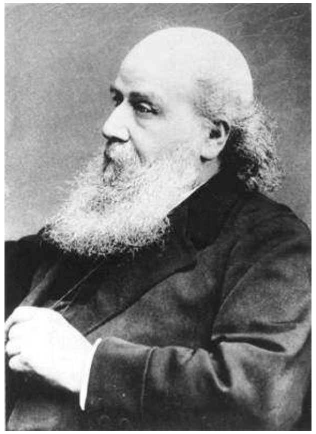
Sylvester's double sums



Carlos D'Andrea

Universitat de Barcelona

Joint work with

Hong Hoon, Teresa Krick & Agnes Szanto

Sturm's Theorem

~ 1829

$$f_0 := F(x)$$

$$f_1 := F'(x)$$

$$f_2 := -\mathit{rem}(f_0, f_1)$$

$$f_3 := -\mathit{rem}(f_1, f_2)$$

$$\vdots$$

Hermite's Quadratic Form

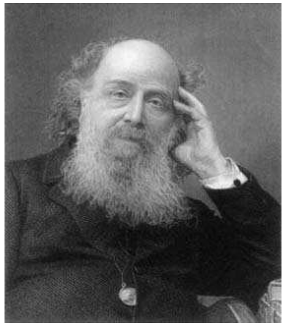
~ 1850

Number of real roots of $F(x)$ related to the study of

symmetrical functions on the roots of F

James Joseph Sylvester

1853



On a theory of syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's function and that of the greatest algebraical common measure

(142 pages!)

“How charming is divine philosophy!

Not harsh and crabbed as dull fools suppose,

But musical as is Apollo’s lute,

And a perpetual feast of nectar’d sweets,

Where no crude surfeit reigns!” -COMUS

Sylvester's work

Sturm's Method		Hermite's Method
Polynomial Remainder sequences of $F(x)$ and $F'(x)$	$\longleftrightarrow$	symmetrical functions in the roots of $F(x)$

General Idea

- **Replace the pair** $(F(x), F'(x))$ **with** $(F(x), G(x))$ **for *any*** $G(x)$
- **Study symmetric functions in the roots of** $F(x)$ **and** $G(x)$.
- **Which of these functions give *prem***$(F(x), G(x))$?

Example: The Resultant

$$F(x) = \prod_{\alpha \in A}(x - \alpha)$$

$$G(x) = \prod_{\beta \in B}(x - \beta)$$

$$prem_{\infty}(F(x), G(x)) = Res(F, G) := \prod_{\alpha \in A,\, \beta \in B} (\alpha - \beta)$$

Sylvester's Double Sums

$$\mathrm{Sylv}^{p,q}(A,B;x)$$

$$:=$$

$$\sum_{\substack{A'\subset A,\, B'\subset B\\ |A'|=p,\, |B'|=q}} R(x,A')\, R(x,B')\, \frac{R(A',B')\, R(A\backslash A', B\backslash B')}{R(A',A\backslash A')\, R(B',B\backslash B')}$$

* $0 \le p \le |A|,\ 0 \le q \le |B|$
* $R(U,V) = \prod_{u\in U,\, v\in V}(u-v)$

If $0 \leq p+q < \min\{|A|,|B|\}$, **then**

$$\mathrm{Sylv}^{p,q}(A,B;x) = (-1)^{p(|A|-d)} \binom{d}{p} \textbf{Sres}_{p+q}(F,G)$$

Sres$_{p+q}(F,G)$ **is the** $(p+q)$**-th subresultant of the polynomials** F **and** G

(one of the *prem*)

(Sylvester, 1853 * Lascoux-Pragacz 2003 * DHKS 2005)

If $p + q = |A| < |B|$, **then**

$$\mathrm{Sylv}^{p,q}(A, B; x) = \binom{|A|}{p} F(x)$$

(Sylvester 1853 * Lascoux-Pragacz 2003 * DHKS 2005)

If $p+q=|A|=|B|$, **then**

$$\mathrm{Sylv}^{p,q}(A,B;x)=\binom{|A|-1}{q}F(x)+\binom{|A|-1}{p}G(x)$$

(Sylvester 1853 * Lascoux-Pragacz 2003)

If $|A| < p + q < |B| - 1$, **then**

$$\mathrm{Sylv}^{p,q}(A, B; x) = 0$$

(Sylvester 1853)

