Effective Pourchet's Theorem

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Gröbner free methods and their applications

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Aula IC [Facultat de Matemàtiques i Informàtica, University of Barcelona]

https://www.ub.edu/arcades/gfm2026.html

Joint work with

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- Teresa Cortadellas Benítez – URL

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"The" question

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Given a polynomial

$$f(x_1, \ldots, x_n) \in \mathbb{R}/\mathbb{Q}[x_1, \ldots, x_n]$$

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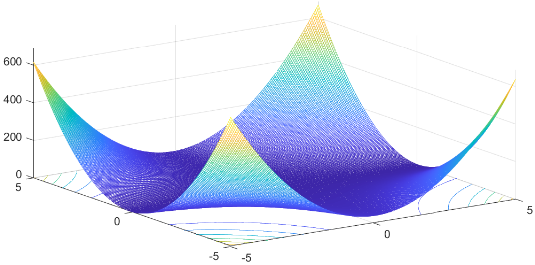
How do I check if $f \geq 0$?



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$$f(t) \geq 0 \; \forall t \in \mathbb{R} \iff$$

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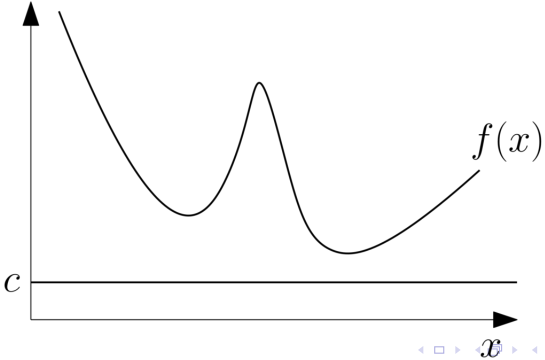
$$f(t) \geq 0 \ \forall t \in \mathbb{R} \iff$$
$$f(x) = f_1(x)^2 + f_2(x)^2$$



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$$f_1(x)^2 + f_2(x)^2 + f_3(x)^2 + f_4(x)^2 + f_5(x)^2$$

Pourchet – 1971

Is 5 optimal?

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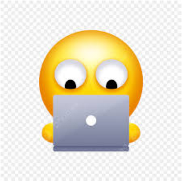
$$x^2 + 7 = x^2 + 2^2 + 1^2 + 1^2 + 1^2$$



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  $f(x) = f_1(x)^2 + \ldots + f_5(x)^2$

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$$\forall p \in \{2, 3, 5, \ldots, \} \cup \{\infty\}$$

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- Local-Global Principle (Hasse-Minkowski)
- Non algorithmic

An algorithm?

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$$p = \infty$$

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$x^2 + ax + b = (x - c)^2 + d^2$ if $a^2 - 4b < 0$

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$$(x - a)^{2k} = ((x - a)^k)^2 + 0^2$$

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$$\boxed{(u^2 + v^2) \cdot (w^2 + z^2) = \alpha^2 + \beta^2}$$

Almost all p

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Five squares are enough if $p = 2$

Sums of 4 squares are computable

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$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = (z_1^2 + z_2^2 + z_3^2 + z_4^2)$$

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Theorem (Pourchet, 71)

$f(x) = f_1^2 + f_2^2 + f_3^2 + f_4^2$ in $K[x]$ $\iff$

- $\mathrm{lc}(f) = a_1^2 + a_2^2 + a_3^2 + a_4^2$ in K, and

Sums of 4 squares are computable

$$(x_1^2+x_2^2+x_3^2+x_4^2)(y_1^2+y_2^2+y_3^2+y_4^2)$$
$$=(z_1^2+z_2^2+z_3^2+z_4^2)$$

Theorem (Pourchet, 71)

$f(x)=f_1^2+f_2^2+f_3^2+f_4^2$ in $K[x]$ $\iff$

- $\mathrm{lc}(f)=a_1^2+a_2^2+a_3^2+a_4^2$ in K, and
- $\forall p(x)$ prime divisor of $f(x)$ with odd multiplicity, there is a non trivial solution of $x_1^2+x_2^2+x_3^2+x_4^2=0$ in $K[x]/(p(x))$

A criteria

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Theorem (Pourchet, 71)

