Tropical Implicitization of Algebraic Plane Curves

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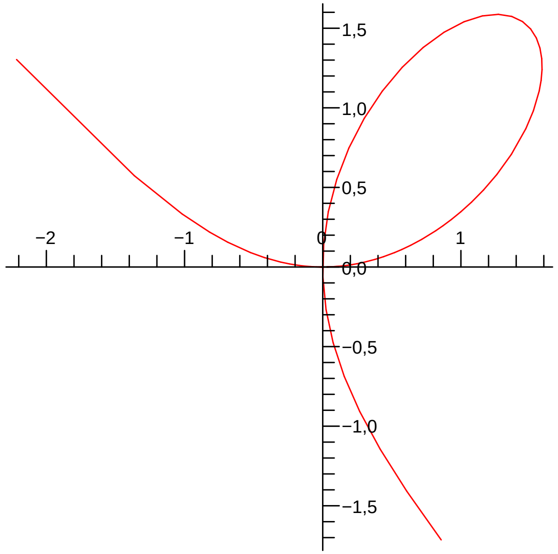
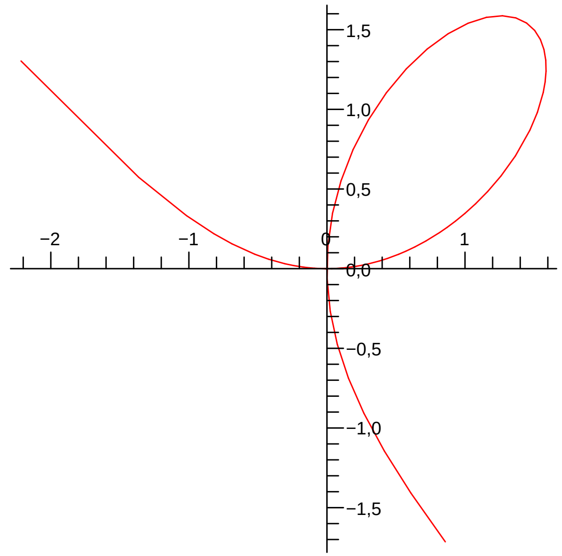
Classical Implicitization



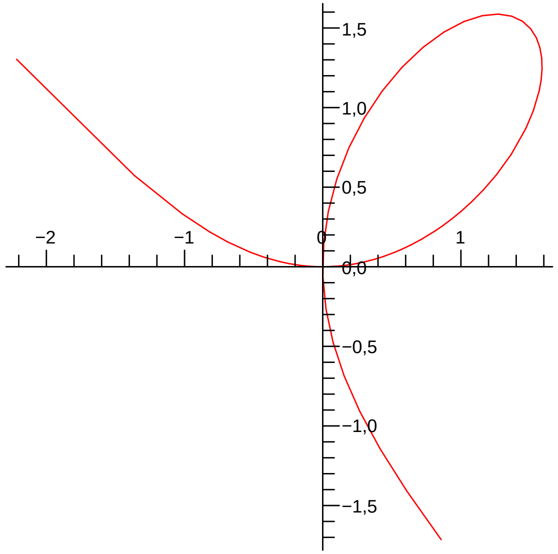
$$\rho : t \longmapsto \left(\frac{3t^2}{t^3+1}, \frac{3t}{t^3+1}\right)$$



The implicit equation of the (closure of the) image of ρ is

$$X^3 + Y^3 - 3XY = 0$$

Tropical Implicitization



The minimal polynomial of ρ is a linear combination of $\{X^3, Y^3, XY, \}$

Classical Implicitization

$$\rho(t) = \left(\frac{f_{\text{num}}(t)}{f_{\text{den}}(t)}, \frac{g_{\text{num}}(t)}{g_{\text{den}}(t)} \right)$$

Eliminate t from

$$f_{\text{den}}(t)\, x - f_{\text{num}}(t) = 0 \quad , \quad g_{\text{den}}(t)\, y - g_{\text{num}}(t) = 0$$

