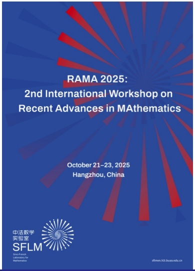
# Towards an effective Pourchet's Theorem

Carlos D'Andrea

Hangzhou, October 23rd 2025

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# BCN Comp Algebra Seminar

- Ana Belén de Felipe – UPC
- Eulàlia Montoro – UB
- Joel Hurtado – UPC
- Teresa Cortadellas Benítez – URL

# “The” Question

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Given a polynomial

$$f(x_1, \ldots, x_n) \in \mathbb{R}/\mathbb{Q}[x_1, \ldots, x_n]$$

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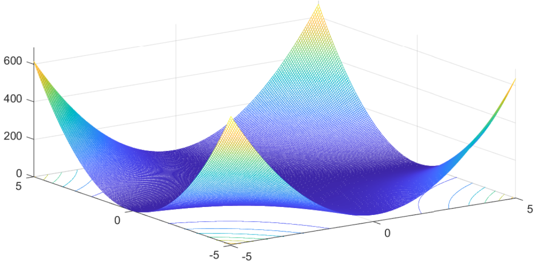
How can we verify/certify if $f \geq 0$?



# Univariate case

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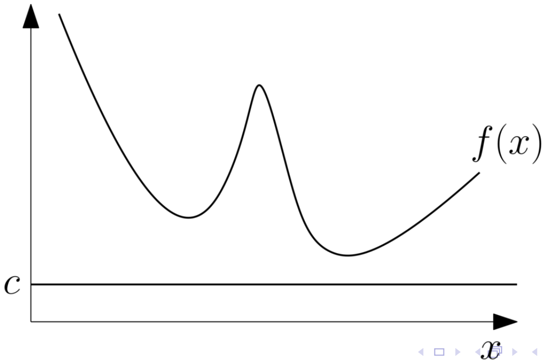
$$f(x) \geq 0 \; \forall x \in \mathbb{R} \iff f(x) = f_1(x)^2 + f_2(x)^2$$



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$$f_1(x)^2 + f_2(x)^2 + f_3(x)^2 + f_4(x)^2 + f_5(x)^2$$

Pourchet – 1971

# Five is sharp

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$$x^2 + 7 = x^2 + 2^2 + 1^2 + 1^2 + 1^2$$

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  $f(x) = f_1(x)^2 + \ldots + f_5(x)^2$

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for all $p \in \{2, 3, 5, \ldots, \} \cup \{\infty\}$

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- Local-global principle
- Highly non-algorithmic

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**Theorem (Easy)**

$f(x) \geq 0 \iff y_1^2 + y_2^2 = f(x)$ can be solved in $\mathbb{R}[x]$

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$$\boxed{(u^2+v^2)\cdot(w^2+z^2) = \alpha^2 + \beta^2}$$

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five squares is enough if $p = 2$

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$$(x_1^2+x_2^2+x_3^2+x_4^2)(y_1^2+y_2^2+y_3^2+y_4^2) \\ =(z_1^2+z_2^2+z_3^2+z_4^2)$$

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**Theorem (Pourchet, 71)**

$f(x) = f_1^2 + f_2^2 + f_3^2 + f_4^2$ in $K[x] \iff$

- $\mathrm{lc}(f)$ is a so4s in $K$, and

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**Theorem (Pourchet, 71)**

$f(x) = f_1^2 + f_2^2 + f_3^2 + f_4^2$ in $K[x] \iff$

- $\mathrm{lc}(f)$ is a so4s in $K$, and
- for all prime divisor $p(x)$ of $f(x)$ of odd multiplicity, there is a nontrivial solution of $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$ in $K[x]/(p(x))$

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*Let $f \in \mathbb{Q}[x] \setminus \{0\}$. TFAE:*