Let $f(x) \in \mathbb{Q}[x] \setminus \{0\}$. *TFA:*

1. $f(x) \in \sum_4 \mathbb{Q}[x]^2$

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Theorem (Pourchet, 71)

Let $f(x) \in \mathbb{Q}[x] \setminus \{0\}$. *TFA:*

1. $f(x) \in \sum_4 \mathbb{Q}[x]^2$
2. $f(t) > 0 \, \forall t \in \mathbb{R}$, *and in* $\mathbb{Q}_2[x]$ *every prime factor of* $f(x)$ *with odd multiplicity has even degree*

Useful criteria

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$$x^2 + 7 = (x - \alpha) \cdot (x + \alpha) \text{ en } \mathbb{Q}_2[x]$$

$$\Longrightarrow \notin \sum\nolimits_4 \mathbb{Q}[x]^2$$

Useful criteria

$$x^2+7=(x-\alpha)\cdot(x+\alpha) \text{ en } \mathbb{Q}_2[x]$$
$$\implies \notin \sum_4 \mathbb{Q}[x]^2$$

$$u\in\mathbb{Q}_2^2 \iff$$
$$u=2^{2a}(8b+1),\ a\in\mathbb{Z},\ b\in\mathbb{Z}_2$$

Algorithm

Pourchet's theorem in action: decomposing univariate nonnegative polynomials as sums of five squares

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Sum of two squares

Algorithm 1 Computing a decomposition of a polynomial as a sum of two squares

Input: A polynomial $f \in \mathbb{Q}[x]$, which is a priori known to be a sum of two squares in $\mathbb{Q}[x]$.

Output: Polynomials $a, b \in \mathbb{Q}[x]$ such that $a^2 + b^2 = f$.

1: Construct the quadratic field extension $\mathbb{Q}(i)/\mathbb{Q}$.

2: Solve the norm equation

$$\mathrm{lc}(f) = N_{\mathbb{Q}(i)/\mathbb{Q}}(x)$$

and denote a solution by $a + bi \in \mathbb{Q}(i)$.

3: Factor f into a product of monic irreducible polynomials

$$f = \mathrm{lc}(f) \cdot p_1^{e_1} \cdots p_k^{e_k}.$$

4: **for** every factor p_j, such that the corresponding exponent e_j is odd **do**

5: Factor p_j over $\mathbb{Q}(i)$ into a product $p_j = g_j \cdot h_j$ with $g_j, h_j \in \mathbb{Q}(i)[x]$.

6: Set

$$a_j := \frac{1}{2} \cdot (g_j + h_j), \qquad b_j := \frac{1}{2i} \cdot (g_j - h_j).$$

7: Update a and b setting:

$$a := aa_j + bb_j \quad \text{and} \quad b := ab_j - ba_j.$$

8: Update a and b setting:

$$a := a \cdot \prod_{j \le k} p_j^{2\lfloor e_j/2 \rfloor} \quad \text{and} \quad b := b \cdot \prod_{j \le k} p_j^{2\lfloor e_j/2 \rfloor}.$$

9: **return** a, b.

Sum of three or four squares

Algorithm 3 Initial solution: modular sum of squares

Input: An irreducible polynomial $f \in \mathbb{Q}[x]$, which is a priori known to be a sum of 3 or 4 squares.

Output: Polynomials h and $g_1, \ldots, g_4$ in $\mathbb{Q}[x]$, such that $\deg h \leq \deg f - 2$ and $fh = g_1^2 + \cdots + g_4^2$.

1: Construct the number fields:

$$K := \mathbb{Q}[x]/(f) \qquad \text{and} \qquad L := K(i).$$

2: Solve the norm equation

$$-1 = N_{L/K}(x)$$

and denote the solution by $\xi = \overline{g}_1 + \overline{g}_2 i$, where $g_1, g_2 \in \mathbb{Q}[x]$ are polynomials of degree strictly less than $\deg f$ and $\overline{g}_j$ denotes the image of g_j under the canonical epimorphism $\mathbb{Q}[x] \twoheadrightarrow K$.

3: Set $g_3 := 1$, $g_4 := 0$ and let $h := (g_1^2 + \cdots + g_4^2)/f$.

4: **return** h, g_1, g_2, g_3, g_4.

How to use this for 5 polynomials?

