Computing Singularities of rational plane curves

Carlos D'Andrea

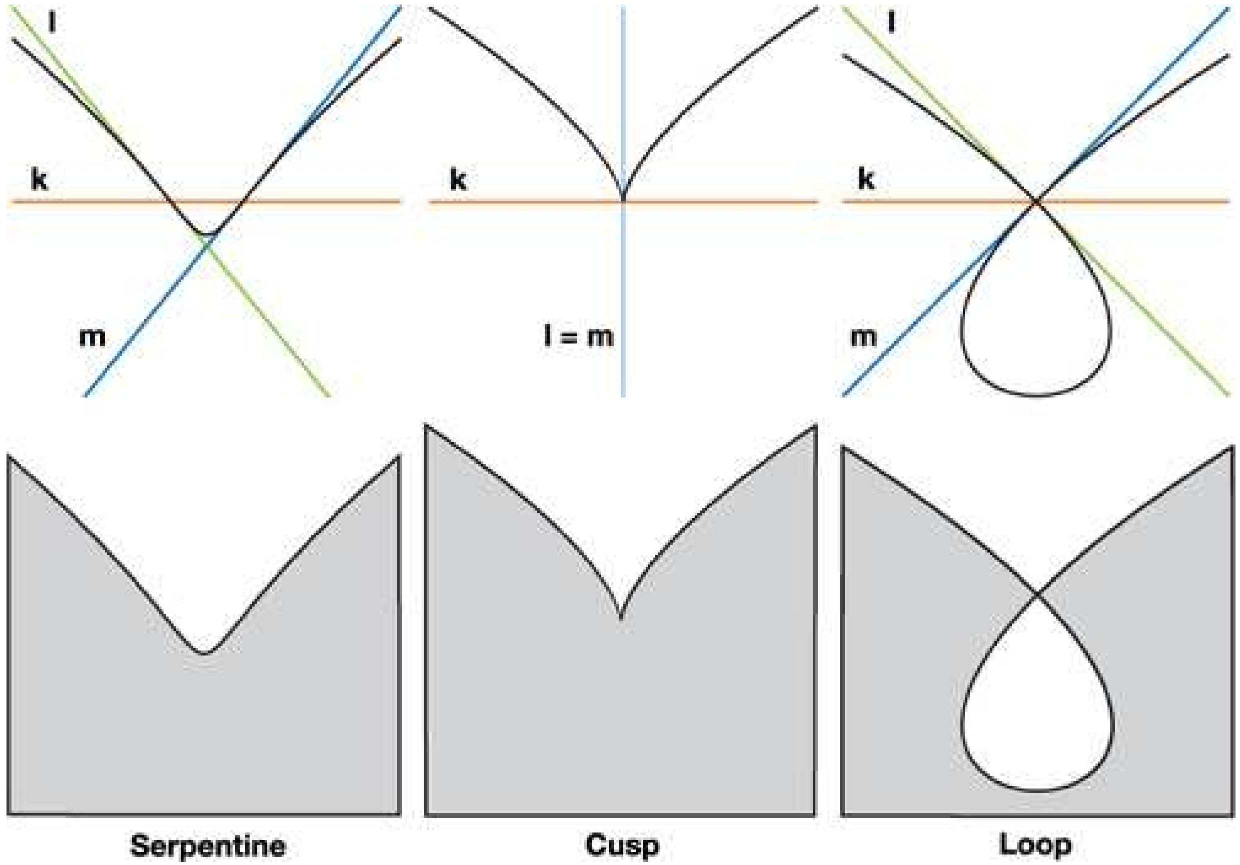
UNIVERSITAT DE BARCELONA

cdandrea@ub.edu

http://atlas.mat.ub.es/personals/dandrea

(Joint work with Laurent Busé at INRIA Sophia-Antipolis)

Singularities of plane curves



Rational Parametrizations

$$\begin{array}{rccc} \phi: & \mathbb{P}^1_{\mathbb{C}} & \to & \mathbb{P}^2_{\mathbb{C}} \\ & (s_0 : v_0) & \mapsto & (a(s_0,v_0) : b(s_0,v_0) : c(s_0,v_0)) \end{array}$$

