RESULTANTS AND SUBRESULTANTS UNIVARIATE VS MULTIVARIATE CASE

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$$\left\{ \begin{array}{rcl} A\mathbf{x}_1 + B\mathbf{x}_2 & = & 0 \\ C\mathbf{x}_1 + D\mathbf{x}_2 & = & 0 \end{array} \right.$$

has a non trivial solution iff

$$AD - BC = 0$$

$$
\begin{cases}
A_{11}\mathbf{x}_1 + A_{12}\mathbf{x}_2 + \ldots + A_{1n}\mathbf{x}_n & = & 0 \\
A_{21}\mathbf{x}_1 + A_{22}\mathbf{x}_2 + \ldots + A_{2n}\mathbf{x}_n & = & 0 \\
\vdots & \vdots & \vdots \\
A_{n1}\mathbf{x}_1 + A_{n2}\mathbf{x}_2 + \ldots + A_{nn}\mathbf{x}_n & = & 0
\end{cases}
$$

$$\begin{cases} f(\mathbf{x}_1, \mathbf{x}_2) & := & A_0\mathbf{x}_1^n + A_1\mathbf{x}_1^{n-1}\mathbf{x}_2 + \ldots + A_n\mathbf{x}_2^n & = & 0 \\ g(\mathbf{x}_1, \mathbf{x}_2) & := & B_0\mathbf{x}_1^m + B_1\mathbf{x}_1^{m-1}\mathbf{x}_2 + \ldots + B_m\mathbf{x}_2^m & = & 0 \end{cases}$$

$$\boxed{\textbf{non-trivial solution} \quad \Longleftrightarrow \quad ???}$$

Proposition

The following are equivalent:

1. $\exists\,(x_0 : y_0) \in \mathbb{P}^1_{\mathbb{K}} :\; f(x_0, y_0) = g(x_0, y_0) = 0.$
2. $\exists h(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{K}[\mathbf{x}_1, \mathbf{x}_2]_{\geq 1} :\; h(\mathbf{x}_1, \mathbf{x}_2)$ divides $f(\mathbf{x}_1, \mathbf{x}_2)$ and $g(\mathbf{x}_1, \mathbf{x}_2)$.
3. Same than above, but with any $K \subset \mathbb{K}$.

$$\begin{array}{rcl} K[\mathbf{x}_1,\mathbf{x}_2]_{m-1} \oplus K[\mathbf{x}_1,\mathbf{x}_2]_{n-1} & \longrightarrow & K[\mathbf{x}_1,\mathbf{x}_2]_{m+n-1} \\ (a,b)) & \longmapsto & af+bg \end{array}$$

$$\mathsf{Res}(f,g) := \det \begin{array}{cc} m+n & \\ \left[\begin{array}{ccccc} A_n & \cdots & A_0 & & \\ & \ddots & & \ddots & \\ & & A_n & \cdots & A_0 \\ \hline B_m & \cdots & B_0 & & \\ & \ddots & & \ddots & \\ & & B_m & \cdots & B_0 \end{array}\right] & \begin{array}{c} m \\ \\ \\ \\ n \end{array} \end{array}$$

$$\deg\left(\gcd(f, g)\right) = 0$$

$$\Updownarrow$$

$$\mathsf{Res}(f, g) \neq 0$$

$$\deg\left(\gcd(f,g)\right) = k$$

$$\Updownarrow$$

???

$$\begin{array}{rcl} K[\mathbf{x}_1,\mathbf{x}_2]_{m-1-k}\oplus K[\mathbf{x}_1,\mathbf{x}_2]_{n-1-k} & \longrightarrow & K[\mathbf{x}_1,\mathbf{x}_2]_{m+n-1-k} \\ (a,b) & \longmapsto & af+bg \end{array}$$

$$S_k^{(j)} := \det \begin{array}{cc} m+n-2k & \\ \left|\begin{array}{ccccc} A_n & \cdots & \cdots & A_{k+1-(m-k-1)} & A_{j-(m-k-1)} \\ & \ddots & & \vdots & \vdots \\ & & A_n \quad \cdots & A_{k+1} & A_j \\ \hline B_m & \cdots & \cdots & B_{k+1-(n-k-1)} & B_{j-(n-k-1)} \\ & \ddots & & \vdots & \vdots \\ & & B_m \quad \cdots & B_{k+1} & B_j \end{array}\right| & \begin{array}{c} m-k \\ \\ \\ \\ n-k \end{array} \end{array}$$

