Bézout's Theorem and Implicitization

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The Implicitization Problem

$$\begin{array}{cccc} \phi : & \mathbb{T}^n & --> & \mathbb{T}^{n+1} \\ & \tau & \mapsto & (\phi_0(\tau), \ldots, \phi_n(\tau)) \end{array}$$

Compute (equations of) the Zariski closure

of the Image of ϕ

- $\mathbb{T}^n = (\mathbb{K}^*)^n$, $\mathbb{K}$ a field (of characteristic zero)
- $\phi_i(t_1, \ldots, t_n) \in \mathbb{K}(t_1, \ldots, t_n) \, \forall i = 0, \ldots, n$

Expected

$P(X_0, X_1, \ldots, X_n)$ irreducible such that

$P\left(\phi(\tau)\right) = 0$ for almost all τ

Generic Implicitization

$$\phi_i(t) := \frac{\sum_{a \in \mathcal{A}_i} \mathbf{u}_{ai} t^a}{\sum_{b \in \mathcal{B}_i} \mathbf{v}_{bi} t^b} = \frac{A_i(t)}{B_i(t)} \quad i = 0, \ldots, n.$$

- $\mathbf{u}_{ai}, \ \mathbf{v}_{bi}$ indeterminates
- $\mathcal{A}_0, \ldots, \mathcal{A}_n, \mathcal{B}_0, \ldots, \mathcal{B}_n \subset \mathbb{N}^n$ finite sets

$$\mathbb{K} = \mathbb{Q}\left(\mathbf{u}_{ai}, \mathbf{v}_{bi}\right)_{\substack{i = 0, \ldots, n \\ a \in \mathcal{A}_i \\ b \in \mathcal{B}_i}}$$

Tropical Generic Implicitization

Given

$$\mathcal{A}_0, \ldots, \mathcal{A}_n, \mathcal{B}_0, \ldots, \mathcal{B}_n \subset \mathbb{N}^n \text{ finite sets}$$

compute

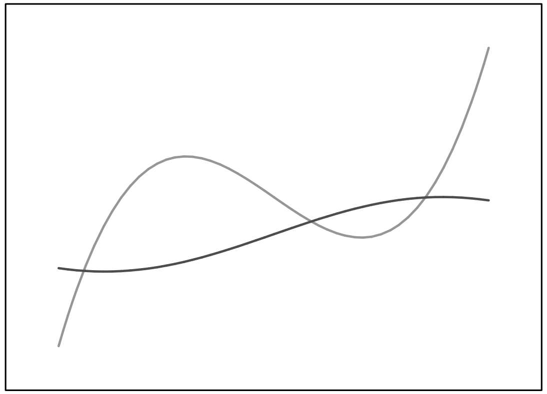
$$d \text{ and } N(P) \subset \mathbb{R}^d$$

the Newton Polytope of the implicit equation

Tropical Implicitization

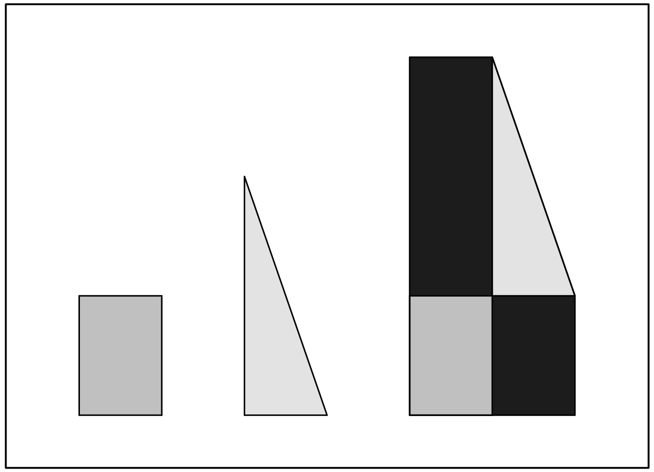
- B. Sturmfels and J.-T. Yu (1992)
- I.Z. Emiris and I.S. Kotsireas (2003-2005)
- B. Sturmfels, J. Tevelev and J. Yu (2006)
- B. Sturmfels and J. Tevelev (2007)
- Khovanskii and Esterov (2007)
- I.Z. Emiris, C. Konaxis and L. Palios (2007)
- P. Huggins, B. Sturmfels and J. Yu (2007)

Bézout's Theorem



The number of solutions of a generic system of n polynomial equations in n unknowns of degrees $d_1, \ldots, d_n$ equals the Bézout Number $d_1 \; \ldots \; d_n$