- $a, b, c \in \mathbb{C}[s,v]_n$
- $\gcd(a,b,c) = 1$
- $\mathcal{C} := \phi(\mathbb{P}^1)$ curve of degree n

The map detects the singularities

$$\boxed{m_P(\mathcal{C}) = \#\phi^{-1}(P)}$$

- Abhyankar
- Sendra and Winkler
- Gutierrez, Rubio and Yie
- Pérez-Diaz

Computation of singularities

$$\begin{cases} A(s,v;t,u) \;:=\; \dfrac{a(s,v)-a(t,u)}{us-tv} \\[2ex] B(s,v;t,u) \;:=\; \dfrac{b(s,v)-b(t,u))}{us-tv} \\[2ex] C(s,v;t,u) \;:=\; \dfrac{c(s,v)-c(t,u)}{us-tv} \end{cases}$$

$$V(A,B,C) = \mathcal{A}_\phi \sqcup \mathcal{B}_\phi \subset \mathbb{P}^1 \times \mathbb{P}^1$$

$$\mathcal{A}_\phi = \{(\alpha,\beta) : \alpha \neq \beta,\ \phi(\alpha) = \phi(\beta)\}$$

$$\mathcal{B}_\phi = \{(\alpha,\alpha) : \nabla\phi(\alpha) = 0\}$$

$$\pi : V(A,B,C) \to \mathbb{P}^1$$

encodes the fibers of the singular points

Elimination theory: resultants

$$\begin{cases} F(s,v) &:= & a_0 s^2 + a_1 sv + a_2 v^2 \\ G(s,v) &:= & b_0 s^2 + b_1 sv + b_2 v^2 \end{cases}$$

$$\boxed{Sylv_{s,v}(F,G) = \begin{pmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{pmatrix}}$$

$$Bez_{s,v}(F,G) = \begin{pmatrix} a_0b_1 - a_1b_0 & a_2b_0 - a_0b_2 \\ a_0b_2 - a_2b_0 & a_2b_1 - a_1b_2 \end{pmatrix}$$

$$Hyb_{s,v}(F,G) = \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ 0 & a_0b_2 - a_2b_0 & a_1b_2 - a_2b_1 \end{pmatrix}$$

$$\mathrm{Res}_{s,v}(f,g) := \det\left(Sylv_{s,v}(f,g)\right)$$
$$= \det\left(Bez_{s,v}(f,g)\right) = \det\left(Hyb_{s,v}(f,g)\right)$$

The Taylor Resultant

(Abhyankar)

$$\boxed{\phi = (a : b : c)}$$

$$\begin{cases} F(s,v;t,u) & := \; a(s,v)c(t,u) - a(t,u)c(s,v) \\ G(s,v;t,u) & := \; b(s,v)c(t,u) - b(t,u)c(s,v) \end{cases}$$

$$P(t,u) := \mathrm{Res}_{(s,v)}\left(\frac{F(s,v;t,u)}{su-tv}, \frac{G(s,v;t,u)}{su-tv}\right)$$

Factorization of $P(t,u)$??

If $c = v^n$ (Abhyankar)

$$\boxed{P(t,u) = \prod_{\substack{i=1,\ldots,r \\ j\in I_i}} (u_{i,j}t - t_{i,j}u)^{\epsilon_{i,j}}}$$

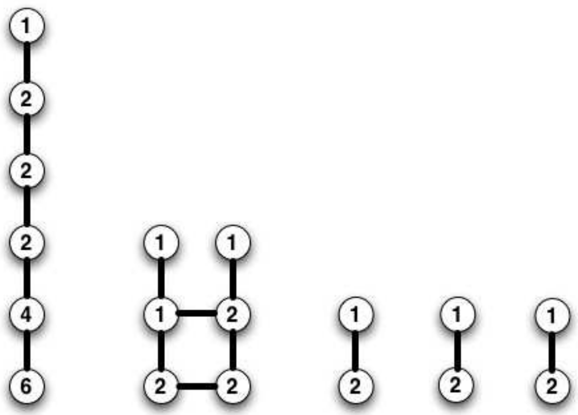
- $\{P_1, \ldots, P_r\}$ the proper multiple points of $\mathcal{C}$
- $\phi(t_{i,j} : u_{i:j}) = P_i,\ j \in I_i$
- $\epsilon_{i,j} = \sum_{h\geq 0} m^i_{j,h}(m_{P^h_{j,h}}(\mathcal{C}) - 1)$