1. *$f$ is a so4s in $\mathbb{Q}[x]$*

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**Theorem (Pourchet, 71)**

*Let* $f \in \mathbb{Q}[x] \setminus \{0\}$. *TFAE:*

1. $f$ *is a so4s in* $\mathbb{Q}[x]$
2. $f > 0$ *and in* $\mathbb{Q}_2[x]$ *every prime factor of odd multiplicity has even degree*

# Useful

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$$x^2 + 7 = (x - \alpha) \cdot (x + \alpha) \text{ in } \mathbb{Q}_2[x]$$

$$\Longrightarrow \text{ it is not a so4s in } \mathbb{Q}[x]$$

$$u \in \mathbb{Q}_2 \text{ is a square } \Longleftrightarrow$$

$$u = 2^{2a}(8b + 1),\ a \in \mathbb{Z},\ b \in \mathbb{Q}_2$$

# Algorithmic approach

**Pourchet's theorem in action: decomposing univariate nonnegative polynomials as sums of five squares**

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ISSAC 2023

# Sums of 2 squares

**Algorithm 1** Computing a decomposition of a polynomial as a sum of two squares

**Input:** A polynomial $f \in \mathbb{Q}[x]$, which is a priori known to be a sum of two squares in $\mathbb{Q}[x]$.

**Output:** Polynomials $a, b \in \mathbb{Q}[x]$ such that $a^2 + b^2 = f$.

1: Construct the quadratic field extension $\mathbb{Q}(i)/\mathbb{Q}$.

2: Solve the norm equation

$$\mathrm{lc}(f) = N_{\mathbb{Q}(i)/\mathbb{Q}}(x)$$

and denote a solution by $a + bi \in \mathbb{Q}(i)$.

3: Factor $f$ into a product of monic irreducible polynomials

$$f = \mathrm{lc}(f) \cdot p_1^{e_1} \cdots p_k^{e_k}.$$

4: **for** every factor $p_j$, such that the corresponding exponent $e_j$ is odd **do**

5: Factor $p_j$ over $\mathbb{Q}(i)$ into a product $p_j = g_j \cdot h_j$ with $g_j, h_j \in \mathbb{Q}(i)[x]$.

6: Set

$$a_j := \frac{1}{2} \cdot (g_j + h_j), \qquad b_j := \frac{1}{2i} \cdot (g_j - h_j).$$

7: Update $a$ and $b$ setting:

$$a := aa_j + bb_j \quad \text{and} \quad b := ab_j - ba_j.$$

8: Update $a$ and $b$ setting:

$$a := a \cdot \prod_{j \le k} p_j^{2\lfloor e_j/2 \rfloor} \quad \text{and} \quad b := b \cdot \prod_{j \le k} p_j^{2\lfloor e_j/2 \rfloor}.$$

9: **return** $a, b$.

# Sums of 3 or 4 squares

**Algorithm 3** Initial solution: modular sum of squares

**Input:** An irreducible polynomial $f \in \mathbb{Q}[x]$, which is a priori known to be a sum of 3 or 4 squares.

**Output:** Polynomials $h$ and $g_1, \dots, g_4$ in $\mathbb{Q}[x]$, such that $\deg h \leq \deg f - 2$ and $fh = g_1^2 + \cdots + g_4^2$.

1: Construct the number fields:

$$K := \mathbb{Q}[x]/(f) \qquad \text{and} \qquad L := K(i).$$

2: Solve the norm equation

$$-1 = N_{L/K}(x)$$

and denote the solution by $\xi = \overline{g}_1 + \overline{g}_2 i$, where $g_1, g_2 \in \mathbb{Q}[x]$ are polynomials of degree strictly less than $\deg f$ and $\overline{g}_j$ denotes the image of $g_j$ under the canonical epimorphism $\mathbb{Q}[x] \twoheadrightarrow K$.

3: Set $g_3 := 1$, $g_4 := 0$ and let $h := {(g_1^2 + \cdots + g_4^2)}/{f}$.

4: **return** $h, g_1, g_2, g_3, g_4$.