Bernstein-Kušnirenko's Theorem



The number of solutions of a generic system of n polynomial equations in n unknowns of supports $\mathcal{D}_1, \ldots, \mathcal{D}_n \subset \mathbb{Z}^n$ equals the Mixed Volume

$$MV_{\mathbb{R}^n}(\mathcal{D}_1, \ldots, \mathcal{D}_n)$$

TFAE

- $Im(\phi)$ is a hypersurface in $\mathbb{T}^{n+1}$
- $MV_{\mathbb{R}^n}(\mathcal{C}_0, \ldots, \widehat{\mathcal{C}_i}, \ldots, \mathcal{C}_n) > 0$ for at least one $i \in \{0, \ldots, n\}$
- There exists a unique essential subfamily of $\{\mathcal{C}_i\}_{i=0,\ldots,n}$

$$\mathcal{C}_i := \mathcal{A}_i \cup \mathcal{B}_i \qquad i = 0, \ldots, n.$$

$$P(X_0,\ldots,X_n)$$

$$=$$

$$Res_{\mathcal{C}_0,\ldots,\mathcal{C}_n}\left(B_0(t)X_0 - A_0(t),\ldots,B_n(t)X_n - A_n(t)\right)$$

- $Res_{\mathcal{C}_0,\ldots,\mathcal{C}_n}$ is the sparse resultant associated with $\{\mathcal{C}_i\}_{i=0,\ldots,n}$
- In particular, $d = \#\left(\text{unique essential subfamily}\right) = \#\{i : MV_{\mathbb{R}^n}(\mathcal{C}_0,\ldots,\widehat{\mathcal{C}_i},\ldots,\mathcal{C}_n) > 0\}$

Example 1

$$(\tau_1, \tau_2) \quad \longmapsto \quad \left(\frac{\mathbf{u}_{10}\tau_1}{\mathbf{v}_{20}\tau_2}, \frac{\mathbf{u}_{11}\tau_1}{\mathbf{v}_{21}\tau_2}, \frac{\mathbf{u}_{12}\tau_1 + \mathbf{u}_{02}}{\mathbf{v}_{22}\tau_2} \right)$$

- $\mathcal{C}_0 = \mathcal{C}_1 = \{(1,0),(0,1)\}$
  $\mathcal{C}_2 = \{(0,0),(1,0),(0,1)\}$
- The unique essential subset is $\{\mathcal{C}_0, \mathcal{C}_1\}$

$$\boxed{P(X_0, X_1) = \mathbf{v}_{20}\mathbf{u}_{11}X_0 - \mathbf{u}_{10}\mathbf{v}_{21}X_1}$$

Example 2

$$(\tau_1, \tau_2) \mapsto$$
$$(\mathbf{u}_{00} + \mathbf{u}_{10}\tau_1^2 + \mathbf{u}_{20}\tau_2^2, \mathbf{u}_{01} + \mathbf{u}_{11}\tau_1^2 + \mathbf{u}_{21}\tau_2^2, \mathbf{u}_{02} + \mathbf{u}_{12}\tau_1^2 + \mathbf{u}_{22}\tau_2^2)$$

- $\mathcal{C}_0 = \mathcal{C}_1 = \mathcal{C}_2 = \{(0,0), (2,0), (0,2)\}$
- The lattice generated by the $\mathcal{C}_i$'s has index 4 in $\mathbb{Z}^2$.

$$\boxed{P(X_0, X_1, X_2) = \pm\det\begin{pmatrix} \mathbf{u}_{00} - X_0 & \mathbf{u}_{10} & \mathbf{u}_{20} \\ \mathbf{u}_{01} - X_1 & \mathbf{u}_{11} & \mathbf{u}_{21} \\ \mathbf{u}_{02} - X_2 & \mathbf{u}_{12} & \mathbf{u}_{22} \end{pmatrix}}$$

Corollary

$$\deg_{X_j}(P) = MV_{\mathbb{L}}(\mathcal{C}_i)_{i \in J \setminus \{j\}}$$

- J is the set of indexes of the unique essential family
- $\mathbb{L}$ is the lattice generated by $\sum_{j \in J} \mathcal{C}_j$
- $MV_{\mathbb{L}}$ is the normalized mixed volume in $\mathbb{L}$

What about $\deg(\phi)$?

$$\phi(\tau) :=$$
$$(\mathbf{U}_0\tau_1 + \mathbf{V}_0, \mathbf{U}_1\tau_1 + \mathbf{V}_1, \mathbf{U}_2\tau_2^{\mathbf{a_2}}, \ldots, \mathbf{U}_n\tau_n^{\mathbf{a_n}})$$