Challenge 1

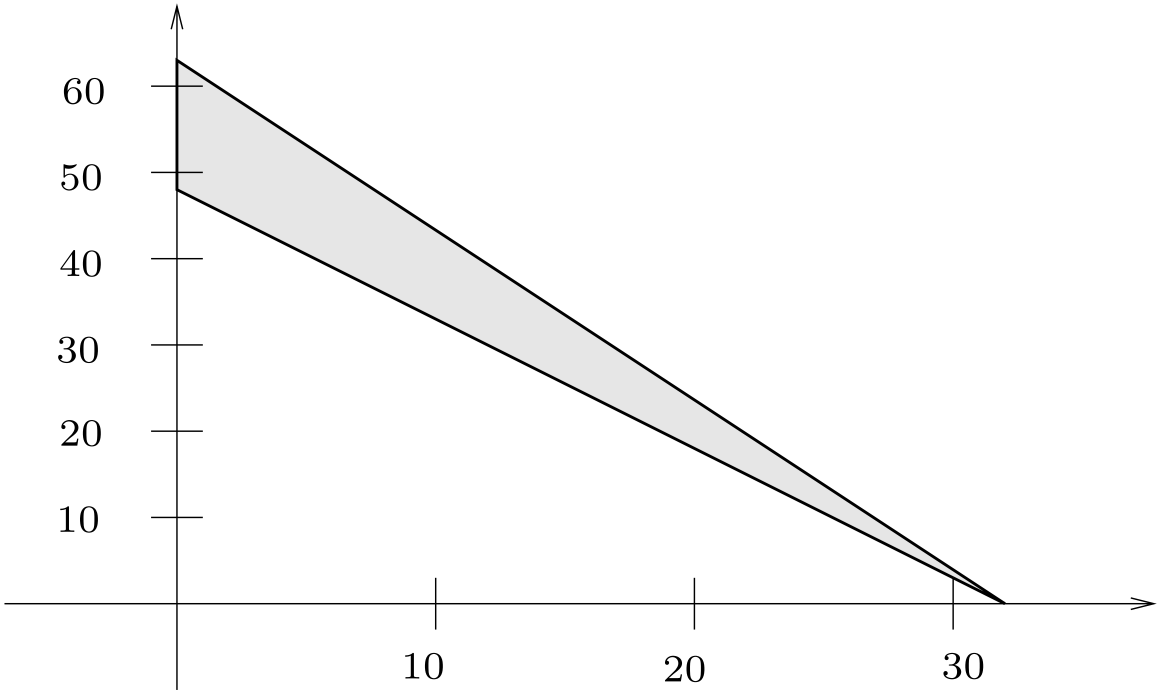
$$\rho : t \mapsto \left(\frac{t(t-1)^a}{(t+1)^{a+1}}, \frac{(t+1)^a}{t(t-1)^{a-1}} \right)$$

$$\{1, X^{a-1}Y^a,\ X^aY^{a+1}\}$$

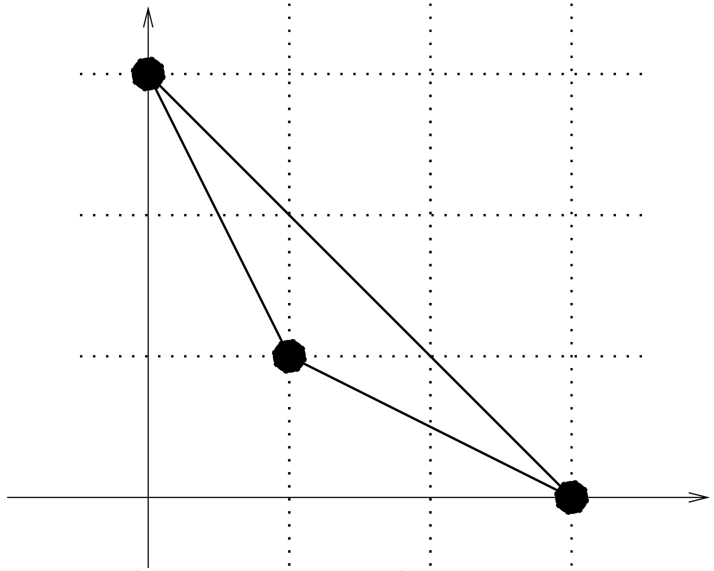
$$\boxed{2 - X^{a-1}Y^a - X^aY^{a+1}}$$

Challenge 2

$$\rho : t \mapsto \left(t^{48} - t^{56} - t^{60} - t^{62} - t^{63}, t^{32}\right)$$