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$$f(t) > 0 \ldots f(t) - \left(\frac{1}{2^\ell}\right)^2 > 0$$

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$$f(x) - \left(\frac{1}{2^\ell}\right)^2 = f_1^2 + f_2^2 + f_3^2 + f_4^2??$$

Algorithm 6

Algorithm 6

Algorithm 6 Reduction to a sum of 4 squares: odd valuation case

Input: A positive square-free polynomial $f = c_0 + c_1 x + \cdots + c_d x^d \in \mathbb{Q}[x]$. The 2-adic valuations of the coefficients of f are $k_j := \operatorname{ord}_2 c_j$ for $0 \le j \le d$. Ensure k_d is odd. It is assumed that f is not a sum of 4 squares.

Output: A polynomial $h \in \mathbb{Q}[x]$ such that $f - h^2$ is a sum of 4 (or fewer) squares.

1: Find a positive number ε such that

$$\varepsilon < \inf\{f(x) \mid x \in \mathbb{R}\}.$$

2: Set $l_1 := \lceil -1/2 \cdot \lg \varepsilon \rceil$.

3: Set $l_2 := \lceil -k_0/2 \rceil + 1$.

4: Set

$$l_3 := \left\lceil \max\left\{ \frac{jk_d - dk_j}{2d - 2j} \mid 0 < j < d \right\} \right\rceil.$$

5: Initialize $l := \max\{l_1, l_2, l_3\}$.

6: **while** $\gcd(d, 2l + k_d) \neq 1$ **do**

7: $\quad l := l + 1$.

8: **return** $h := 2^{-l}$.

Sum of 6 squares

Algorithm 8 Decomposition of a nonnegative univariate rational polynomial into a sum of 6 squares

Input: A nonnegative polynomial $f \in \mathbb{Q}[x]$.
Output: Polynomials $f_1, \ldots, f_6 \in \mathbb{Q}[x]$ such that $f_1^2 + \cdots + f_6^2 = f$.

if f is a square **then**
  return $f_1 := \sqrt{f}, f_2 := \cdots f_6 := 0$.
if f is a sum of 2 squares {Use Observation 8 to check it} **then**
  Execute Algorithm 1 to obtain $f_1, f_2 \in \mathbb{Q}[x]$ such that $f_1^2 + f_2^2 = f$.
  return f_1, f_2 and $f_3 := \cdots f_6 := 0$.
if f is a sum of 4 squares {Use [36, Theorem 17.2] to check it} **then**
  Execute Algorithm 5, to obtain $f_1, \ldots, f_4 \in \mathbb{Q}[x]$ such that $f_1^2 + \cdots + f_4^2 = f$.
  return $f_1, \ldots, f_4$ and $f_5 := f_6 := 0$
Compute the square-free decomposition of $f = g \cdot h^2$, where $g, h \in \mathbb{Q}[x]$ and g is square-free.
Execute Algorithm 7 with g as an input to obtain $g_1, g_2 \in \mathbb{Q}[x]$ such that $g - g_1^2 - g_2^2$ is a sum of 4 squares in $\mathbb{Q}[x]$.
Execute Algorithm 5 to decompose $g - g_1^2 - g_2^2$ into a sum of 4 squares in $\mathbb{Q}[x]$. Denote the output by $g_3, \ldots, g_6$.
return $f_1 := g_1 h, \ldots, f_6 := g_6 h$.

Conjecture

Algorithm 9 Reduction to a sum of 4 squares

Input: A positive square-free polynomial $f = c_0 + c_1 x + \cdots + c_d x^d \in \mathbb{Q}[x]$.

Output: A polynomial $h \in \mathbb{Q}[x]$ such that $f - h^2$ is a sum of 4 (or fewer) squares.

```
if f is a sum of 4 squares then
    return h := 0.
Set f_* := c_d + c_{d-1} x + ... + c_0 x^d.
Find a positive number ε such that
    ε < inf{ f(x) | x ∈ ℝ }  and  ε < inf{ f_*(x) | x ∈ ℝ }.
Initialize l := ⌈−1/2 · lg ε⌉.
while True do
    if f − 2^{−2l} is irreducible in ℚ_2[x] then
        return h := 2^{−l}.
    if f − 2^{−2l} x^d is irreducible in ℚ_2[x] then
        return h := 2^{−l} x^{d/2}.
    l := l + 1.
```