$(j \leq k)$

$$\mathsf{Sres}(f,g) := \sum_{j=0}^{k} S_k^{(j)} \mathbf{x}_1^j \mathbf{x}_2^{k-j}$$

$$\boxed{\begin{gathered} \deg\left(\gcd(f,g)\right) = k \\ \Updownarrow \\ S_\ell^{(\ell)} = 0 \text{ for } 0 \le \ell < k \text{ and } S_k^{(k)} \neq 0 \end{gathered}}$$

$$\gcd(f,g) = \mathsf{Sres}(f,g)$$

MULTIVARIATE RESULTANTS

$$\left\{\begin{array}{lclcl} f_1(\mathbf{X}) & := & \sum_{|\alpha|=d_1} A_{1\alpha}\mathbf{X}^\alpha & = & 0 \\ f_2(\mathbf{X}) & := & \sum_{|\alpha|=d_2} A_{2\alpha}\mathbf{X}^\alpha & = & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_3(\mathbf{X}) & := & \sum_{|\alpha|=d_n} A_{n\alpha}\mathbf{X}^\alpha & = & 0 \end{array}\right.$$

$$\mathbf{X}^\alpha = {\mathbf{x}_1}^{\alpha_1}{\mathbf{x}_2}^{\alpha_2}\ldots{\mathbf{x}_n}^{\alpha_n}$$

$$\boxed{\texttt{non-trivial solution} \iff \text{???}}$$

Theorem

$\exists$ Res $\in \mathbb{Z}[A_{i,\alpha}]$ which vanishes iff there is a solution of the system in $\mathbb{P}^{n-1}_{\overline{K}}$

$$\begin{array}{ccc} \bigoplus_{i=1}^n K[\mathbf{X}]_{t-d_i} & \longmapsto & K[\mathbf{X}]_t \\ (A_1, \ldots, A_n) & \longmapsto & \sum_{i=1}^n A_i f_i \end{array}$$

$$(t > (d_1 - 1) + \ldots + (d_n - 1))$$

MULTIVARIATE SUBRESULTANTS

$$S_k^{(j)} = 0$$

$$\Updownarrow$$

$$\left\{ \begin{array}{c} \mathbf{x}_1^{m+n-k-1}, \mathbf{x}_1^{m+n-k-2}\mathbf{x}_2, \ldots, \mathbf{x}_1^{m+n-k-j}\mathbf{x}_2^{j-1}, \\ \mathbf{x}_1^{m+n-k-j-2}\mathbf{x}_2^{j+1}, \ldots, \mathbf{x}_1^{m+n-2k-1}\mathbf{x}_2^{k} \end{array} \right\}$$

is linearly dependent in

$$K[\mathbf{x}_1, \mathbf{x}_2]/\langle f(\mathbf{x}_1, \mathbf{x}_2), g(\mathbf{x}_1, \mathbf{x}_2)\rangle$$

MULTIVARIATE SUBRESULTANTS

$$S(T) := \frac{\prod_{i=1}^{s}(1-T^{d_i})}{(1-T)^n} = s_0 + s_1 T + \dots$$

$$(s \leq n)$$

$$\mathcal{S} = \{\mathbf{X}^{\alpha_1}, \dots, \mathbf{X}^{\alpha_{s_t}}\} \subset K[\mathbf{X}]_t$$

Theorem

$\exists\, \Delta_{\mathcal{S}} \in \mathbb{Z}[A_{i,\alpha}]$ such that $\Delta_{\mathcal{S}} = 0$ iff

$\mathcal{S}$ is l.d. in $K[\mathbf{X}]/\langle f_1(\mathbf{X}), \dots, f_s(\mathbf{X})\rangle$