The Newton Polygon of a polynomial



$$N(X^3 + Y^3 - 3XY)$$

Tropical Implicitization Problem

(Plane rational curves)

Given $\rho : \mathbb{K} \dashrightarrow \mathbb{K}^2$ a rational parametrization, compute $N(\rho)$

Tropical Implicitization

- Convex Geometry
- Combinatorics
- Intersection Theory
- Tropical Geometry

Preliminary approach

(Sturmfels, J.-T Yu, 92)

Given

$$N(f_{\mathrm{num}}),\ N(f_{\mathrm{den}}),\ N(g_{\mathrm{num}}),\ N(g_{\mathrm{den}})$$

compute $N(\rho)$

- I.Z. Emiris and I.S. Kotsireas (2003-2005)
- B. Sturmfels, J. Tevelev and J. Yu (2006)
- B. Sturmfels and J. Tevelev (2007)
- Khovanskii and Esterov (2007)

Approach with Multiplicities

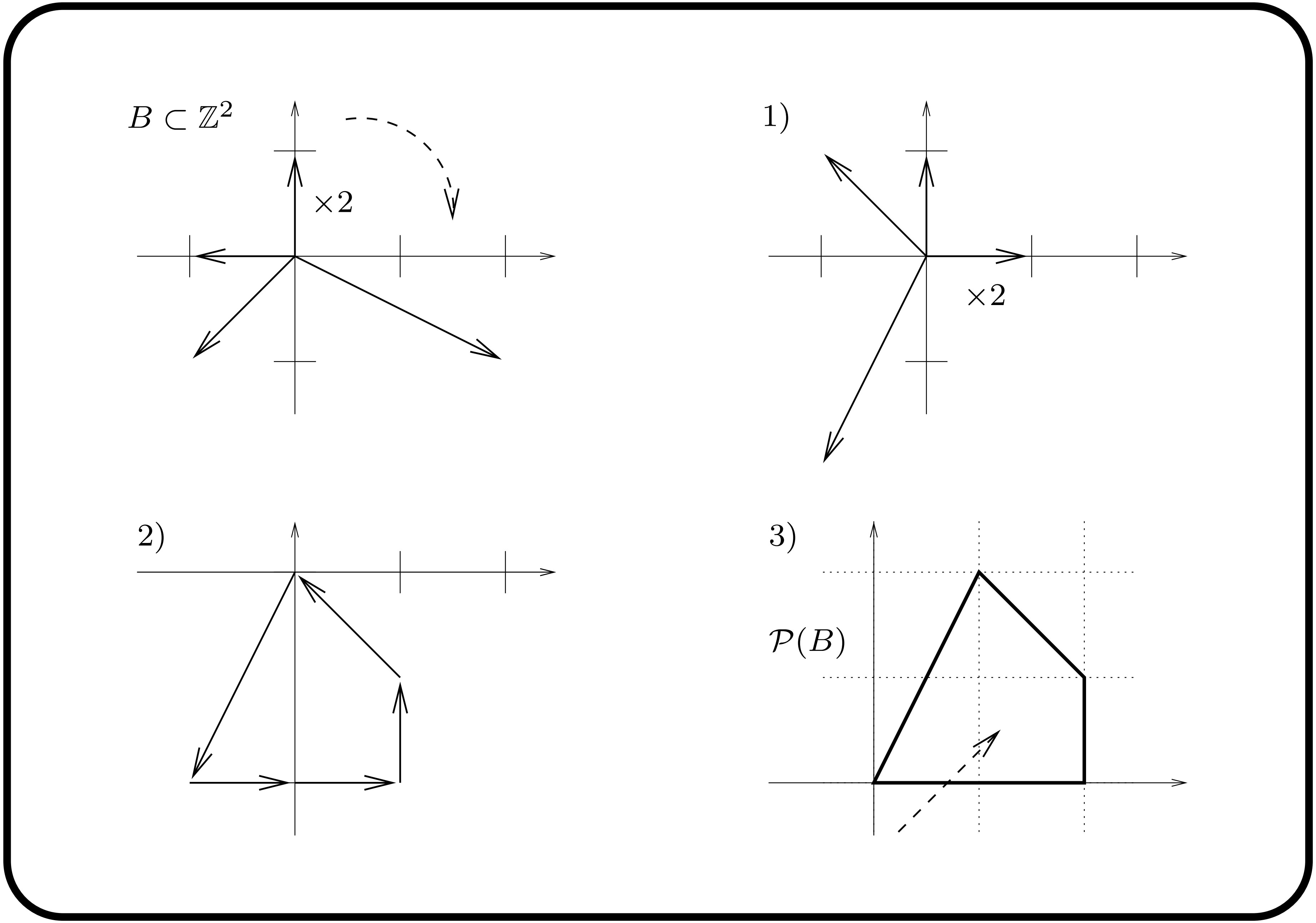
$$\rho : \mathbb{K} \dashrightarrow \mathbb{K}^2 \quad , \quad t \mapsto (f(t), g(t))$$

