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Incorporating fuzzy information in pricing substandard annuities

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ABSTRACT

There is a growing interest in the insurance industry in offering substandard annuities. These annuities, based on medical underwriting, provide a greater pay out than the standard ones to those individuals who are expected to have a lower than average life expectancy. Medically underwritten annuities often involve imprecise or vague information about the individuals such as health status and lifestyle. To address this issue, this paper proposes two approaches based on Fuzzy Sets Theory tools. Firstly, in order to determine substandard annuity payments, fuzzy mortality factors (also known as mortality multipliers) are introduced. These fuzzy mortality factors, modelled by means of triangular fuzzy numbers, can be estimated using conventional statistical confidence intervals. Secondly, by designing a fuzzy inference system, we demonstrate how to obtain the substandard annuity payment based on imprecise or vague personal information about annuitants. Numerical applications based on Spanish mortality data are provided for illustration.

1. Introduction

Traditionally, it has been a common practice for insurance companies that sell annuities to offer the same price to all policyholders of the same age. It is assumed that each policyholder has an "average" health condition based on their age. In such a way, no risk classification is considered when pricing standard annuities. Thus, they are expensive and unfair for those who have a life expectancy below average due to their poor health condition and/or lifestyle. To improve marketing, there is a growing interest in offering substandard annuities (also known as underwritten or enhanced annuities) on a medically underwritten basis. A substandard annuity contract uses insureds' health and lifestyle information to offer a greater pay out to those individuals who are expected to have a lower than average life expectancy.

Substandard annuities have been the subject of several papers. Gatzert and Klotzki (2016) discussed three different research focuses of substandard annuities: description of market characteristics, analysis of the implications of introducing substandard annuities and pricing and underwriting techniques. The market characteristics includes development, size, products, customer behaviour, key success factors/drivers of profitable business, underwriting methods and processes in the UK and the US/Canada. A comprehensive description of some of these areas can be found in LIMRA International and Ernst & Young (2006). Papers on the effects of introducing substandard annuities discussed how they might affect in, for example: the insurer's profits, the standard annuity market, distribution channels, reinsurance, the so-called annuity puzzle, adverse selection and the demand for annuities. Several authors, as Hoermann and Ruß (2008), Gatzert, Schmitt-Hoermann, and Schmeiser (2012) and Olivieri and Pitacco (2016) concluded that, in general, the practice of individual underwriting always increases insurer's profitability and that offering substandard annuities would have a beneficial effect on the insurer's risk profile. Papers on pricing and underwriting techniques of substandard annuities, as Gatzert et al. (2012) and Meyricke and Sherris (2013), dealt with medical underwriting, pricing factors and risk classification. Further, Ramsay and Oguledo (2019) analized doubly enhanced annuities that provide not only greater annual benefits to insureds with shorter than average life expectancies, but also long-term care benefits and a bequest motive through a death benefit. Other papers related to substandard annuities are provided in Table 1.

The development of the annuity market is likely to make substandard annuities an important provision saving asset (Gatzert & Klotzki, 2016). This leads to a change in the annuity market from the age-based pricing to a medically underwritten pricing, allowing for heterogeneity among an age class. This phenomenon is already happening in some countries (such as the UK and the US) where substandard annuities are present and it has given rise to the consideration of different risk factors, including postal code, smoking condition,

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Table 1
Overview of the literature on substandard annuities and FST insurance applications.

