Toric Euler-Jacobi vanishing theorem and zeros at infinity

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Complex local residues

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The (local) residue of $\varphi(z)\,dz$ at ξ

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is a rational function of the coefficients of f and g

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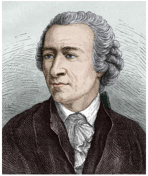
- $\mathrm{Res}_g\left(\frac{p(z)}{f(z)}dz\right) = 0$ if $f(z) \mid p(z)$

Euler-Jacobi Vanishing Theorem

If $\deg(p(z)) < \deg(f(z)) - 1$

then $\text{Res}_g\left(\frac{p(z)}{f(z)}dz\right) = 0$



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with equality generically

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- $\mathrm{Res}_g\left(\frac{p\ dz}{f_1\ldots f_n}\right)$ is computable

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$\Longrightarrow$ effective membership tests

Dickenstein-Sessa 1990

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then

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Compactifications and Stokes Theorem

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$

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$$\dim_{\mathbb{K}} \left(\mathbb{K}[z_0, \ldots, z_n] / \langle z_0, f_1^h, \ldots, f_n^h \rangle \right)_{d_1 + \ldots + d_n - n} = 1$$

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$z_0 \,|\, p^h \implies$ Euler-Jacobi

Limitations

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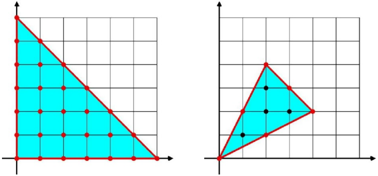
$$\#V(f) = d_1 \dots d_n$$

is a very **strong** condition



"Sparse" philosophy: $f = \sum_{a \in \mathcal{A}} c_a t^a$

$\mathcal{A} \subset \mathbb{Z}^n$ finite

Newton polytopes

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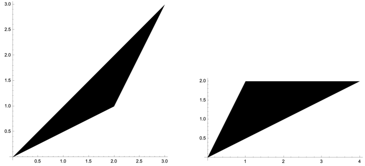
$$f_2 = a' + b'xy^2 + c'x^4y^2$$



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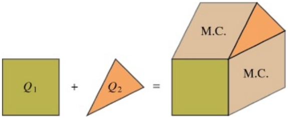
If $f_1, \ldots, f_n \in \mathbb{K}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ and $V(\boldsymbol{f}) \subset (\mathbb{K}^\times)^n$ finite then

$$\# V(\boldsymbol{f}) \le MV(N(f_1), \ldots, N(f_n))$$

with generic equality



the mixed volume

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local and global residues are defined as usual:

$$\mathrm{Res}_g\left(\frac{h}{f_1 \ldots f_n}\frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_n}{t_n}\right) := \sum_{\xi \in V(f)} \mathrm{Res}_\xi\left(\frac{h}{f_1 \ldots f_n}\right)\frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_n}{t_n}$$

Euler-Jacobi in the torus

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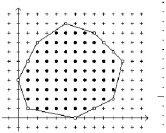
If $\boxed{\#V(f) = \text{MV}(N(f_1), \ldots, N(f_n))}$

and

$$N(t_1 \ldots t_n\, p) \subset (N(f_1) + \ldots + N(f_n))^\circ$$



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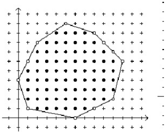

$$\Longrightarrow \boxed{\mathrm{Res}_g\left(\frac{p}{f_1 \ldots f_n}\frac{dt}{\mathrm{t}}\right) = 0}$$

“Homogeneous” polynomials

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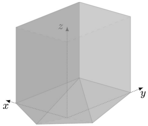
$P := N(f_1) + N(f_2) + \ldots N(f_n)$ The **Cox ring** is $\mathbb{K}[x_1, x_2, \ldots, x_N]$ (one variable per facet)



and a “homogenization” rule

$$f_i \mapsto f_i^h \in \mathbb{K}[x_1, \ldots, x_N]$$

BKK bound and Toric residues

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$$\# V(f_1, \ldots, f_n) = \mathrm{m} \implies$$
$$\exists f_0^h : \; V_{X_P}(f_0^h, f_1^h, \ldots, f_n^h) = \emptyset$$