$$ord_v(\rho) := (ord_v(f), ord_v(g)) \in \mathbb{Z}^2$$

for every $v \in \mathbb{P}^1$

A balanced family

- $ord_v(\rho) = (0,0)$ for almost all $v \in \mathbb{P}^1$
- $\sum_{v \in \mathbb{P}^1} ord_v(\rho) = (0,0)$


$B \subset \mathbb{Z}^2$
$\times 2$
1)
$\times 2$
2)
3)
$\mathcal{P}(B)$

Tropical Implicitization of rational curves

Dickenstein-Feitchner-Sturmfels-Tevelev 07

D-Sombra 07

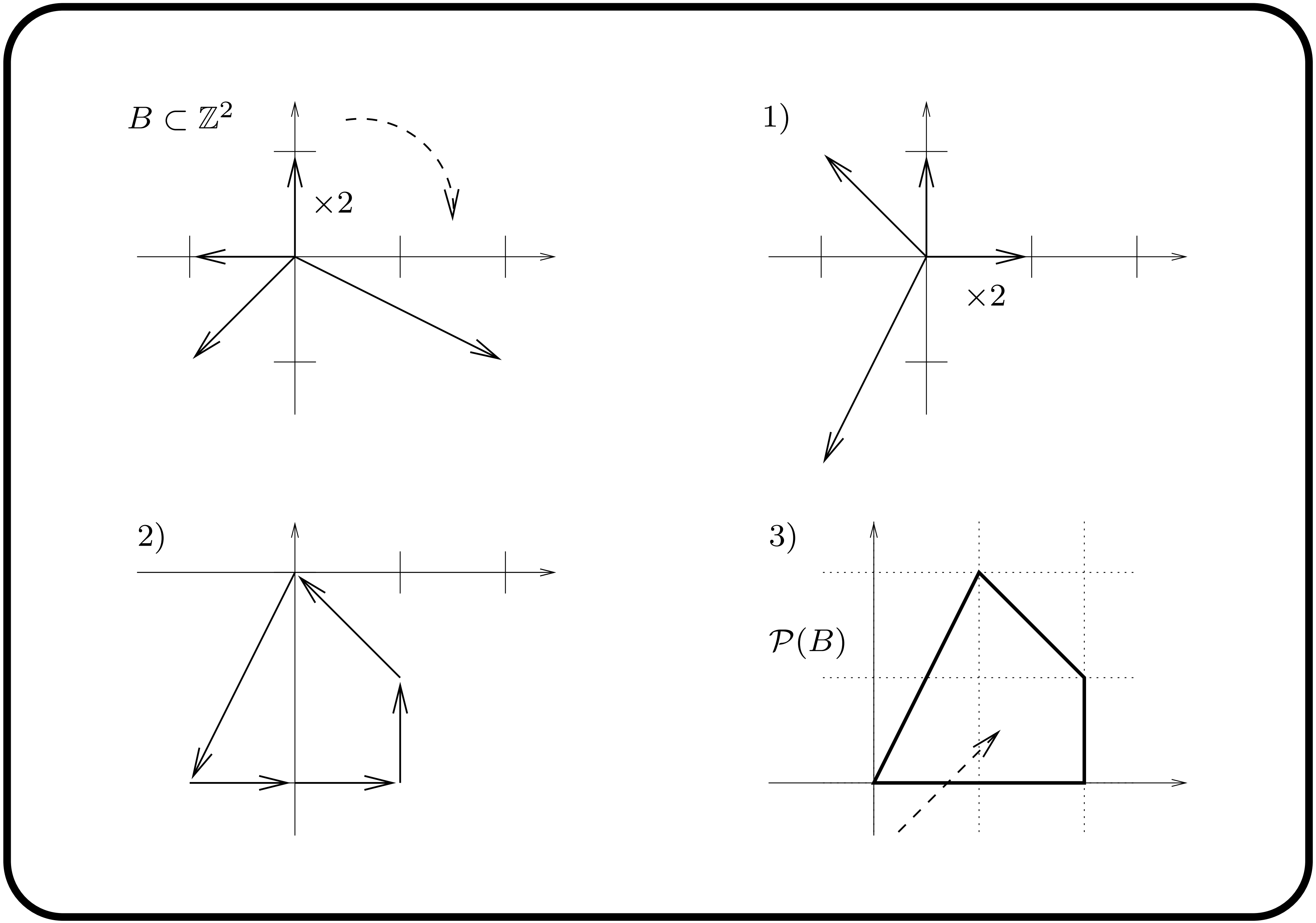
$$\deg(\rho) N\Big(\rho\Big) = P\Big((ord_v(\rho))_{v\in\mathbb{P}^1}\Big)$$