Actuarial topic	Papers (chronological order)
Substandard annuities	Brown and McDaid (2003), Kwon and Jones (2006), LIMRA International and Ernst & Young (2006), Hoermann and Ruß (2008), Becker and Hurley (2011), Telford et al. (2011), Gatzert et al. (2012), Ridsdale (2012), Charrington (2013), Meyricke and Sherris (2013), Woo (2013), Kling et al. (2014), Fong (2015), Gatzert and Klotzki (2016), OECD (2016), Olivieri and Pitacco (2016) and Ramsay and Oguledo (2019).
Fuzzy classification	Lemaire (1990), Ostaszewski (1993), Derrig and Ostaszewski (1995), Verrall and Yakoubov (1999), Subudhi and Panigrahi (2017) and Majhi, Bhatachharya, Pradhan, and Biswal (2019).
Fuzzy underwritting	De Wit (1982), Lemaire (1990), Young (1993), Horgby et al. (1997), Jablonowski (1997), Horgby (1998), Shapiro and Koissi (2015), Esfandabadi and Esfahani (2018), Torbati and Sayadi (2018).
Fuzzy projected liabilities	Cummins and Derrig (1993), Dalkilic, Tank, and Kula (2009), Huang, Zhao, and Tang (2009), Heberle and Thomas (2014), de Andrés-Sánchez (2016), Kemaloglu, Shapiro, Tank, and Apaydin (2018) and Apollinaire, Le Doux, and Ronald (2019),
Fuzzy future and present values	Buckley (1987), Lemaire (1990), Ostaszewski (1993), Betzuen et al. (1997), Jiménez and Rivas (1998), Mircea and Covrig (2015) and Vernic (2016).
Fuzzy insurance pricing	Lemaire (1990), Young (1996), Cummins and Derrig (1997), Koissi and Shapiro (2006), Shapiro (2013), de Andrés-Sánchez and González-Vila (2012, 2017, 2019), Anzilli, Facchinetti, & Pirotti (2018) and Subartini et al. (2018).
Asset allocation, cash flows and investments with FST	Derrig and Ostaszewski (1997), de Andrés-Sánchez and Terceño (2003), Chang (2003), Zhang et al. (2011) and Wang (2019).

blood pressure level, cholesterol value and body-mass-index.

This paper contributes to current pricing and underwriting techniques to price substandard annuities by using Fuzzy Sets Theory (FST). Traditional actuarial methodologies have been built upon classical or binary logic and probabilistic models, which are the cornerstones of the Actuarial Science. However, a significant amount of information that insurance companies use is imprecise, vague or does not have a clear definition and thus, cannot be modelled using traditional methods. For this purpose, FST provides a useful way of formally treating such imprecise information. Since the paper by De Wit (1982) that first used FST in an actuarial context, FST has been applied to a wide range of actuarial areas (see Table 1). Although actuarial quantitative analysis is essentially based on statistical methods, it is nowadays accepted that FST is a useful complemental tool to statistics (Derrig & Ostaszewski, 2004).

To the best of our knowledge, little research has been done using FST in pricing substandard annuities. However, as it has been previously discussed, pricing this type of annuity requires collecting individual policyholder's medical information that may be imprecise or vague. For instance, information like "high blood pressure", "overweight", "very good health status", can be hard to define precisely. So, using fuzzy sets to represent this vague information and constructing a Fuzzy Inference System (FIS) based on these fuzzy sets is very suitable to price substandard annuities. As stated in LIMRA International and Ernst & Young (2006, pp. 32–34), once the health information of the prospective policyholder is collected (usually by a questionnaire and/or by interviews with the policyholder's doctor), the insurance company will determine the reduction of individual's life expectancy and, from it, the corresponding annuity payment. This payment can be based on a mortality factor applied to the mortality base table used by the insurance company, thus obtaining modified mortality probabilities. This paper models both mortality factors and modified mortality probabilities by means of fuzzy numbers (FNs), the main instrument used in FST to represent uncertain quantities. The intrinsic nature of some information collected with an unclear definition makes it reasonable to consider both fuzzy mortality factors and fuzzy probabilities related to mortality (or survival) rates as discussed in several papers (Lemaire, 1990; Koissi & Shapiro, 2006 and Shapiro, 2013).

The remainder of this paper is organised as follows. Section 2 provides some basic notations and definitions of FST used in the paper. Section 3 describes a framework for pricing substandard annuities with a fuzzy mortality factor and proposes several ways to adjust this parameter with a triangular FN (TFN). Section 4 develops an FIS that allows determining the level payment of a substandard annuity. Conclusions and further extensions are summarized in Section 5.

