Certificates of positivity for real polynomials

Carlos D'Andrea

Atlanta, February 6th 2026

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$f(x_1, \ldots, x_n) \in \mathbb{R}$ (or) $\mathbb{Q}[x_1, \ldots, x_n]$

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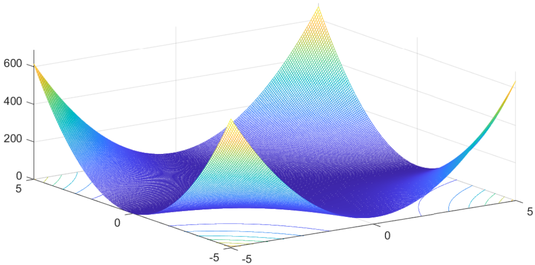
How can we verify/certify if $f \geq 0$?



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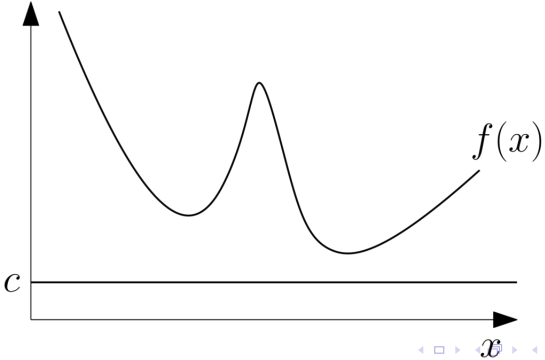

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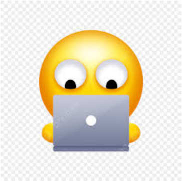
$$f(x) =$$
$$f_1(x)^2 + f_2(x)^2 + f_3(x)^2 + f_4(x)^2 + f_5(x)^2$$

Pourchet – 1971

5 is optimal

$$x^2 + 7 = x^2 + 2^2 + 1^2 + 1^2 + 1^2$$



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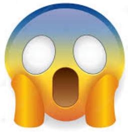

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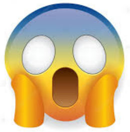

Negative solution to Hilbert's 17th Problem

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$$f(x_1,\ldots,x_n)\geq 0 \iff$$
$$\exists h, f_1,\ldots,f_k\in\mathbb{R}[x_1,\ldots,x_n],\ h\neq 0,$$
with
$$h^2\,f = f_1^2+\ldots+f_k^2$$

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$f(x_1, \ldots, x_n)$ is a sos $\iff$
$\exists B \in \mathbb{R}^{N\times N},\ B^t = B$, positive semidefinite, such that

$$f = (1\, x_1\, \ldots x^\alpha \ldots)\cdot B\cdot(1\, x_1\, \ldots x^\alpha \ldots)^t$$

Testing s.o.s

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Testing s.o.s

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- Computing if a given semialgebraic set is empty or not (Powers)
- Solving a semidefinite linear problem (Laserre)

Reals versus racionals

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$$40x_0^4 + 8x_0^2x_1^2 + 32x_0^2x_1x_2 + 64x_0^2x_1x_3$$
$$+16x_0^2x_2^2 + 16x_0^2x_2x_3 + 32x_0^2x_3^2 + 2x_1^4$$
$$+8x_1^2x_2^2 + 8x_1^2x_2x_3 + 16x_1x_2x_3^2$$
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but cannot written as a sos with polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$

Bounds for Artin's representation

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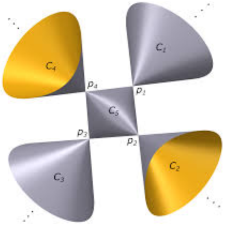
$$\deg(p, f_1, \ldots, f_k) \leq 2^{2^{2^{d^{4^n}}}}$$

Lombardi-Perrucci-Roy 2000

A more general problem

How can I verify if $f \geq 0$ in $\{f_1 \geq 0,\ f_2 \geq 0, \ldots f_s \geq 0\}$?