Example 1

$$\rho(t) = \left(\frac{1}{t(t-1)}, \frac{t^2 - 5t + 2}{t}\right)$$

$$1 - 16X - 4X^2 - 9XY - 2X^2Y - XY^2$$

- $ord_0(\rho) = (-1, -1)$
- $ord_1(\rho) = (-1, 0)$
- $ord_\infty(\rho) = (2, -1)$
- for $v^2 - 5v + 2 = 0$ $ord_v(\rho) = (0, 1)$

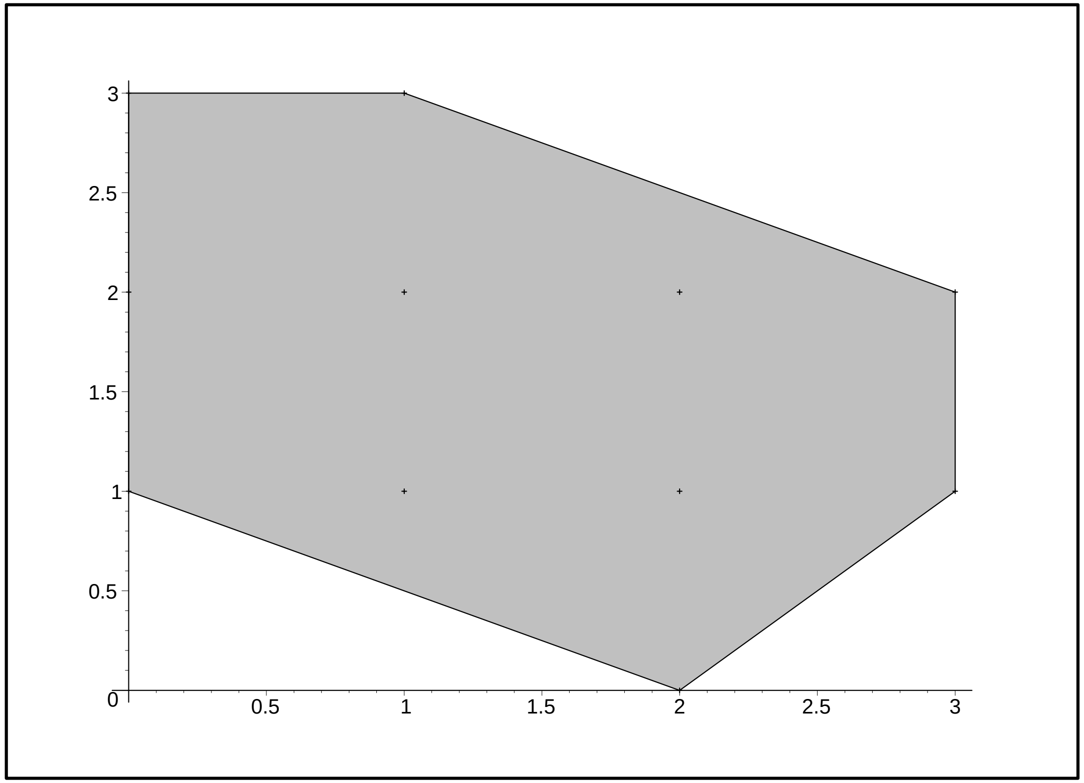

$B \subset \mathbb{Z}^2$
$\times 2$
1)
$\times 2$
2)
3)
$\mathcal{P}(B)$

Example 2

$$\rho(t) = \left(\frac{t(t+1)^2}{(t-1)(t+2)}, \frac{(t-1)t^2}{t-2} \right)$$

$$8Y - 88XY - 24Y^2 + 18Y^3 - 16X^2 + 80X^2y + 22XY^2$$
$$-12X^3Y - 22X^2Y^2 - 4XY^3 + 4X^3Y^2$$

- $ord_0(\rho) = (1, 2)$
- $ord_1(\rho) = (-1, 1)$
- $ord_{-1}(\rho) = (2, 0)$
- $ord_2(\rho) = (0, -1)$
- $ord_{-2}(\rho) = (-1, 0)$