2. Basic notation on fuzzy sets

Throughout this paper, a fuzzy set defined over a reference set X is denoted by $\widetilde{A}$, being its membership function $\mu_{\widetilde{A}}: X \to [0, 1]$, i.e. $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) | x \in X\}$. Furthermore, the α-cuts of $\widetilde{A}$ are A_α with $A_\alpha = \{x \in X | \mu_{\widetilde{A}}(x) \geq \alpha\}$, for any $\alpha \in (0, 1]$ with the convention that $A_{\alpha=0}$ is the closure of the support[1] of $\widetilde{A}$. An FN can be interpreted as a fuzzy quantity approximately equal to the real number for which the membership function takes the value 1.

A triangular fuzzy number (TFN) over the set of real numbers $\Re$, is denoted as $\widetilde{A} = (A_l, A_c, A_u)$, being A_c the core (also known as center or mode) of the FN with $\mu_{\widetilde{A}}(A_c) = 1$, whereas $[A_l, A_u]$ is its support and $A_\alpha = [\underline{A}(\alpha), \overline{A}(\alpha)] = [A_l + (A_c - A_l)\alpha, A_u - (A_u - A_c)\alpha]$ are their α-cuts. The core A_c can be understood as the most reliable value of the TFN. TFNs are often used to quantify imprecise quantitative predictions. For example, a physician's statement as "Risk factor X increases mortality probabilities by about 100%–150%" can be quantified in a natural way as (100%, 125%, 150%).

Let be f be a continuous real-valued function of n-real variables x_j, $j = 1, 2, \cdots, n$. If $x_j s$ are not crisp numbers, but FNs $\widetilde{A}_j$ with α–cuts $A_{j\alpha} = [\underline{A}_j(\alpha), \overline{A}_j(\alpha)]$, $j = 1, 2, \cdots, n$, an FN $\widetilde{B}$ is induced via f such that $\widetilde{B} = f(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_n)$. It is often difficult to obtain a closed expression for the membership function of $\widetilde{B}$. However, if f increases with respect to x_j, $j = 1, 2, \ldots, m$, $m \leq n$, and decreases in the last $n - m$ variables, Buckley and Qu (1990) demostrated that the closed expression of B_α is

$$\begin{aligned} B_\alpha &= [\underline{B}(\alpha), \overline{B}(\alpha)] \\ &= [f(\underline{A}_1(\alpha), \underline{A}_2(\alpha), \cdots, \underline{A}_m(\alpha), \overline{A}_{m+1}(\alpha), \overline{A}_{m+2}(\alpha), \cdots \overline{A}_n(\alpha)), \\ &\quad f(\overline{A}_1(\alpha), \overline{A}_2(\alpha), \cdots \overline{A}_m(\alpha), \underline{A}_{m+1}(\alpha), \underline{A}_{m+2}(\alpha), \cdots, \underline{A}_n(\alpha))] \end{aligned} \tag{1}$$

It is worth noting that the result $\widetilde{B}$ of evaluating a non-linear f with the TFNs $\widetilde{A}_j$ is not necessarily a TFN. However, the FN $\widetilde{B}$ can be approximated by a TFN, through *secant approach* (SA). It builds up the shape of the approximate FN $\widetilde{B}'$ on $\widetilde{B}$ by means of the secant lines that unite the 0-cut and the 1-cut of $\widetilde{B}$. Such that $\widetilde{B}'$ is a TFN as $(B_l, B_c, B_u) = (\underline{B}(0), \underline{B}(1) = \overline{B}(1), \overline{B}(0))$. This approximation, as shown in Kaufmann (1986), works relatively well for nonlinear functions of TFNs like product, division, power, etc. In that paper, it has shown that this approach fitted fuzzy net present values with triangular fuzzy parameters. Other applications of the SA in financial and insurance contexts can be found in Jiménez and Rivas (1998) and Heberle and Thomas (2014). Notice that this approximating method only requires evaluating f at three different points: the lower value B_l, the upper value B_u and the most feasible value B_c.

[1] The support of $\widetilde{A}$ is the set of all $x \in X$ with $\mu_{\widetilde{A}}(x) > 0$.