ELIMINATION THEORY

- A commutative ring. $A[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ with $\deg(\mathbf{x}_i) = 1$
- $\mathcal{M} := \langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle$
- $I := \langle f_1, \ldots, f_s \rangle \subset \mathcal{M}$
- $B := A[\mathbf{X}]/I$

$$Proj(B) \subset \mathbb{P}_A^{n-1} := Proj\left(A[\mathbf{X}]\right)$$

$$\begin{aligned} \mathcal{U} \quad := \quad & Ker\left(A = \Gamma\left(Spec(A), \mathcal{O}_{Spec(A)}\right) \stackrel{can}{\to} \Gamma\left(Proj(B), \mathcal{O}_{Proj(B)}\right)\right) \\ & (I :_{A[\mathbf{x}]} \mathcal{M}^{\infty}) = H^0_{\mathcal{M}}(B)_0 \end{aligned}$$

$$H^0_{\mathcal{M}}(B) := \bigcup_{k \in \mathbb{N}} \left(0 :_B \mathcal{M}^k\right) = \{p \in B : \exists k \in \mathbb{N} \text{ such that } \mathcal{M}^k p = 0\}$$

ELIMINATION THEOREM

$\rho : A \longrightarrow K$ a specialization map.

The following are equivalent:

1. $\rho(\mathcal{U}) = 0$
2. $\exists\, K \hookrightarrow L$ and a non-trivial zero of I in L^n
3. I has a non-trivial zero in $\overline{K}^n$

SOLUTION OF OVER-DETERMINED SYSTEMS

(González Vega-Szanto)

"In general", n polynomials in $n-1$ variables do not have common zeroes

By using relations of the form

$$\Delta_{\{\mathbf{X}^\beta\}}\mathbf{X}^\alpha \pm \Delta_{\{\mathbf{X}^\alpha\}}\mathbf{X}^\beta \in \langle f_1, \ldots, f_n \rangle$$

We can recover the common zeroes of the input system

INTEGRATION OF RATIONAL FUNCTIONS

(Lazard $\&$ Rioboo - Trager)

$$\int \frac{P(t)}{Q(t)}dt = \sum_{b:\, S(b)=0} b \log(R_i(t,b))$$

- $S(y) :=$ $\text{Res}_t(P(t) - yQ'(t), Q(t))$
- $R_i(t,y) :=$the i-th subresultant polynomial

COMPUTATION OF RESIDUES

(Jouanolou - D & Khetan)

$f_1(\mathbf{X}), \ldots, f_n(\mathbf{X})$ with finite zeroes in $\mathbb{C}^n$.

$$\mathsf{R}_{\mathbf{f}}(\mathbf{X}^\alpha) := \left(\frac{1}{2\pi i}\right)^n \int_{|f_1(\mathbf{X})|=\epsilon} \cdots \int_{|f_n(\mathbf{X})|=\epsilon} \frac{\mathbf{X}^\alpha}{f_1(\mathbf{X}) \ldots f_n(\mathbf{X})} d\mathbf{x}_1 \wedge \ldots \wedge d\mathbf{x}_n$$

$$\boxed{\mathsf{R}_{\mathbf{f}}(x^\alpha) = \frac{\Delta_{\{\mathbf{x}^\alpha\}}}{\mathsf{Res}(f_1, \ldots, f_n)}}$$

SINGULARITIES OF THE RESULTANT

(Jouanolou)

Rational singularities of $\{\text{Res} = 0\}$

$$=$$

$$\{\Delta_{\{\mathbf{X}^\alpha\}} = 0, |\alpha| = d_1 + \ldots + d_n - n\}$$

Formulas in roots

$$\begin{cases} A_0\mathbf{x}_1^n + A_1\mathbf{x}_1^{n-1} + \ldots + A_n & = & A_0 \prod_{i=1}^n (\mathbf{x}_1 - \alpha_i) \\ B_0\mathbf{x}_1^m + B_1\mathbf{x}_1^{m-1} + \ldots + B_m & = & B_0 \prod_{j=1}^m (\mathbf{x}_1 - \beta_j) \end{cases}$$

$$\text{Res}(f,g) = {A_0}^m \prod_{i=1}^{n} g(\alpha_i, 1) = {A_0}^m {B_0}^n \prod_{i,j} (\alpha_i - \beta_j)$$

Similar results in the Multivariate case (ongoing work with Hong, Krick, Szanto)

OPEN QUESTIONS

A more geometric understanding of subresultants:

- for which $\mathcal{S}$ is $\Delta_{\mathcal{S}}$ identically zero?
- Which $\Delta_{\mathcal{S}}$'s are irreducible?
- If $\Delta_{\mathcal{S}}$ factorices, is it true that it factors as a product of subresultants?