- $\mathbf{a_2}, \ldots, \mathbf{a_n} \in \mathbb{N}$
- $P(X_0, X_1) = \mathbf{U}_1 X_0 - \mathbf{U}_0 X_1 + \mathbf{V}_1 \mathbf{U}_0 - \mathbf{U}_1 \mathbf{V}_0$
- $\deg(\phi) = \mathbf{a_2} \ldots \mathbf{a_n}$

There exists $\mathbf{D} \in \mathbb{N}$ such that

$$\deg_{\mathcal{C}_j}\left(Res_{\mathcal{C}_0,\ldots,\mathcal{C}_n}{}^{\mathbf{D}}\right) = MV_{\mathbb{R}^n}(\mathcal{C}_i)_{i=0,\ldots,j-1,j+1,\ldots,n}$$

$$\boxed{\deg(\phi) = \mathbf{D}}$$

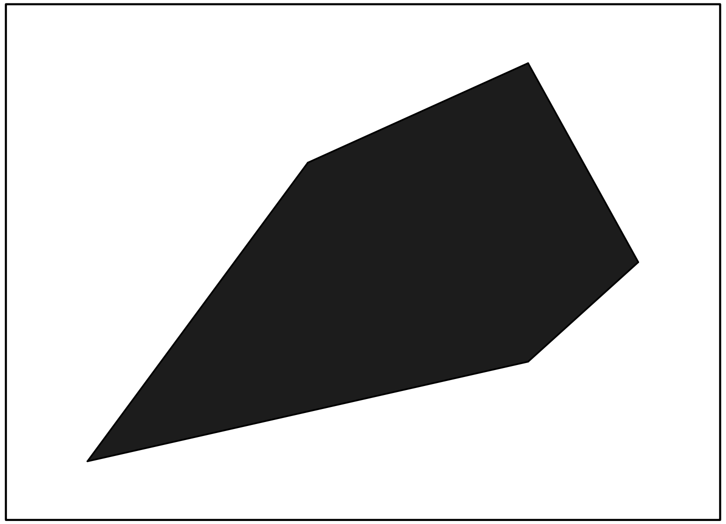
$\mathbf{D} = [Z^n : \mathbb{L}]$ if $\{\mathcal{C}_i\}_{0\leq i\leq n}$ is essential

Sketch of the Proof

$$\begin{array}{ccccc} \mathbb{T}^n & \longrightarrow & \mathbb{T}^{n+1} & \longrightarrow & \mathbb{T}^n \\ \tau & \longmapsto & \phi(\tau) & \longmapsto & \left(\phi_1(\tau), \ldots, \widehat{\phi_{\mathbf{j}}(\tau)}, \ldots, \phi_n(\tau)\right) \end{array}$$

- The degree of the composition is $\deg(\phi)\ \deg_{X_{\mathbf{j}}}(P)$
- $\deg_{X_{\mathbf{j}}}(P) = MV_{\mathbb{L}}(\mathcal{C}_i)_{i \in J \setminus \{\mathbf{j}\}}$
- The number of solutions of $B_i(\tau)X_i = A_i(\tau)\ i \neq \mathbf{j}$ equals $MV_{\mathbb{R}^n}(\mathcal{C}_i)_{i=0,\ldots,\mathbf{j}-1,\mathbf{j}+1,\ldots,n}$

The Newton Polytope of P



$N(P)$ equals the mixed fiber polytope $\sum_\pi(\mathcal{C}_0, \ldots, \mathcal{C}_n)$

associated with $\{\mathcal{C}_i\}$

(Sturmfels & Tevelev – Khovanskii & Esterov)

Computation of N(P)?

Problem posted by

- B. Sturmfels and J.-T. Yu (1992)

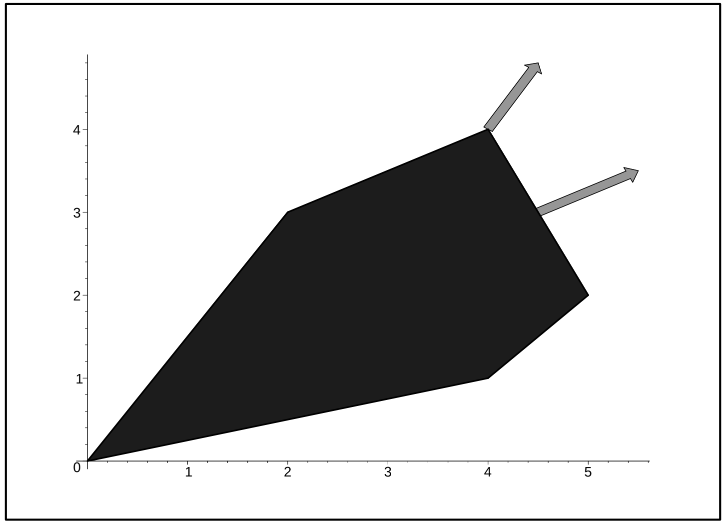
Use of the Newton Poltyope of the resultant