Understanding the exponents

- $\mathfrak{z}_j^i,\ j \in I_i$, the irreducible branch-curves of $\mathcal{C}$ at P_i
- $(P_{j,h}^i)_{0\le h}$ the neighboring point sequence of $\mathfrak{z}_j^i$ at P_i
- $(m_{j,h}^i)_{0\le h}$ the multiplicity sequence of $\mathfrak{z}_j^i$ at P_i
- $(\sim_h)_{0\le h}$ the equivalence relations of the multiplicity graph of $\mathcal{C}$
- $m_{P_{j,h}^i}(\mathcal{C}) := \sum_{j' \sim_h j} m_{j',h}^i$

Example

$$\begin{cases} a = s^2 (2s+v)^2 (s+v)^6 \\ b = s^3 (2s+v)^5 (3s^2+2sv+v^2) \\ c = -(s+v)^{10} \end{cases}$$



Working with resultant matrices

The maximal minors of $Sylv^{2n}_{(s,v)}(F,G)$, $Hyb^{2n-k}_{(s,v)}(F,G)$ or $Bez^{n}_{(s,v)}(F,G)$ encode the fibers of the multiple points

(Chiohn and Sederberg)

$$Sylv^{2n}_{(s,v)}(F,G)\cdot\begin{pmatrix} s_0^{2n-1} & {s_0'}^{2n-1} & \dots & {s_0^{(k)}}^{2n-1} \\ s_0^{2n-2}v_0 & {s_0'}^{2n-2}v_0' & \dots & {s_0^{(k)}}^{2n-2}v_0^{(k)} \\ \vdots & \vdots & \dots & \vdots \\ v_0^{2n-1} & {v_0'}^{2n-1} & \dots & {v_0^{(k)}}^{2n-1} \end{pmatrix}$$

has rank one for $\phi(s_0:v_0)=\phi(s_0':v_0)=\ldots=\phi(s_0^{(k)},v_0^{(k)})$

Smaller resultant matrices via μ-bases

- $a, b, c \in \mathbb{C}[s, v]_n$
- $\gcd(a, b, c) = 1$

The Syzygy module $Syz(a, b, c)$ is free of rank 2

(Hilbert-Burch)

$$\boxed{0 \to \mathbb{C}[s,v]^2 \xrightarrow{(\mathbf{p},\mathbf{q})} \mathbb{C}[s,v]^3 \xrightarrow{(a,b,c)} \mathbb{C}[s,v] \to (a,b,c) \to 0}$$

$\{\mathbf{p},\ \mathbf{q}\}$ is called a μ-basis with $\mu = \min\{\deg(\mathbf{p}),\ \deg(\mathbf{q})\}$

$$\big(\deg(\mathbf{p}) + \deg(\mathbf{q}) = n\big)$$

In our example

$$\begin{cases} a &= s^2\,(2\,s+v)^2\,(s+v)^6 \\ b &= s^3\,(2\,s+v)^5\,\left(3\,s^2+2\,sv+v^2\right) \\ c &= -\,(s+v)^{10} \end{cases}$$

$$\mathbf{p} = \left((s+v)^4, 0, s^2(2s+v)^2\right)$$

$$\mathbf{q} = \left(s(3\,s^2+2\,sv+v^2)(2\,s+v)^3, -(s+v)^6, 0\right)$$

Smaller resultant matrices

- $\mathbf{p} = (p_1, p_2, p_3)$, $\mathbf{q} = (q_1, q_2, q_3)$
- $p_\phi = p_1(s,v)a(t,u) + p_2(s,v)b(t,u) + p_3(s,v)c(t,u)$
- $q_\phi = q_1(s,v)a(t,u) + q_2(s,v)b(t,u) + q_3(s,v)c(t,u)$