# How do you tackle 5 polynomials?

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$$f(x) > 0 \ldots f(x) - \left(\frac{1}{2^{\ell}}\right)^2 > 0$$

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for $\ell \gg 0$

$$f(x) - \left(\frac{1}{2^\ell}\right)^2 = f_1^2 + f_2^2 + f_3^2 + f_4^2??$$

# Algorithm 6

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**Algorithm 6** Reduction to a sum of 4 squares: odd valuation case

**Input:** A positive square-free polynomial $f = c_0 + c_1 x + \cdots + c_d x^d \in \mathbb{Q}[x]$. The 2-adic valuations of the coefficients of $f$ are $k_j := \operatorname{ord}_2 c_j$ for $0 \le j \le d$. Ensure $k_d$ is odd. It is assumed that $f$ is not a sum of 4 squares.

**Output:** A polynomial $h \in \mathbb{Q}[x]$ such that $f - h^2$ is a sum of 4 (or fewer) squares.

Find a positive number $\varepsilon$ such that

$$\varepsilon < \inf\{f(x) \mid x \in \mathbb{R}\}.$$

Set $l_1 := \lceil -1/2 \cdot \lg \varepsilon \rceil$.

Set $l_2 := \lceil -k_0/2 \rceil + 1$.

Set

$$l_3 := \left\lceil \max\left\{ \frac{jk_d - dk_j}{2d - 2j} \mid 0 < j < d \right\} \right\rceil.$$

Initialize $l := \max\{l_1, l_2, l_3\}$.

**while** $\gcd(d, 2l + k_d) \neq 1$ **do**

  $l := l + 1$.

**return** $h := 2^{-l}$.

# Sum of 6 squares

**Algorithm 8** Decomposition of a nonnegative univariate rational polynomial into a sum of 6 squares

**Input:** A nonnegative polynomial $f \in \mathbb{Q}[x]$.
**Output:** Polynomials $f_1, \dots, f_6 \in \mathbb{Q}[x]$ such that $f_1^2 + \cdots + f_6^2 = f$.

1: **if** $f$ is a square **then**
2: **return** $f_1 := \sqrt{f}, f_2 := \cdots f_6 := 0$.
3: **if** $f$ is a sum of 2 squares {Use Observation 8 to check it} **then**
4: Execute Algorithm 1 to obtain $f_1, f_2 \in \mathbb{Q}[x]$ such that $f_1^2 + f_2^2 = f$.
5: **return** $f_1, f_2$ and $f_3 := \cdots f_6 := 0$.
6: **if** $f$ is a sum of 4 squares {Use [36, Theorem 17.2] to check it} **then**
7: Execute Algorithm 5, to obtain $f_1, \dots, f_4 \in \mathbb{Q}[x]$ such that $f_1^2 + \cdots + f_4^2 = f$.
8: **return** $f_1, \dots, f_4$ and $f_5 := f_6 := 0$
9: Compute the square-free decomposition of $f = g \cdot h^2$, where $g, h \in \mathbb{Q}[x]$ and $g$ is square-free.
10: Execute Algorithm 7 with $g$ as an input to obtain $g_1, g_2 \in \mathbb{Q}[x]$ such that $g - g_1^2 - g_2^2$ is a sum of 4 squares in $\mathbb{Q}[x]$.
11: Execute Algorithm 5 to decompose $g - g_1^2 - g_2^2$ into a sum of 4 squares in $\mathbb{Q}[x]$. Denote the output by $g_3, \dots, g_6$.
12: **return** $f_1 := g_1 h, \dots, f_6 := g_6 h$.

# Conjectural Algorithm

**Algorithm 9** Reduction to a sum of 4 squares

**Input:** A positive square-free polynomial $f = c_0 + c_1 x + \cdots + c_d x^d \in \mathbb{Q}[x]$.

**Output:** A polynomial $h \in \mathbb{Q}[x]$ such that $f - h^2$ is a sum of 4 (or fewer) squares.

```
if f is a sum of 4 squares then
    return h := 0.
Set f_* := c_d + c_{d-1} x + ... + c_0 x^d.
Find a positive number ε such that
    ε < inf{ f(x) | x ∈ ℝ }   and   ε < inf{ f_*(x) | x ∈ ℝ }.
Initialize l := ⌈−1/2 · lg ε⌉.
while True do
    if f − 2^{−2l} is irreducible in ℚ_2[x] then
        return h := 2^{−l}.
    if f − 2^{−2l} x^d is irreducible in ℚ_2[x] then
        return h := 2^{−l} x^{d/2}.
    l := l + 1.
```