- $ord_\infty(\rho) = (-1, -2)$



3
2.5
2
1.5
1
0.5
0
0.5
1
1.5
2
2.5
3

The tropical proof

Dickenstein-Feitchner-Sturmfels-Tevelev 07

Study the "tropicalization" of

$$\rho : \mathbb{T}^1 \dashrightarrow \mathbb{T}^2$$

- $\mathcal{T}_{\rho^*(\mathbb{T}^1)} = \bigcup_{v \in \mathbb{P}^1} (\mathbb{R}_{\geq 0})\, ord_v(\rho)$,
- $m_{\rho^*(\mathbb{T}^1)}(b) = \sum_{v:\, ord_v(\rho) \in (\mathbb{R}_{>0}) b} \ell(ord_v(\rho))$ for $b \in \mathcal{T}^0_{\rho^*(\mathbb{T}^1)}$

Alternative Proof

(D-Sombra)

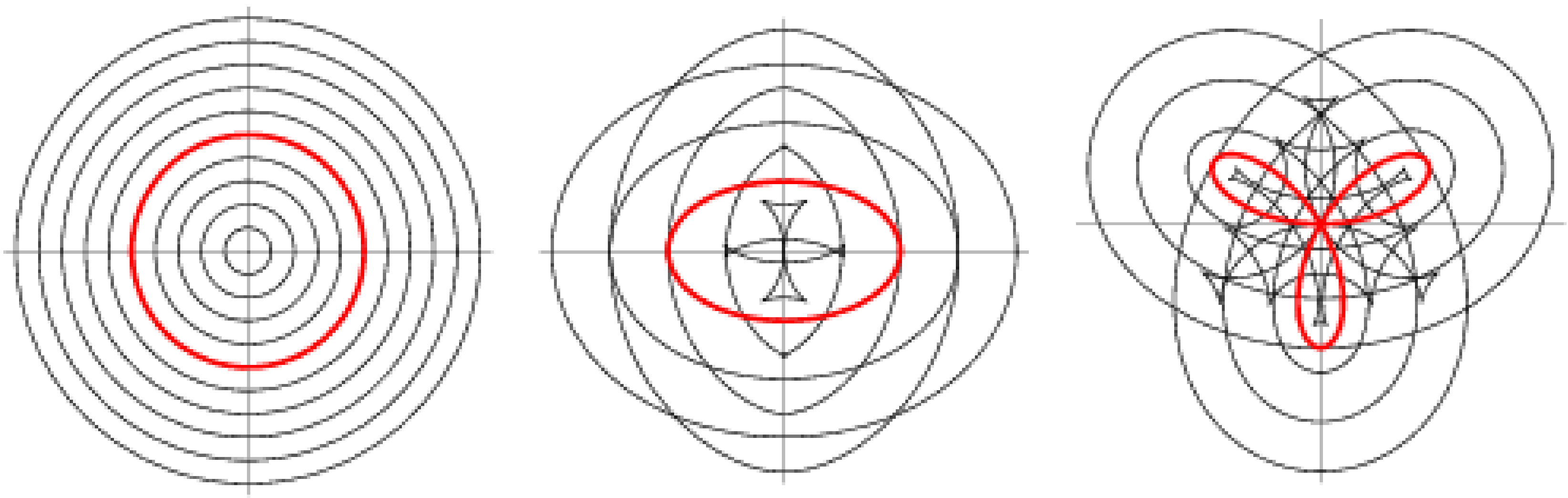
- Intersection theory in $\mathbb{K} \times \mathbb{T}$
- A refinement of Kušnirenko-Bernštein's formula

And if the curve is not rational?