The maximal minors of $Hyb^{n-\mu}_{(s,v)}\left(p_\phi, q_\phi\right)$ encode the fibers of the multiple points (Chen, Wang and Liu)

Stronger fact

(Chen, Wang and Liu)

The invariant factors of $Hyb_{s,v}\big(p_\phi, q_\phi\big)$ give a stratification of the singularities of $\mathcal{C}$

$$Hyb_{s,v}(p_\phi, q_\phi) \sim \text{diag}\big(\alpha_1, \ldots, \alpha_{n-\mu-1}, 0\big)$$

with $\alpha_i \mid \alpha_{i+1}$

$d_i := \frac{\alpha_i}{\alpha_{i-1}}$, the singular factors of the parametrization

Singular factors

(Chen, Wang and Liu)

- $P_i = \phi(t_0 : u_0)$ has order k iff $d_{n-\mu-k}(t_0 : u_0) = 0$ and $d_{n-\mu-j}(t_0 : u_0) \neq 0 \, \forall j > k$
- $h_k \,|\, d_{n-\mu-k}$, with $h_k =$ the product of all inversion formulas of points of order k
- $h_k = d_{n-\mu-k} \, \forall k \iff \mathcal{C}$ has no virtual points

Extraneous factors

What is $\dfrac{d_{n-\mu-k}}{h_k}$??

(we know that $\{d_{n-\mu-k} = 0\} = \{h_k = 0\}$)

Some conjectures given by Chen, Wang and Liu in terms of the virtual points of $\mathcal{C}$

Theorem

(Busé - D)

$$d_{n-\mu-k}(t,u) = \prod_{i=1,\dots,r,\ j\in I_i} (u_{i,j}t - t_{i,j}u)^{\epsilon_{i,j}^k}$$

$$\epsilon_{i,j}^k = \sum_{h \text{ such that } m_{P_{j,h}^i}(\mathcal{C})=k} m_{j,h}^i$$

(arXiv:0912.2723v1)

In particular

$$d_{n-\mu-k}(t,u) = h_k(t,u) \prod_{\substack{i=1,\ldots,r \\ j\in I_i}} (u_{i,j}t - t_{i,j}u)^{\overline{\epsilon}^k_{i,j}}$$

with

$$\overline{\epsilon}^k_{i,j} = \sum_{h>0 \text{ such that } m_{P^i_{j,h}}(\mathcal{C})=k} m^i_{j,h} = \epsilon^k_{i,j} - m^i_{j,0}$$

In our Example

$$(u = 1)$$

$$d_1(t) = (t+1)^6,\ d_2(t) = 1$$

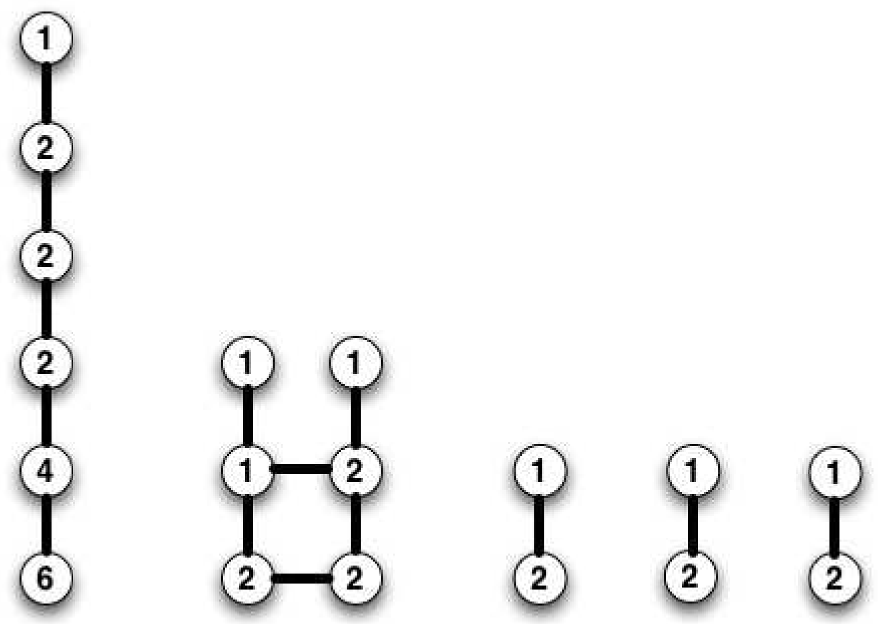
$$d_3(t) = \frac{1}{4}\,(2\,t+1)^2\,(t+1)^4\,t^2$$