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(C**D**DHM)

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- The conjectural algorithm works if $\deg(f(x)) = 4k$

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- The conjectural algorithm works if $\deg(f(x)) = 4k$
- Fails for this family:

$$\boxed{f_{k,N}(x) = \frac{4x^{2(2k+1)} + x^{2k+1} + 4}{N^2}}$$

$k = 0, 1, \ldots, \; N \in \mathbb{N}$ odd, $N > 64$

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**Theorem**

If $f(x) \in \mathbb{Q}[x]$ of degree $d = 2(2k+1),\ k \in \mathbb{N}, \ell \in \mathbb{N}$ such that $f(t) - \frac{1}{2^{2\ell}}(t^2+t+1)^{2k}t^2 > 0 \forall t,$

# An extension

(CDDHM)

**Theorem**

If $f(x) \in \mathbb{Q}[x]$ of degree $d = 2(2k+1),\ k \in \mathbb{N},\ \ell \in \mathbb{N}$ such that $f(t) - \frac{1}{2^{2\ell}}(t^2+t+1)^{2k}t^2 > 0 \forall t$, then $f(x) - \frac{1}{2^{2\ell}}(x^2+x+1)^{2k}x^2$ is a so4s iff $f(0)$ is not a square in $\mathbb{Q}_2$

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What to do if $f(0)$ is a square in $\mathbb{Q}_2$?

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$$4x^6 + 4x^3 + 9 = (1 + 2x^3)^2 + 8$$

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$$f \geq 0 \iff f = \sum_{i=1}^{N} f_i^2$$

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- semidefinitive optimization (over the reals)
- over the rationals
  Baldo-Krick-Mourrain 2025

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Not true anymore:

$$f(x_1, x_2) = 1 + x_1^2 x_2^2 (x_1^2 + x_2^2 - 3) \geq 0$$

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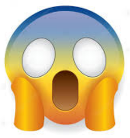

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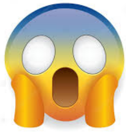

Negative solution to Hilbert's 17th Problem

# Reals versus racionals

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$$
\begin{aligned}
&40x_0^4 + 8x_0^2x_1^2 + 32x_0^2x_1x_2 + 64x_0^2x_1x_3 \\
&+16x_0^2x_2^2 + 16x_0^2x_2x_3 + 32x_0^2x_3^2 + 2x_1^4 \\
&+8x_1^2x_2^2 + 8x_1^2x_2x_3 + 16x_1x_2x_3^2 \\
&+8x_2^2x_3^2 + 8x_3^4 = f_1^2 + f_2^2 + f_3^2 + f_4^2
\end{aligned}
$$

# Reals versus racionals

$$40x_0^4 + 8x_0^2x_1^2 + 32x_0^2x_1x_2 + 64x_0^2x_1x_3$$
$$+16x_0^2x_2^2 + 16x_0^2x_2x_3 + 32x_0^2x_3^2 + 2x_1^4$$
$$+8x_1^2x_2^2 + 8x_1^2x_2x_3 + 16x_1x_2x_3^2$$
$$+8x_2^2x_3^2 + 8x_3^4 = f_1^2 + f_2^2 + f_3^2 + f_4^2$$

but cannot written as a sos with polynomials in $\mathbb{Q}[x_0, x_1, x_2, x_3]$

# References

# References


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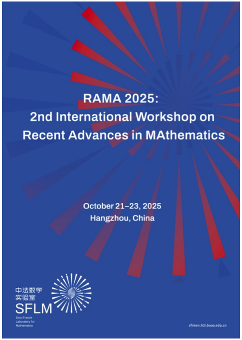


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