Our results

(C**D**DHM)

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- The conjecture works if $\deg(f(x)) = 4k$

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- The conjecture works if $\deg(f(x)) = 4k$
- It does not work for this family:

$$f_{k,N}(x) = \frac{4x^{2(2k+1)} + x^{2k+1} + 4}{N^2}$$

$k = 0, \ldots,$ $N \in \mathbb{N}$ odd, $N > 64$

Extension of the method

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Theorem

If $f(x) \in \mathbb{Q}[x]$ of degree $d = 2(2k+1),\ k \in \mathbb{N}, \ell \in \mathbb{N}$ such that $f(t) - \frac{1}{2^{2\ell}}(t^2+t+1)^{2k}t^2 > 0\ \forall t \in \mathbb{R}$

Extension of the method

(C**D**DHM)

Theorem

If $f(x) \in \mathbb{Q}[x]$ of degree $d = 2(2k+1), \ k \in \mathbb{N}, \ell \in \mathbb{N}$ such that $f(t) - \frac{1}{2^{2\ell}}(t^2+t+1)^{2k}t^2 > 0 \ \forall t \in \mathbb{R}$ then $f(x) - \frac{1}{2^{2\ell}}(x^2+x+1)^{2k}x^2 \in \sum_4 \mathbb{Q}[x]^2$ if $f(0) \notin \mathbb{Q}_2^2$

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Conjeture

If $f(x) \in \mathbb{Q}_2[x]$ is square free and verifies that $f(\mathbb{Q}_2^2) \subset \mathbb{Q}_2^2$ then it is a product of irreducibles of even degree in $\mathbb{Q}_2[x]$

Sketch of the proof

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$$(*)\ 2^{2\ell} f(x) - (x^2 + x + 1)^{2k} x^2 = \prod_{j=1}^{N} P_j(x)$$

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$$[x^2 + x + 1]^{2k} = \prod_{j_1} [P_{j_1}(x)]$$ *and*

$$[x]^2 = \prod_{j_2} [P_{j_2}(x)]$$

Sketch of the proof

$$(*)\ 2^{2\ell}f(x)-(x^2+x+1)^{2k}x^2=\prod_{j=1}^{N}P_j(x)$$

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$[x]^2=\prod_{j_2}[P_{j_2}(x)]\implies P_{j_1}(x)$ of even degree

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$[x]^2=\prod_{j_2}[P_{j_2}(x)] \implies P_{j_1}(x)$ of even degree and $P_{j_2}(x)$ also iff $(*)$ has no roots $\iff f(0)\notin\mathbb{Q}_2^2$

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You do not need 5 or 6 polinomials to certify nonegativity:

$$f \geq 0 \iff f = \sum_{i=1}^{N} f_i^2$$

- Semidefinite optimization (sobre $\mathbb{R}$)
- Over $\mathbb{Q}$ (Baldo-Krick-Mourrain 2025)

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Positivity vs sums of squares

$$f(x_1, x_2) = 1 + x_1^2 x_2^2 (x_1^2 + x_2^2 - 3) \geq 0$$

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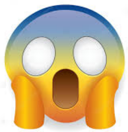

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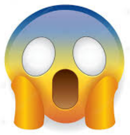

Hilbert's 17 Problem

$\mathbb{R}$ vs $\mathbb{Q}$

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$$40x_0^4 + 8x_0^2x_1^2 + 32x_0^2x_1x_2 + 64x_0^2x_1x_3$$
$$+16x_0^2x_2^2 + 16x_0^2x_2x_3 + 32x_0^2x_3^2 + 2x_1^4$$
$$+8x_1^2x_2^2 + 8x_1^2x_2x_3 + 16x_1x_2x_3^2$$
$$+8x_2^2x_3^2 + 8x_3^4 = f_1^2 + f_2^2 + f_3^2 + f_4^2$$

$\mathbb{R}$ vs $\mathbb{Q}$

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$$+8x_2^2x_3^2 + 8x_3^4 = f_1^2 + f_2^2 + f_3^2 + f_4^2$$

over $\mathbb{R}$ but cannot be written as any sum of squares over $\mathbb{Q}$

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Thanks!

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