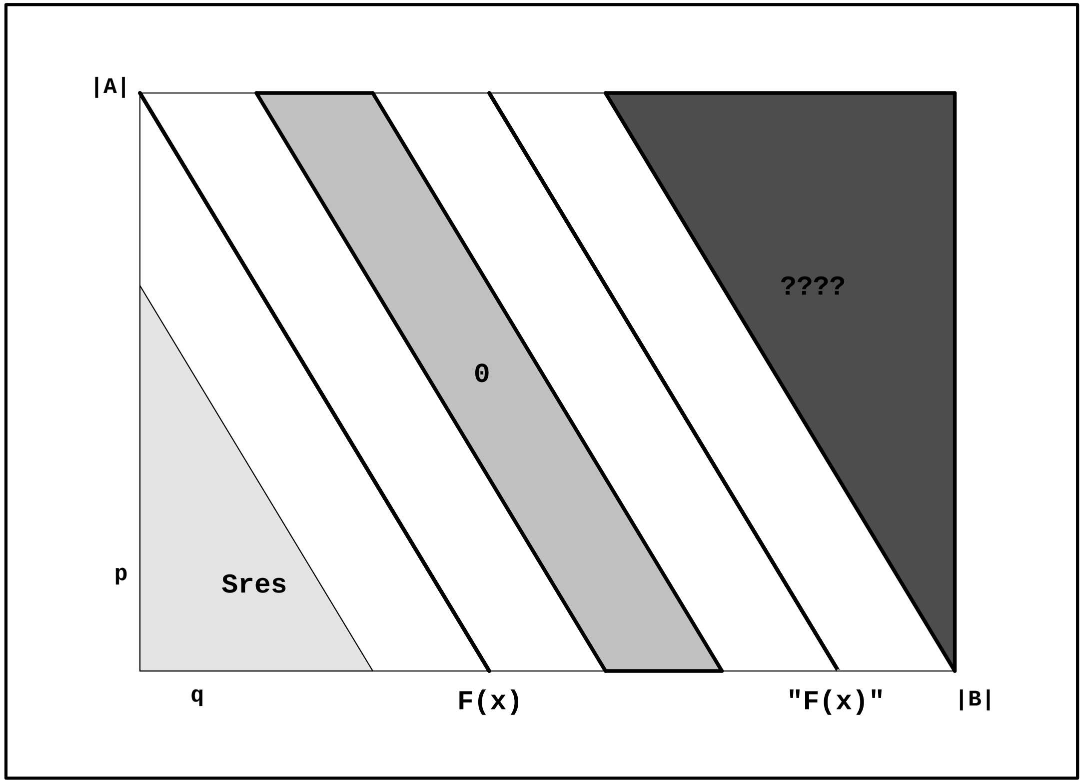
If $|A| < p + q = |B| - 1$**, then**

$$\mathrm{Sylv}^{p,q}(A, B; x)$$

is a "numerical multiplier" of $F(x)$

(Sylvester 1853)

What's left?



The “antithetical relation”

(Sylvester, Arts 23 & 25)

$$\mathrm{Sylv}^{p,q}(A,B;x) \;\leftrightarrow\; \mathrm{Sylv}^{|A|-p,|A|-q}(A,B;x)$$

$$Res(F,G) \;\leftrightarrow\; F(x)\,G(x)$$

`For every form of degree` $\leq |A|$`, there is a sort of conjugate form between` $|B|$ `and` $|A|+|B|$

“ To enter into a further or more detailed examination of the values assumed by $\mathrm{Sylv}^{p,q}(A, B; x)$ for the most general values of p, q would be to transcend the limits I have proposed to myself in drawing up the present memoir.”

(Sylvester, Art. 25)

Generating functions for $\mathrm{Sylv}^{p,q}$

(DHKS 2007)

$$\langle p(t),\Gamma\rangle_v := \begin{array}{cc} & u \\ & \boxed{\begin{matrix} p(\gamma_1) & \dots & p(\gamma_u) \\ \vdots & & \vdots \\ \gamma_1^{v-1}p(\gamma_1) & \dots & \gamma_u^{v-1}p(\gamma_u) \end{matrix}} \end{array} \; v$$

$$\Gamma = \{\gamma_1,\, \gamma_2, \dots, \gamma_u\}$$

Theorem

(DHKS 2007)

For every $0 \leq d \leq |A| + |B|$:

$$\sum_{p=0}^{d} (-1)^{(d-p)(|A|-p)} \,\mathrm{Sylv}^{p,d-p}\; T^{|A|-p}$$

$$=$$

$$\frac{\det \begin{array}{cc|c} |B| & |A| & \\ \langle 1, B\rangle_{|A|+|B|-d} & \langle T, A\rangle_{|A|+|B|-d} & |A|+|B|-d \\ \langle x-t, B\rangle_{d} & \langle x-t, A\rangle_{d} & d \end{array}}{\mathcal{V}(A)\,\mathcal{V}(B)}$$