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$$\#V(f_1,\ldots,f_n)=\mathrm{m}\implies$$
$$\exists f_0^h:\ V_{X_P}(f_0^h,f_1^h,\ldots,f_n^h)=\emptyset$$

There is a trace map

$$\left(\mathbb{K}[x_1,\ldots,x_N]/\langle f_0^h,f_1^h,\ldots,f_n^h\rangle\right)_\rho\to\mathbb{K}$$

giving the global residue

Cox 96

Cattani, Cox, Dickenstein 97

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Cox 96

Cattani, Cox, Dickenstein 97

"Combinatorial" jacobian

Cattani, Cox, Dickenstein 97

D, Khetan 2003

Khetan, Soprunov 05

Dimension one?

$$1 \leq \dim \left(\mathbb{K}[x_1, \ldots, x_N]/\langle f_0^h, f_1^h, \ldots, f_n^h\rangle\right)_\rho \leq 1 + \square$$

with $\square$ being described combinatorially

Cox, Dickenstein 05

Our questions

w/ A. Dickenstein

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Is it true that

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Euler-Jacobi $\iff$ $\#V(f) = \mathrm{m}$?

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Euler-Jacobi $\iff$ $\#V(f) = \mathrm{m}$?

$J_{f_0^h,\ldots,f_n^h} \in \langle f_0^h, \ldots, f_n^h \rangle \iff$

$V(f_h^0, \ldots, f_n^h) \neq \emptyset$?

Euler-Jacobi

Euler-Jacobi

Is it true that if $\#V(f) < \text{m?}$ then $\exists p: \; N(t_1 \ldots t_n p) \subset (N(f_1) + \ldots + N(f_n))^{\circ}$ such that $\mathrm{Res}_g\left(\frac{p}{f_1 \ldots f_n}\frac{dt}{t}\right) \neq 0$?

Answer: "complete" your system

$\dim\left(\mathbb{K}[x_1,\ldots,x_N]/\langle f_0^h, f_1^h, \ldots, f_n^h\rangle\right)_\rho$ could **never** be one

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$$\begin{cases} f_1 &= 2 - t_1 + t_1^2 + t_2 + 2t_1 t_2 + t_2^2 \\ f_2 &= 1 + t_1 + t_2 \\ f_3 &= 2 - 3t_3 + t_3^2 \end{cases}$$

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$$N(f_1) = 2\Delta_2 \times \{0\}$$
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has a monomial of nonzero global residue

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D-Dickenstein 2025

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D-Dickenstein 2025

Assume that $N(f_1), \ldots, N(f_n)$ is **"completable"**. If $\#V(f) < \mathrm{m}$ and $V_X(f^h)$ is finite (there are zeroes at X_∞)then $\exists p : \; N(t_1 \ldots t_n\, p) \subset (N(f_1) + \ldots + N(f_n))^\circ$ such that

$$\mathrm{Res}_g\left(\frac{p}{f_1 \ldots f_n} \frac{d\mathrm{t}}{\mathrm{t}}\right) \neq 0$$

Second question

D-Dickenstein 2025

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If generically

$$\dim\left(\mathbb{K}[x_1,\ldots,x_N]/\langle F_0^h,F_1^h,\ldots,F_n^h\rangle\right)_\rho=1$$

Second question

D-Dickenstein 2025

If generically

$$\dim\left(\mathbb{K}[x_1,\ldots,x_N]/\langle F_0^h, F_1^h,\ldots,F_n^h\rangle\right)_\rho = 1 \text{ but}$$

$$\emptyset \neq V_X(f_0^h, f_1^h,\ldots,f_n^h)$$

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D-Dickenstein 2025

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In progress (Dickenstein & Soprunov)

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- Description of $\left(\mathbb{K}[x_1,\ldots,x_N]/\langle f_0^h, f_1^h,\ldots,f_n^h\rangle\right)_\rho$ when the dimension is larger than 1

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- Description of $\left(\mathbb{K}[x_1, \ldots, x_N]/\langle f_0^h, f_1^h, \ldots, f_n^h \rangle\right)_\rho$ when the dimension is larger than 1
- Computing "canonical" elements of residue 1
- Computing "canonical" elements of residue 0

Thanks!

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