An Algorithmic Approach to Codimension Theorems for Complete Toric Varieties

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UNIVERSITAT DE BARCELONA

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- $\dim_{\mathbb{C}} \left(\mathbb{C}[x_0, \dots, x_n]/\langle f_0, \dots, f_n\rangle\right)_\rho = 1$
- The residue map $\mathrm{res}_{\boldsymbol{f}} \left(\mathbb{C}[x_0, \dots, x_n]/\langle f_0, \dots, f_n\rangle\right)_\rho \to \mathbb{C}$ is an isomorphism

What is the residue map?

Algebraic definition

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$$\begin{array}{rlcl} \mathrm{res}_f : & (\mathbb{C}[x_0, \ldots, x_n]/\langle f_0, \ldots, f_n \rangle)_\rho & \to & \mathbb{C} \\ & [J_f] & \mapsto & d_0 \ldots d_n \end{array}$$

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- $0 \to \mathbb{Z}^n \to \mathbb{Z}^N \to A_{n-1}(X) \to 0$

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If all the α_i's are ample

Cattani-Cox-Dickenstein 97 – Cattani-Dickenstein 97

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Cox-Dickenstein 05

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The family $\{P_{\alpha_i}\}$is *essential* $\Longleftrightarrow$

$$\forall J \subsetneq \{0, \ldots, n\}$$

$$\dim\left(\sum_{j \in J} P_{\alpha_j}\right) \geq j$$

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Cox 96 - Cattani-Cox-Dickenstein 97

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Cox 96 - Cattani-Cox-Dickenstein 97

It is not identically zero if P_{α_i} is a lattice polytope $\forall i$ and $\{P_{\alpha_i}\}$is essential

Khetan-Soprunov 05

Elements of non-zero residue?

- Toric jacobians

  Cox 96, Cattani-Dickenstein 97

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  Cox 96, Cattani-Dickenstein 97

- Discrete/combinatorial versions

  Cattani-Cox-Dickenstein 97 – **D**-Khetan 05 – Khetan-Soprunov 05

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- Δ has residue 1 wrt $\{f_i,\ i \notin J\}$

Example

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$$\{y_0^{l-i-2} y_1^{l} J_x(f_0, f_2),\ x_0^{k-2-i} x_1^{i} J_y(f_1, f_2)\}_{i,l}$$

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D-Dickenstein-Soprunov

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and also

$$1+\prod_{\dim(P_j)=1}\left(\#(P_j\cap\mathbb{Z}^n)-1\right)$$

D-Dickenstein-Soprunov

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- $\Delta := \sum_a c_a \, x^a$

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- Compute bases of $\left(\mathbb{C}[x_{I_j}]/\langle F_i,\ i \in I_j\rangle\right)_{\rho_{I_j}}$
- Describe $\dim_{\mathbb{C}}\left(\mathbb{C}[x_1,\dots,x_N]/\langle f_0,\dots,f_n\rangle\right)_\rho$ in combinatorial terms

Last but not least

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MATEMAX

Computational Algebra, Algebraic Geometry and Applications II

Buenos Aires, Argentina, December 15-17 2025

Celebrating Alicia Dickenstein's contributions to Math and beyond

TOPICS OF THIS CONFERENCE INCLUDE BUT ARE NOT LIMITED TO

Algebraic Geometry, Toric Geometry, Tropical Geometry, and their Applications to Biology

Thanks!

SIAM 2025 Conference on Applied Algebraic Geometry

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