Effective Methods in Algebraic Geometry

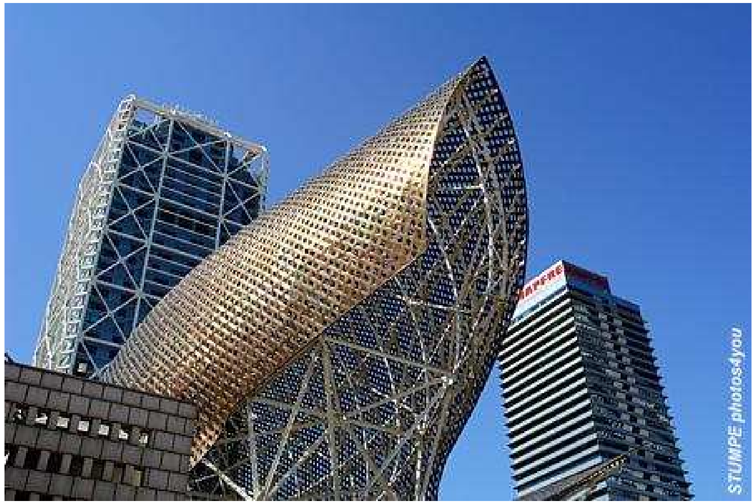
Effective
Effektive
Эффективные
Effettivi
Effectives
Efectivos

Methods
Methoden
Методы
Metodi
Méthodes
Métodos

MEGA 09

in Algebraic
in Algebraischer
в Алгебраической
Algebrica
Algébrique
Algebraica

Geometry
Geometrie
Геометрии
in Geometria
en Géométrie
en Geometria



Barcelona, June 15–19 2009

`http://www.imub.ub.es/mega09`

(CoCoA school: June 9-13)

Subresultants in Multiple roots

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Ongoing work with

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- **Agnes Szanto** *(North Carolina)*

The Euclidean Algorithm

Given $f(x),\ g(x) \in \mathbb{K}[x]$

There exist $a(x),\ b(x) \in \mathbb{K}[x]$

such that

$$a(x)f(x) + b(x)g(x) = \gcd(f, g)$$

Effective Euclidean Algorithm

$$a(x)f(x) + b(x)g(x) = \gcd(f, g)$$

with

$$\deg(a) < \deg(g) - \deg(\gcd(f, g))$$

$$\deg(b) < \deg(f) - \deg(\gcd(f, g))$$

Computational Linear Algebra

$$\deg(f) = n, \ \deg(g) = m, \ \deg(\gcd(f,g)) = d$$

$$\gcd(f,g) \in \mathbb{K}\langle f(x), g(x)\rangle$$

generated by

$$f(x), \ xf(x), \ \ldots, \ x^{m-d-1}f(x),$$

$$g(x), \ xg(x), \ \ldots, x^{n-d-1}g(x)$$

The “Matrix” of this vector space

- $f(x) = f_0 + f_1 x + \ldots + f_n x^n$
- $g(x) = g_0 + g_1 x + \ldots + g_m x^m$

$$M_d(f,g) := \begin{array}{cc} \overset{m+n-d}{} & \\ \begin{array}{|ccccc|} \hline f_0 & \cdots & f_n & & \\ & \ddots & & \ddots & \\ & & f_0 & \cdots & f_n \\ \hline g_0 & \cdots & g_m & & \\ & \ddots & & \ddots & \\ & & g_0 & \cdots & g_m \\ \hline \end{array} & \begin{array}{c} \\ m-d \\ \\ \\ n-d \\ \\ \end{array} \end{array}$$

Theorem

There is a polinomial of degree d in $\mathbb{K}\langle f(x), g(x)\rangle_{m+n-d-1}$ iff the principal minor of $M_d(f,g)$ is different than zero

$$M_d(f,g) := \begin{array}{c} m+n-d \\ \left[\begin{array}{ccccc} f_0 & \cdots & f_n & & \\ & \ddots & & \ddots & \\ & & f_0 & \cdots & f_n \\ \hline g_0 & \cdots & g_m & & \\ & \ddots & & \ddots & \\ & & g_0 & \cdots & g_m \end{array}\right] \end{array} \begin{array}{l} m-d \\ \\ \\ n-d \end{array}$$

Moreover

$$S_d(f,g) := \det \begin{array}{cc} m+n-2d & \\ \begin{array}{|ccccc|} \hline f_0 & \cdots & f_n & & f(x) \\ & \ddots & & \ddots & \vdots \\ & & f_0 & \cdots & x^{m-d-1}f(x) \\ \hline g_0 & \cdots & g_m & & g(x) \\ & \ddots & & \ddots & \vdots \\ & & g_0 & \cdots & x^{n-d-1}g(x) \\ \hline \end{array} & \begin{array}{c} m-d \\ \\ \\ \\ n-d \end{array} \end{array}$$

is a polynomial of the form $a(x)f(x) + b(x)g(x)$ with

$$\deg(a) < m-d,\ \deg(b) < m-d$$

Definition

$S_d(f,g)$ is the d- subresultant associated to (f,g)