- I.Z. Emiris and I.S. Kotsireas (2003-2005)
- I.Z. Emiris, C. Konaxis and L. Palios (2007)

Tropical Geometry

- B. Sturmfels, J. Tevelev and J. Yu (2006)
- P. Huggins, B. Sturmfels and J. Yu (2007)

Mixed Volumes and Weighted Degrees



$$\deg_\sigma(P) := \max_{p\in N(P)}\langle \sigma, p\rangle$$

Theorem

(D-Sombra)

$$\deg_\sigma(P) = \frac{MV_{\mathbb{R}^{2n+1}}(\Delta,\ldots,\Delta,\mathcal{D}_0,\ldots,\mathcal{D}_n)}{\deg(\phi)} + \sum_{\sigma_i<0}\sigma_i\ell_i$$

- $\mathcal{D}_i := \mathcal{B}_i \times \{\sigma_i\,\mathbf{e}_i\} \cup \mathcal{A}_i \subset \mathbb{Z}^{n+1} \times \mathbb{Z}^n$
- $\Delta := \{0, \mathbf{e}_0, \ldots, \mathbf{e}_n\} \subset \mathbb{Z}^{n+1}$
- $\ell_j = MV_{\mathbb{L}}(\mathcal{C}_i)_{i\in J\setminus\{j\}}$

Sketch of the Proof

$$\begin{array}{ccccccc}
G(\sigma) & \subset & \mathbb{T}^n \times \mathbb{T}^{n+1} & \xrightarrow{(\tau,x)\mapsto(\tau,x^\sigma)} & \mathbb{T}^n \times \mathbb{T}^{n+1} & \supset & G \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\
V(P(X^\sigma)) & \subset & \mathbb{T}^{n+1} & \xrightarrow{x\longrightarrow x^\sigma} & \mathbb{T}^{n+1} & \supset & V(P)
\end{array}$$

- $G(\sigma) := V\left(B_i(t)X_i^{\sigma_i} - A_i(t),\ i = 0, \ldots, n\right)$
- $G := G\left((1, 1, \ldots, 1)\right)$
- $P(X^\sigma)$ is reduced
- $\deg_\sigma(P) = \deg\left(P(X^\sigma)\right) = \deg\left(V(P(X^\sigma))\right) - \sum_{\sigma_i<0} \sigma_i \ell_i$

Applications

- If there is i such that $\mathcal{A}_i = \mathcal{B}_i$ then

  $N(P) = \prod_{j=0}^{n+1} [0, \ell_j]$

- If there is i such that $\mathcal{A}_i \subset \mathcal{B}_i$ then

  $N(P) \supset \prod_{j=0}^{i-1} [0, \ell_j] \times \{\ell_i\} \times \prod_{j=i+1}^{n+1} [0, \ell_j]$

Idea:

Study the differentiability of

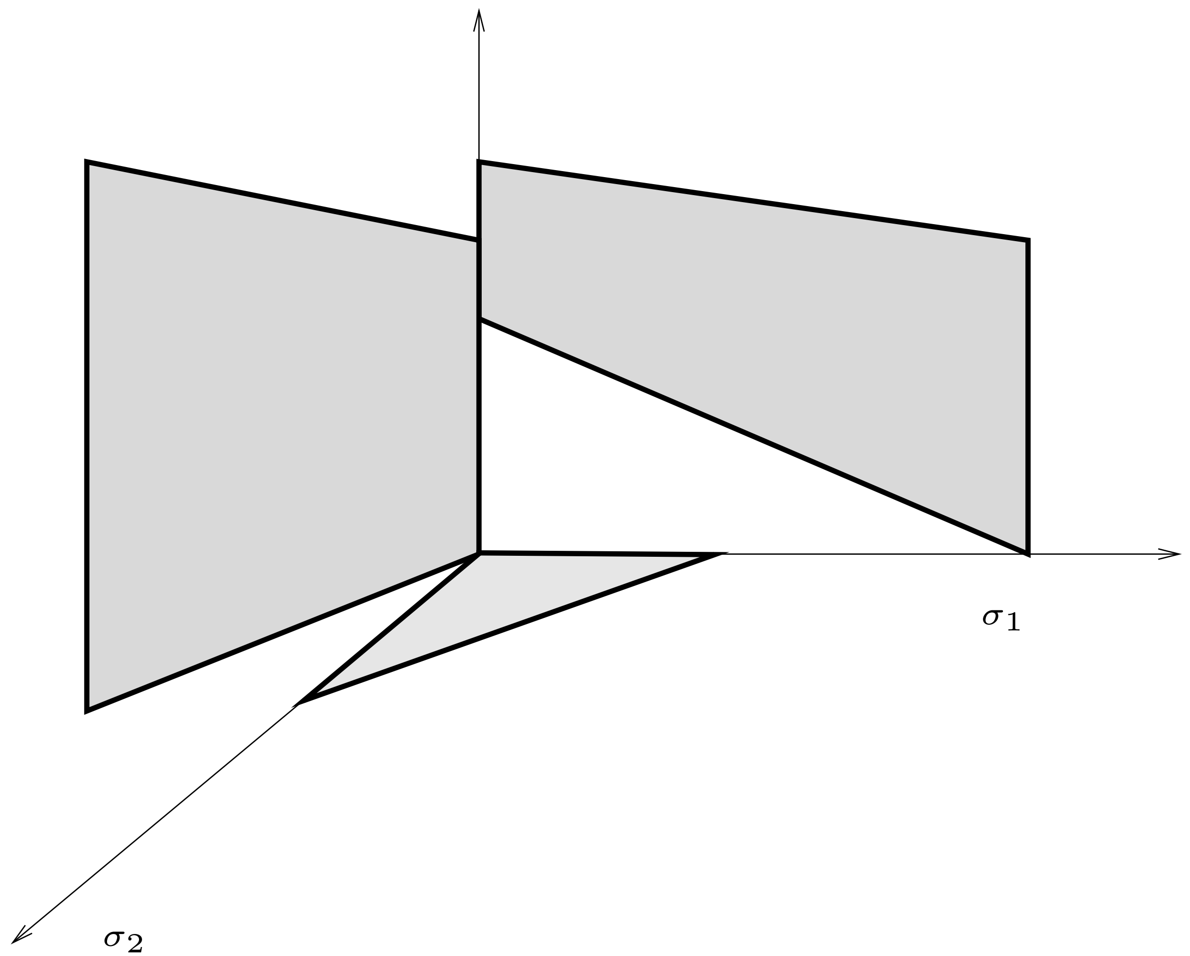
$$\deg_\sigma(P) = \frac{MV(\sigma)}{\deg(\phi)} + \sum_{\sigma_i < 0} \sigma_i \ell_i = \langle \sigma, p_\sigma \rangle$$

Rational Parametric Curves

$$X_1 = \frac{p(t)}{q(t)} \quad X_2 = \frac{r(t)}{s(t)}$$

- $p(t), q(t), r(t), s(t)$ are supported in the integer points of $[p_0, p_1]$, $[q_0, q_1]$, $[r_0, r_1]$, $[s_0, s_1]$

Computation of the Mixed Volume



- If $\sigma_1 > 0,\ \sigma_2 > 0:$

$$MV(\sigma) =$$

$$\max\left(\langle\sigma,(s_1,p_1)\rangle,\langle\sigma,(s_1,q_1)\rangle,\langle\sigma,(r_1,q_1)\rangle\right) +$$

$$\max\left(\langle\sigma,(-s_0,-p_0)\rangle,\langle\sigma,(-s_0,-q_0)\rangle,\langle\sigma,(-r_0,-q_0)\rangle\right)$$