3. Pricing substandard annuities with fuzzy parameters

3.1. Modelling fuzzy heterogeneity in substandard annuity payments

As pointed out by Pitacco (2019), modeling mortality heterogeneity commonly follows two steps. Firstly, a biometric function (presented, e.g. as a mortality base table) to represent the average age-pattern of mortality for a given population (e.g. population of a country, members of a pension fund, etc.) is set. Secondly, a "specific" age-pattern of mortality (in particular, mortality of people in poorer or better conditions than the average) must be expressed as a transformation of that biometric function. Olivieri (2006) outlines that it is commonplace in actuarial mathematics capturing mortality heterogeneity by modifying standard yearly mortality rates as

$$q'_x = d \cdot q_x + c \tag{2}$$

$$q'_x = q_{x+t},\ t = 1, 2, \cdots \tag{3}$$

Here q_x is the mortality probability of dying between age x and $x+1$ in the base table, the superscript " ′ " symbolizes modified rates, the coefficient d is the mortality factor (also known in the insurance industry as "the multiplier") and c is a parameter that, following Olivieri (2006), models extra-mortality due to accidents (related either to occupation or to extreme sports) in certain insurance contracts. As stated in LIMRA International and Ernst & Young (2006), one of the most common practices of insurance companies is to choose a mortality base table, to perform prospective insured's individual underwriting and to collect their medical information. Then, medical underwriters must estimate, at least, one of the following two parameters:

- Mortality factor (or multiplier), d, that must be applied in (2) on the mortality base table to obtain modified mortality probabilities. For example, an extra-mortality of 100% implies multiplying annual mortality probabilities for each age x, q_x, by $d = 2$. From the modified mortality probabilities, the insured's modified life expectancy can be obtained.
- Insured's modified life expectancy that will allow determining their modified age group and obtaining (3).

Olivieri (2006) indicated that in the field of substandard annuities (2) is often used. So, this paper focuses on the modification of mortality probabilities of a base table by a mortality factor. It is also often assumed in the insurance industry that, in (2), $c = 0$ and $d = 1 + \gamma$, with $\gamma > 0$ for the case of substandard annuities. For example, in Gatzert et al. (2012), Hoermann and Ruß (2008) and Kling, Ritcher, and Ruß (2014) each person is characterized by a factor d (referred to as frailty factor) and their individual mortality probabilities are given by d times the mortality probabilities from a standard mortality base table. Therefore, an insured (annuitant) with an individual factor $d > 1$ has an above-average mortality rate (or, equivalently, a below-average life expectancy). Obviously, this factor cannot be greater than $\frac{1}{q_x}$ as $q'_x \leq 1$ for all age x.

In practice, as stated by Pitacco (2019), γ can be adjusted by using the credit-debit method, where each factor that affects an insured's life expectancy negatively (positively) generates a debit (credit). A debit increases mortality probabilities whereas a credit diminishes them. A common and intuitive implementation of this method is the *numerical rating system*. For m factors, γ can be expressed as $\gamma = \sum_{j=1}^{m} \rho_j$, where ρ_j is a debit (positive value) or credit (negative value) for the jth factor, and so

$$d = 1 + \gamma = 1 + \sum_{j=1}^{m} \rho_j \tag{4}$$

Alternatively, it is possible to use the estimate of modified life expectancy for a person aged x, e'_x, with repect to the standard life expectancy, e_x. So, the modified life expectancy for a person aged x can be defined as

$$e'_x = f \cdot e_x \tag{5}$$

where $f < 1$ for substandard annuities. Therefore, $f = 1 - \phi$, for $0 < \phi < 1$, with ϕ quantifying the deduction rate on the standard curtate life expectancy e_x for an individual aged x, whereas $\phi \cdot e_x$ is the number of the expected years of life lost due to the personal circumstances that originate ϕ. Recall that the standard curtate life expectancy is $e_x = \sum_{t=1}^{\omega-x} {}_tp_x$, where ${}_tp_x$ are the probabilities of surviving until t (for $1 \leq t \leq \omega - x$, with ω being the maximum attainable age) and can be obtained from a standard mortality base table. If we write ${}_tp'_x = (1-\phi)\ {}_tp_x$, then the substandard curtate life expectation is