It is a polynomial of the form

$a(x)f(x)+b(x)g(x)$ with

$\deg(a)<m-d,\ \deg(b)<m-d$

Example

$$S_0(f,g) = \det \begin{array}{cc} \overset{m+n}{} & \\ \left[\begin{array}{cccccc} f_0 & \cdots & f_n & & & \\ & \ddots & & & \ddots & \\ & & f_0 & & \cdots & f_n \\ \hline g_0 & \cdots & g_m & & & \\ & \ddots & & & \ddots & \\ & & g_0 & & \cdots & g_m \end{array}\right] & \begin{array}{c} m \\ \\ \\ n \end{array} \end{array}$$

is the resultant $Res(f,g)$

Algorithmic criteria

$$S_d(f,g) = S_d^d x^d + S_d^{d-1} x^{d-1} + \ldots + S_d^0$$

$S_0^0 = S_1^1 = \ldots = S_{d-1}^{d-1} = 0$ and $S_d^d \neq 0$, then $S_d(f,g) = \gcd(f,g)$.

Applications

- Computation of gcd's
- Solution of overdetermined systems
- Search of fewnomials in ideals of polynomials
- Study of singularities of plane curves
- Symbolic computation of residues and integrals
- . . .

Formulas in roots

- $f(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$
- $g(x) = (x - b_1)(x - b_2) \ldots (x - b_m)$

$$S_0 = \prod_{i=1}^{m} f(b_j) = \prod_{i,j=1}^{m,n} (a_i - b_j)$$

Poisson type formulas

(Jouanolou, Hong,..)

$$S_d(f,g) = \frac{\begin{vmatrix} (x-a_1)a_1^0 & \dots & (x-a_n)a_n^0 \\ \vdots & & \vdots \\ (x-a_1)a_1^{d-1} & \dots & (x-a_n)a_n^{d-1} \\ g(a_1)a_1^0 & \dots & g(a_n)a_n^0 \\ \vdots & & \vdots \\ g(a_1)a_1^{n-d-1} & \dots & g(a_n)a_n^{n-d-1} \end{vmatrix}}{\begin{vmatrix} a_1^0 & \dots & a_n^0 \\ \vdots & & \vdots \\ a_1^{n-1} & \dots & a_n^{n-1} \end{vmatrix}}$$

Some interesting computations

$$n - 1 \leq m$$

$$S_{n-1}(f,g) = \sum_{i=1}^{n} g(a_i) \left(\prod_{j \neq i} \frac{x - a_j}{a_i - a_j} \right)$$

$$S_1(f,g) = \sum_{i=1}^{n} \left(\prod_{j \neq i} \frac{g(a_j)}{a_j - a_i} \right) (x - a_i)$$

Formulas in the two groups of roots

(Sylvester)

- $R(X, Y) := \prod_{x \in X, y \in Y} (x - y)$
- $A := \{a_1, \ldots, a_n\}$
- $B := \{b_1, \ldots, b_m\}$
- $p + q = d$

Formulas in the two groups of roots

(Sylvester)

$$\binom{d}{p} S_d(f,g)$$

$$=$$

$$\sum_{A',B'} R(x,A')R(x,B')\frac{R(A',B'),R(A\setminus A',B\setminus B')}{R(A',A\setminus A'),R(B',B\setminus B')}$$

$$A' \subset A,\ |A'| = p,\ B' \subset B,\ |B'| = q$$

Multiple roots?

- $f(x) = (x - a_1)^{n_1}(x - a_2)^{n_2} \ldots (x - a_k)^{n_k}$
- $g(x) = (x - b_1)^{m_1}(x - b_2)^{m_2} \ldots (x - b_j)^{m_j}$

Generalized Vandermonde

$$A := \{(a_1, m_1), \ldots, (a_k, m_k)\}$$

$$V(A) = [W_{m_1}(a_1)\; W_{m_2}(a_2)\; \ldots\; W_{m_k}(a_k)]$$

$$W_k(x) = \begin{array}{cc} m_1+\ldots+m_k & \\ \boxed{\begin{matrix} 1 & 0 & 0 & \ldots & 0 \\ x & 1 & 0 & \ldots & 0 \\ x^2 & 2x & 1 & & \\ \vdots & \vdots & & \ddots & \vdots \\ x^{k-1} & (x^{k-1})' & \frac{(x^{k-1})''}{2!} & \ldots & \frac{(x^{k-1})^{(n-1)}}{(n-1)!} \end{matrix}} & k \end{array} .$$