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$$f|_{V(f_1, \ldots f_k)} = 0 \iff f^\ell \in \langle f_1, \ldots, f_k \rangle$$

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$$f^{2\ell} + g_1^2 + \ldots + g_r^2 \in \langle f_1, \ldots, f_k \rangle$$

Krivine–1964

Algebraic and semialgebraic sets

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$$\boxed{\begin{array}{c} S(f_1,\ldots,f_n) \\ = \\ \{x\in\mathbb{K}^n:\ f_i(x)\geq 0,\ 1\leq i\leq k\} \end{array}}$$

The preordering & Positivstellensatz

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$$PO(f_1, \ldots, f_k)$$
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Positivstellensatz (Stengle 1974)

$$f|_{S(f_1,\ldots,f_k)} > 0 \iff$$
$$s\,f = 1 + t,\ s, t \in PO(f_1, \ldots, f_k)$$

Nichtnegativstellensatz

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(Stengle 1974)

$$f|_{S(f_1,\ldots,f_k)} \geq 0 \iff sf = f^{2\ell} + t,\ \ell \in \mathbb{N},\ s,t \in PO(f_1,\ldots,f_k)$$

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Computational aspects are hard!

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$$M(f_1, \ldots, f_k) :=$$
$$\{g_0 + g_1 f_1 + \ldots + g_k f_k\}$$
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$$M(f_1, \ldots, f_k) \subset PO(f_1, \ldots, f_k)$$

$f \in M(f_1, \ldots, f_k)$ can be easily tested

Quadratic modules and positivity

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When do we have

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$M(f_1,\ldots,f_k)$ is called **archimedean** **if** $N - x_1^2 - \ldots - x_n^2 \in M(f_1,\ldots,f_k)$ for some $N > 0$

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Putinar's Positivstellensatz

If $M(f_1,\dots,f_k)$ is archimedean and $f|_{S(f_1,\dots,f_n)} > 0$ then $f \in M(f_1,\dots,f_k)$

Cases of applications/extensions

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- Polytopes Powers 2004

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- Polytopes Powers 2004
- Cylinders Escorcielo–Perrucci 2020
- Zero-dimensional ideals
  Krick-Mourrain-Szanto 2020 –
  Baldi-Krick-Mourrain 2024

Barcelona Computational Algebra

- Ana Belén de Felipe – UPC
- Eulàlia Montoro – UB
- Joel Hurtado – UPC
- Teresa Cortadellas Benítez – URL

Recent contributions

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Given $f_1, \ldots, f_k \in \mathbb{Q}[x_1, \ldots, x_n]$

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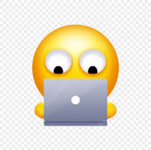
Given $f_1, \ldots, f_k \in \mathbb{Q}[x_1, \ldots, x_n]$, how to certify if $M(f_1, \ldots, f_k)$ is archimedean?



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Joint work with Joel Hurtado

Related question

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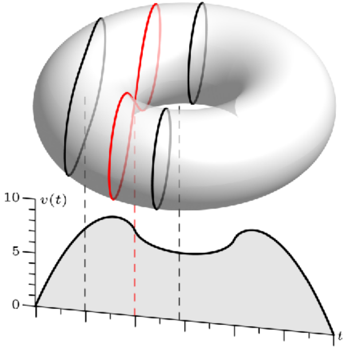
Given $f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_n]$

Related question

Given $f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_n]$, how to certify if $S(f_1, \ldots, f_k)$ is compact?



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compute a bound for its diameter

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- Basu-Roy 2010
- **D**-Hurtado 2026

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  $f(x) \in \mathbb{Q}[x], \;\; f(t) \geq 0 \forall t \in \mathbb{R}$

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- **Input:**
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- **Output:** $f_1(x), \ldots, f_5(x) \in \mathbb{Q}[x]$,
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arXiv:2511.11783

References

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Thanks!

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