Recovering Sylvester's results (I)

$$
\det \begin{array}{cc|c}
|B| & |A| & \\
\begin{array}{|c|c|}
\hline
\langle 1, B\rangle_{|A|+|B|-d} & \langle T, A\rangle_{|A|+|B|-d} \\
\hline
\langle x-t, B\rangle_d & \langle x-t, A\rangle_d \\
\hline
\end{array} & & \begin{array}{c} |A|+|B|-d \\ d \end{array}
\end{array}
$$

$$=$$

$$(-1)^{d|A|}\mathcal{V}(A)\,\mathcal{V}(B)\,\mathsf{Sres}_d(F,G)\,T^{|A|-d}(T-1)^d$$

if $d < |A| \le |B|$ or $d = |A| < |B|$

Recovering Sylvester's results (II)

$$\det \begin{array}{|c|c|l} \multicolumn{1}{c}{|B|} & \multicolumn{1}{c}{|A|} & \\ \hline \langle 1, B\rangle_{|A|+|B|-d} & \langle T, A\rangle_{|A|+|B|-d} & |A|+|B|-d \\ \hline \langle x-t, B\rangle_d & \langle x-t, A\rangle_d & d \\ \hline \end{array}$$

$$=$$

$$0$$

if $|A| < d < |B|-1$

Recovering Sylvester's results (III)

$$
\begin{array}{cc|c|c|l}
 & |B| & |A| & \\
\det & \langle 1, B\rangle_{|A|+|B|-d} & \langle T, A\rangle_{|A|+|B|-d} & |A|+|B|-d \\
 & \langle x-t, B\rangle_{d} & \langle x-t, A\rangle_{d} & d
\end{array}
$$

$$=$$

$$\pm\mathcal{V}(A)\,\mathcal{V}(B)\,F(x)\,(T-1)^{|A|}$$

if $|A| < d = |B| - 1$

Recovering Sylvester's results (IV)

$$
\det\;
\begin{array}{|c|c|l}
\multicolumn{1}{c}{|B|} & \multicolumn{1}{c}{|A|} & \\
\cline{1-2}
\langle 1, B\rangle_{|A|+|B|-d} & \langle T, A\rangle_{|A|+|B|-d} & |A|+|B|-d \\
\cline{1-2}
\langle x-t, B\rangle_d & \langle x-t, A\rangle_d & d \\
\cline{1-2}
\end{array}
$$

$$=$$

$$\mathcal{V}(A)\,\mathcal{V}(B)\,\mathrm{Res}(F,G)\,F(x)\,G(x)$$

if $d = |A| + |B|$

Remaining cases???

$$|A| \leq |B| \leq d < |A| + |B|$$

- $Sres_d(F, G) = \mathtt{f}_d(x)F(x) + \mathtt{g}_d(x)G(x)$
- $d' := |A| + |B| - d$

Theorem

(DHKS 2007)

$$\begin{array}{rccl}
 & |B| & |A| & \\
\det & \langle 1, B\rangle_{|A|+|B|-d} & \langle T, A\rangle_{|A|+|B|-d} & |A|+|B|-d \\
 & \langle x - t, B\rangle_d & \langle x - t, A\rangle_d & d
\end{array}$$

$$= \pm \mathcal{V}(A)\mathcal{V}(B)\left(\mathtt{f}_{d'-1}(x)F(x) + T\mathbf{g}_{d'-1}(x)G(x)\right)$$

$$(T-1)^{d'-1}$$

if $|A| \leq |B| \leq d < |A| + |B|$

Theorem

(DHKS 2007)

$$\boxed{\begin{gathered}\mathrm{Sylv}^{p,q}(A,B;x) \\ = \\ \pm\left(\binom{d'-1}{m-p}\mathtt{f}_{d'-1}(x)F(x) - \binom{d'-1}{n-q}\mathtt{g}_{d'-1}(x)G(x)\right)\end{gathered}}$$

if $|A| \leq |B| \leq d < |A| + |B|$

References

- Lascoux-Pragacz 2003 “Double Sylvester sums for subresultants and multi-Schur functions”. J. Symbolic Comput. 35 (2003), no.6, 689-710.
- DHKS 2005 “A simple proof of Sylvester’s double sums for subresultants”. J. Symbolic Comput. 42 (2007), 290-297.
- DHKS 2007 “Sylvester’s double sums: the general case”. `math.AC/0701721`

Effective
Effektive
Эффективные
Effettivi
Effectives
Efectivos

Methods
Methoden
Методы
Metodi
Méthodes
Métodos

E
M 09 A
G

in Algebraic
in Algebraischer
в Алгебраической
Algebrica
Algébrique
Algebraica

Geometry
Geometrie
Геометрии
in Geometria
en Géométrie
en Geometria

Barcelona, 15-19 June 2009