathbb{K}[x_1, \ldots, x_N]/\langle x^\alpha, f_1^h, \ldots, f_n^h\rangle\right)_\rho = 1$$

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$$\left\{\begin{array}{lcl} f_1 & = & 2 - t_1 + t_1^2 + t_2 + 2t_1t_2 + t_2^2 \\ f_2 & = & 1 + t_1 + t_2 \\ f_3 & = & 2 - 3t_3 + t_3^2 \end{array}\right.$$

Answer: look at the codimension

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$$N(f_1) = 2\Delta_2 \times \{0\}$$
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fails the codimension 1 property

Only obstruction (so far):

D-Dickenstein 2025

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Assume that $N(f_1),\ldots,N(f_n)$ is **“completable”**.

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Assume that $N(f_1),\ldots,N(f_n)$ is **"completable"**. If $\#V(\boldsymbol{f}) < \mathrm{m}$ and $V_X(\boldsymbol{f}^h)$ is finite

Only obstruction (so far):

D-Dickenstein 2025

Assume that $N(f_1), \ldots, N(f_n)$ is **"completable"**. If $\#V(f) < \mathrm{m}$ and $V_X(f^h)$ is finite then $\exists p: \; N(t_1 \ldots t_n\, p) \subset (N(f_1) + \ldots + N(f_n))^{\circ}$ such that

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Second question

D-Dickenstein 2025

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If generically

$$\dim \left(\mathbb{K}[x_1, \ldots, x_{n+r}] / \langle F_0^h, F_1^h, \ldots, F_n^h \rangle \right)_\rho = 1$$

Second question

D-Dickenstein 2025

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$$\dim\left(\mathbb{K}[x_1,\ldots,x_{n+r}]/\langle F_0^h, F_1^h,\ldots,F_n^h\rangle\right)_\rho = 1 \text{ but}$$

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D-Dickenstein 2025

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and $V_X(f_1^h, \ldots, f_n^h)$ is finite

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D-Dickenstein 2025

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and $V_X(f_1^h,\ldots,f_n^h)$ is finite then

$$J_{f_0^h,\ldots,f_n^h} \in \langle f_0^h,\ldots,f_n^h\rangle$$

In progress (Dickenstein & Soprunov)

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- Description of $\left(\mathbb{K}[x_1, \ldots, x_N]/\langle f_0^h, f_1^h, \ldots, f_n^h\rangle\right)_\rho$ when the codimension is larger than 1

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- Computing "canonical" elements of residue 1
- Computing "canonical" elements of residue 0

To learn more

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- Cox, D.; Little, J.; Schenck, H. *Toric varieties.* Graduate Studies in Maths 2011
- Cattani, E.; Cox, D.; Dickenstein, A. *Residues in toric varieties.* Compositio Math. 97
- Cox, D.; Dickenstein, A. *Codimension theorems for complete toric varieties.* Proceedings AMS 05
- Sombra, M.; Yger, A. Bounds for multivariate residues and for the polynomials in the elimination theorem. Moscow Math J. 2021

Thanks!

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