- If $\sigma_1 < 0,\ \sigma_2 > 0:$

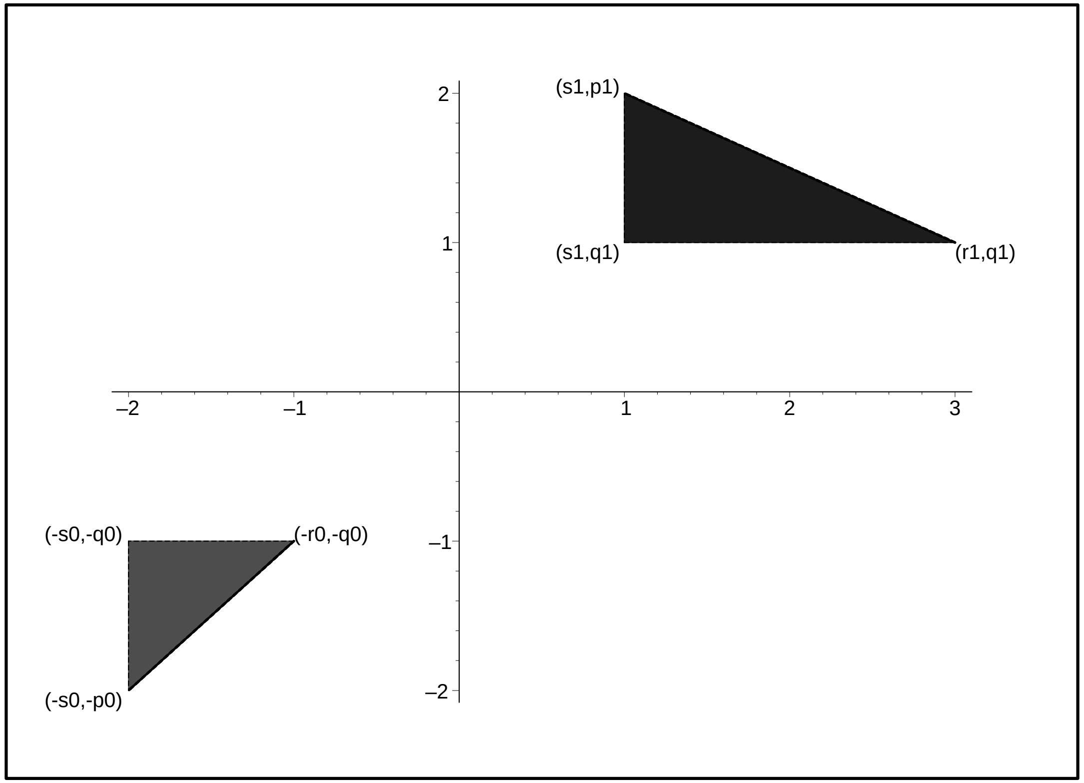
$$MV(\sigma) =$$

$$\max\left(\langle\sigma,(-s_1,q_1)\rangle,\langle\sigma,(-s_1,p_1)\rangle,\langle\sigma,(-r_1,p_1)\rangle\right) +$$

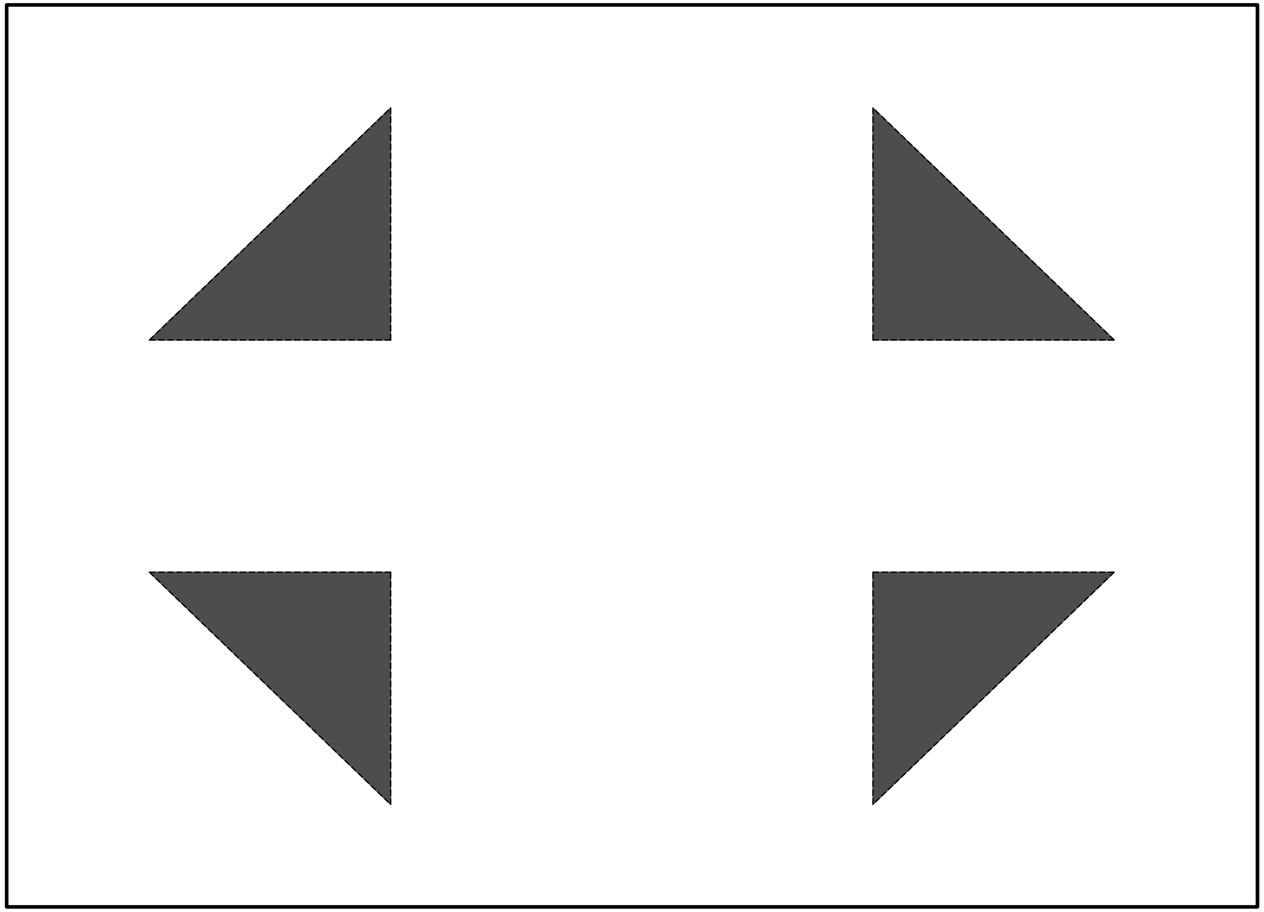
$$\max\left(\langle\sigma,(s_0,-q_0)\rangle,\langle\sigma,(s_0,-p_0)\rangle,\langle\sigma,(r_0,-p_0)\rangle\right)$$