Theorem

$$\det(V(A)) = \prod_{1 \le u < v \le j} (a_u - a_v)^{m_u m_v}$$

Poisson like formulas

$$S_d(f,g) = \frac{\begin{vmatrix} (x-a_1)a_1^0 & (x-a_1)0 & \dots & (x-a_k)0 \\ \vdots & \vdots & & \vdots \\ (x-a_1)a_1^{d-1} & (x-a_1)(d-1)a_1^{d-2} & \dots & (x-a_j)\binom{d-1}{n_k}a_k^{d-n_k} \\ g(a_1)a_1^0 & g(a_1)0 & \dots & g(a_k)0 \\ \vdots & \vdots & & \vdots \\ g(a_1)a_1^{n-d-1} & (tg(t))'(a_1)a_1^{n-d-1} & \dots & (tg(t))^{(n_k-1)}a_j^{n-d-1} \end{vmatrix}}{\det(V(A))}$$

For instance...

$$n - 1 \leq m$$

$$S_{n-1}(f, g) = \sum_{i=1}^{k} \sum_{u=1}^{n_i} g(a_i) L_{i,u}(x)$$

with $L_{i,u}(x)$ being the basis of the generalized Lagrange system given by A

This formulas can be extended

Poisson like formulae for multivariate polynomials

(D- Krick - Szanto, J. Algebra, 2006)

Formulas in the two groups of roots?

The answer is given in terms of

$$\frac{\partial}{\partial a_1}\left(\sum_{A',B'} R(x,A')R(x,B')\frac{R(A',B'),R(A\setminus A',B\setminus B')}{R(A',A\setminus A'),R(B',B\setminus B')}\right)\Big|_{a_1=a_2}$$

$$A'\subset A,\ |A'|=p,\ B'\subset B,\ |B'|=q$$

Example

- $f(x) = (x - a_1)^{n_1} \ldots (x - a_k)^{n_k}$
- $g(x) = (x - b)^m$

$$S_1(f,g) = \sum_{\ell_1+\cdots+\ell_k=m-2} \binom{n_1+m-3}{n_1-1} \cdots \binom{n_k+m-3}{n_k-1} (b-a_1)^{(m-1)n_1-\ell_1} \cdots (b-a_r)^{(m-1)n_k-\ell_r}$$

$$+ \sum_{j_1+\cdots+j_k=n-1} \binom{n_1+n-2}{n_1-1} \cdots \binom{n_k+m-2}{n_k-1} (b-a_1)^{(m-1)n_1-j_1} \cdots (b-a_r)^{(n-1)n_k-j_k} (x-b)$$

Challenge

Find a nice Combinatorial description of

$$\frac{\partial}{\partial a_1}\left(\frac{R(A',B'),R(A\backslash A',B\backslash B')}{R(A',A\backslash A'),R(B',B\backslash B')} - \frac{R(\overline{A}',B'),R(A\backslash\overline{A}',B\backslash B')}{R(\overline{A}',A\backslash\overline{A}'),R(B',B\backslash B')}\right)\Big|_{a_1=a_2}$$

$$A' \setminus \overline{A}' = \{a_1\},\ \overline{A}' \setminus A' = \{a_2\}$$

THANKS...

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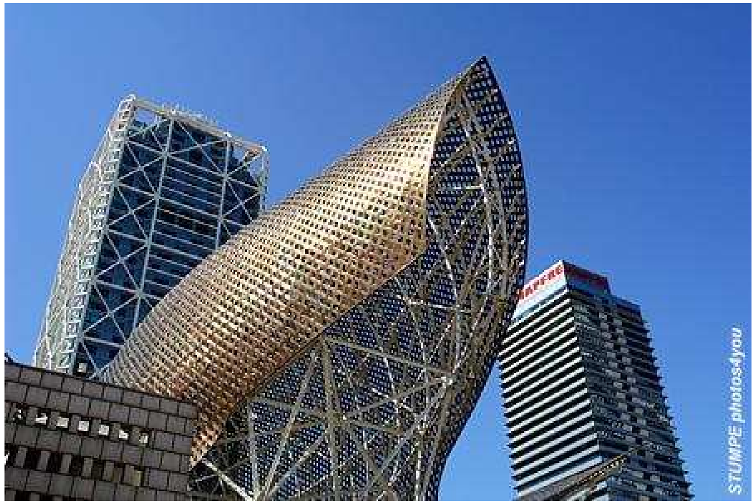

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