Joint work with

- Fernando San Segundo
- Rafael Sendra
- Martín Sombra

Offsets to plane curves



Parametric equation of the offset

(to a parametric curve)

$$O_\rho(t) = \rho(t) \pm d\frac{N(t)}{\|N(t)\|}$$

- $d \in \mathbb{R}$ is the distance
- $N(t)$ is a normal field to the curve

Parametric equations of the offset

$$\begin{cases} X^{\pm}(t) = \dfrac{A_1(t) \pm \sqrt{h(t)}\,B_1(t)}{D_1(t)} \\ Y^{\pm}(t) = \dfrac{A_2(t) \pm \sqrt{h(t)}\,B_2(t)}{D_2(t)} \end{cases}$$

“Theorem”

(D-San Segundo-Sendra-Sombra)

$$\deg(\rho) N(\rho) = P\Big(\big(ord_v(\rho)\big)_{v \in \mathbb{P}^1}\Big)$$

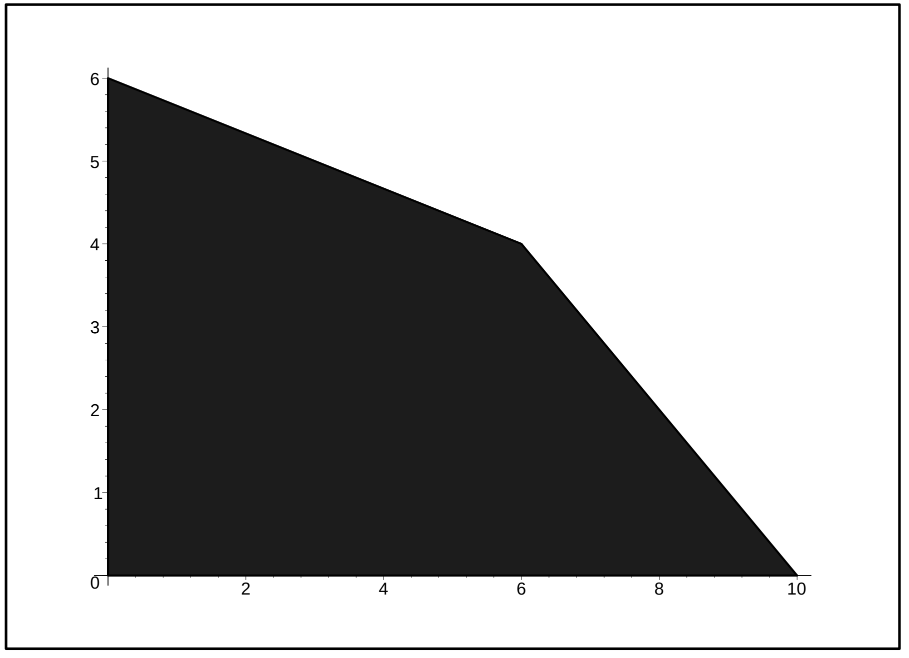
Example

$$\rho(t) = (t, t^3) \;\; d = 1$$

$$X^{\pm}(t) = t \mp \frac{3t^2}{\sqrt{9t^4+1}}, \quad Y^{\pm}(t) = t^3 \mp \frac{1}{\sqrt{9t^4+1}}$$

$$X^{\pm}(t) = \frac{t\,(9\,t^4+1) \mp 3\,t^2\sqrt{9\,t^4+1}}{9\,t^4+1}$$

$$Y^{\pm}(t) = \frac{t^3\,(9\,t^4+1) \pm \sqrt{9\,t^4+1}}{9\,t^4+1}$$



In general...

The tropical implicitization theorem works for complete intersections of the form

$$\begin{cases} P(t, X) = 0 \\ Q(t, Y) = 0 \end{cases} \quad \begin{cases} P(t, X) = 0 \\ Q(t, X, Y) = 0 \end{cases}$$

Ingredients of the Proof

- Kapranov's Theorem in $\mathbb{K}^3$ plus counting multiplicities
- A projection formula for both the tropical variety in $\mathbb{K}^3$(Dickenstein - Feichner - Sturmfels) and the multiplicity map (Sturmfels-Tevelev)

Muchas Gracias

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