$$\cdots$$

“Visualization” of the cuts

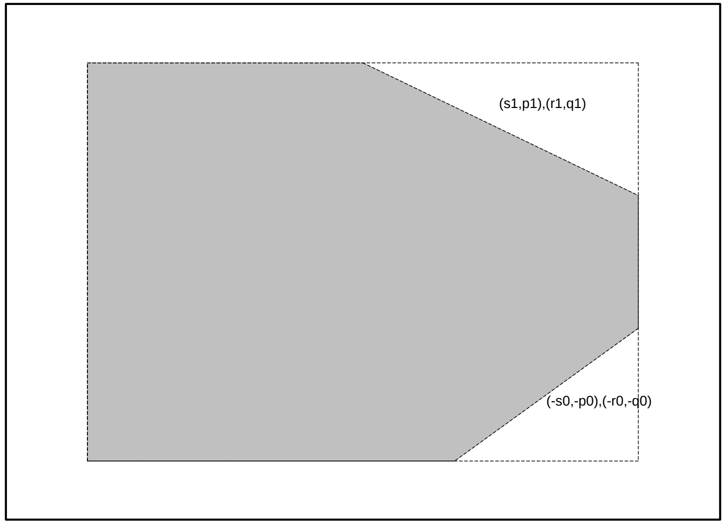


Possible configurations



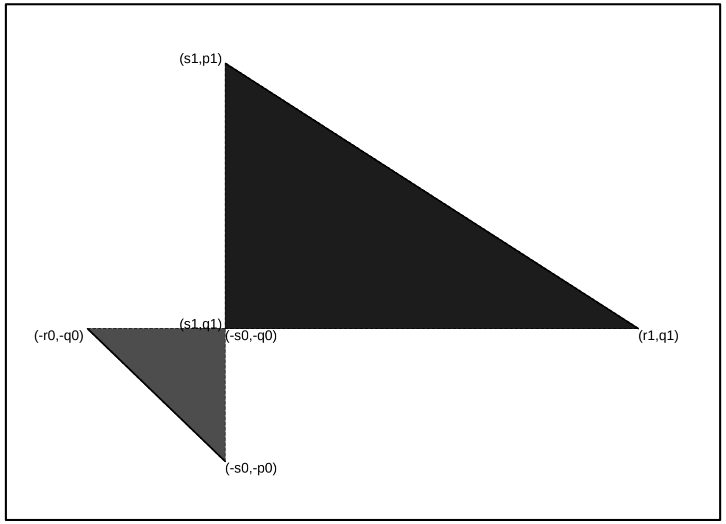
Theorem

(D-S)