$$d_4(t) = \frac{1}{4}\,(2\,t+1)^2\,t$$

$$d_5(t) =$$

$$\frac{1}{43}\,(43\,t^6 + 74\,t^5 + 71\,t^4 + 48\,t^3 + 21\,t^2 + 6\,t + 1)\,(t+1)^6$$

We can build the multiplicity graph from the singular factors!



(not always the case)

Some tools

The singular factors of ϕ do not depend neither on the choice of the μ-basis nor the coordinates of $\mathbb{P}^2$

Some tools (2)

(Busé-D)

$$\text{coker}\Big(Sylv_{(s,v)}(p_\phi, q_\phi)\Big)$$

$$\simeq$$

$$\text{coker}\Big(Hyb_{(s,v)}(p_\phi, q_\phi)\Big) \quad \text{as } \mathbb{C}[t,u]\text{-modules}$$

$$\simeq$$

$$\text{coker}\Big(Bez_{(s,v)}(p_\phi, q_\phi)\Big)$$

Idea of the proof

- induction on the length of the resolution of singularities
- Thompson's Theorem: relates the invariant factors of $A, B, C \in \mathbb{C}[t]^{n\times n}$ when $AB = C$

By-Product

(Busé - D)

A complete factorization of the Taylor resultant:

$$\boxed{\mathrm{Res}_{(s,v)}\left(\frac{a(s,v)c(t,u)-a(t,u)c(s,v)}{su-tv}, \frac{b(s,v)c(t,u)-b(t,u)c(s,v)}{su-tv}\right) = c(t,u)^{n-1}d_n^{n-1}d_{n-1}^{n-2}\dots d_2}$$

By-Product(2)

(Busé - D)

A complete factorization of

$$P(t) := \mathrm{Res}_s\left(\frac{a(s)d(t)-a(t)d(s)}{s-t}, \frac{b(s)c(t)-b(t)c(s)}{s-t}\right)$$

in terms of the invariants of $\left(\frac{a}{d}, \frac{b}{c}\right), \left(\frac{a}{d}, \frac{c}{b}\right), \left(\frac{d}{a}, \frac{b}{c}\right), \left(\frac{d}{a}, \frac{c}{b}\right)$

(Gutierrez-Rubio-Yu)

By-product (3)

Good definition of the Taylor Resultant for rational functions:

$$\Delta_i := d_1^i d_2^{i-1} \ldots d_i$$

$$\Delta_{n-\mu-1} = SRes_1(p_\phi, q_\phi) = \prod_{\substack{i=1,\ldots,r \\ j \in I_i}} (u_{i,j}t - t_{i,j}u)^{\epsilon_{i,j}}$$

(cf. Busé)

Principal subresultants vs minors

- $S_0,\ S_1,\ S_2,\ \ldots$ with $S_0 = \mathrm{Res}_{s,v}(F,G)$
- $\Delta_0,\ \Delta_1, \ldots,$ with
  Δ_i “ $=''$ $\mathrm{gcd}\left((n-i)-\right.$ minors of $\left. Bez_{s,v}(F,G)\right)$
- both “measure” the degree of the gcd of two polynomials
- Is there a connection between them?

In our example

- $S_0 = \Delta_0 = 0$
- $S_1 = \Delta_1 = d_1^5 d_2^4 d_3^3 d_2^2 d_1$ (Abhyankar)
- $S_2 = \Delta_2(192t^{11} + 2369t^{10} + \ldots)$
- $S_3 = \Delta_3(256t^{10} + 3008t^9 + \ldots)$
- $S_4 = \Delta_4(2t+3)^2(2t^2+5t+4)^2$

Conjecture

$$\Delta_j = \gcd(S_j, S_{j-1}, \ldots, S_0)$$

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Thanks!!