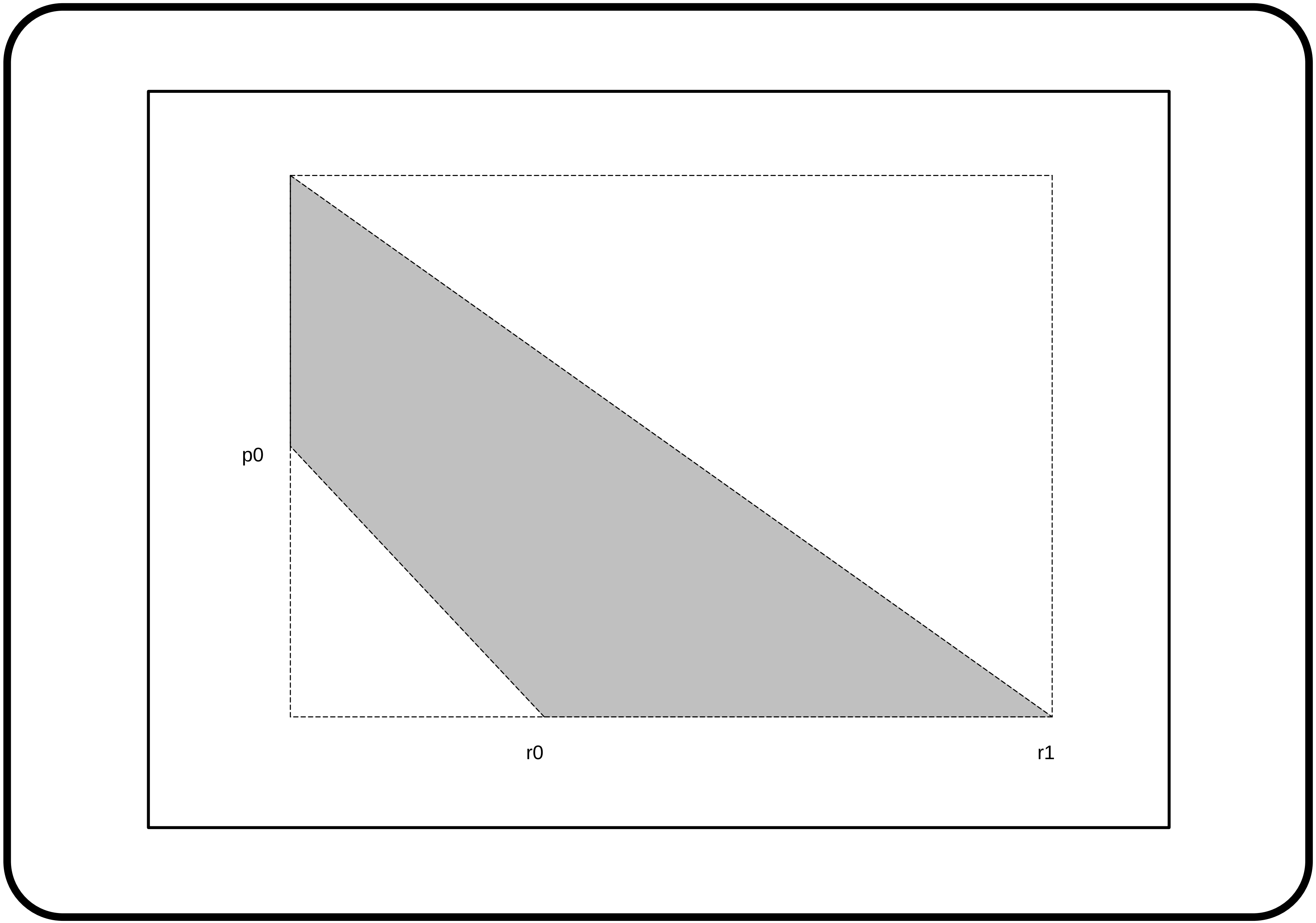


$N(P)$ is a (possibly degenerated) hexagon made by cutting two edges: $(s_1 - r_1, p_1 - q_1)$ and $(s_0 - r_0, p_0 - q_0)$

Polynomial parametrizations



- $q_0 = q_1 = s_0 = s_1 = 0$
- $\ell_1 = r_1$
- $\ell_2 = p_1$



p0
r0
r1

Future Work

- Surfaces
- Understanding the structure of mixed fiber polytopes
- What else can we say about the shape of $N(P)$?
  - When is $N(P) = \prod_{j=0}^{n+1} [0, \ell_j]$?
  - Which is the class of polytopes that can be $N(P)$?
  - $\ldots$