$$e'_x = \sum_{t=1}^{\omega-x} {}_tp'_x \tag{6}$$

In the sequel, a non-deferred immediate annuity to be paid to an annuitant aged x at the end of each living year is considered. The insured pays a fixed single premium, Π, in exchange for the annuity. Further, it is assumed that the insurer uses a mortality base table that is modified with the parameter d and an interest rate, i. So, the annual payment C that the insured will receive can be obtained by using a standard annuity pricing formula (see e.g. Gerber, 1997, pp. 35–47)

$$C = \frac{\Pi}{\sum_{t=1}^{\omega-x} (1+i)^{-t} \prod_{k=0}^{t-1} (1 - q'_{x+k})} \tag{7}$$

where $q'_{x+k} = \min\{1, d \cdot q_x\}$ and d is the mortality factor as in (2), with $d = 1$ for a standard annuity and $d > 1$ for a substandard annuity.

Alternatively, if a survival factor $f = 1 - \phi$ (as in (5)) is considered, the annual substandard annuity payment, C, is given by

$$C = \frac{\Pi}{\sum_{t=1}^{\omega-x} (1+i)^{-t} f \cdot {}_tp_x} = \frac{\Pi}{\sum_{t=1}^{\omega-x} (1+i)^{-t}\ {}_tp'_x} \tag{8}$$

The mortality factor d (or the survival factor f) can be modelled stochastically, e.g. by means of a gamma distribution (Hoermann & Ruß, 2008; Gatzert et al., 2012; Kling et al., 2014 and Olivieri & Pitacco, 2016). However, some information of the risk factors considered in the underwriting process of substantdard annuities is often ill defined. Indeed, as pointed out by Horgby, Lohse, and Sittaro (1997) and Horgby (1998), judgements as "high blood pressure", "overweight" or "very high level of cholesterol" could be considered as linguistic variables with unclear borders and could be well-modelled by fuzzy sets, instead of being represented by a single (crisp) value. Therefore, this paper proposes modelling the multiplier d (or the survival factor f) by means of TFNs thus generating both fuzzy mortality and fuzzy survival probabilities. Notice that fuzzy mortality and survival probabilities were also used in Lemaire (1990), Koissi and Shapiro (2006), Shapiro (2013) and de Andrés-Sánchez and González-Vila (2019).

We are now in a position to discuss how to extend the standard actuarial formulas (7) and (8) to the use of a fuzzy mortality factor $\tilde{d}$, estimated as a TFN. In our model, instead of using a crisp mortality (survival) factor equal to d (f), we use "approximately equal to d (f)", and so working with fuzzy mortality (survival) probabilities is necessary. Therefore, if instead of using the crisp value d for the mortality factor, it is substituted by the fuzzy mortality factor $\tilde{d} = (d_l, d_c, d_u)$, with α-cuts $d_\alpha = [\underline{d}(\alpha), \overline{d}(\alpha)]$, the guaranteed income C in (7) is then replaced by an FN, $\tilde{C}$, whose α-cuts, $C_\alpha = [\underline{C}(\alpha), \overline{C}(\alpha)]$, can be obtained by using (1) and considering that (7) is an increasing function with respect to d.

Let us define the fuzzy modified probability of death by

$$\tilde{q}'_{x+k} = \tilde{d} \cdot q_{x+k}$$

Noting that $\tilde{q}'_{x+k}$ cannot take values greater than 1 and that two positive numbers are multiplied, it can be got

$$q'_{x+k\alpha} = [\underline{q'_{x+k}}(\alpha), \overline{q'_{x+k}}(\alpha)] = [\min\{1, \underline{d}(\alpha)\cdot q_{x+k}\}, \min\{1, \overline{d}(\alpha)\cdot q_{x+k}\}]$$
$$= [\min\{1, (d_l + (d_c - d_l)\alpha)\cdot q_{x+k}\}, \min\{1, (d_u - (d_u - d_c)\alpha)\cdot q_{x+k}\}] \quad (9)$$

And so

$$\tilde{q}'_{x+k} \approx (q'_{x+kl}, q'_{x+kc}, q'_{x+ku})$$
$$= (\min\{1, d_l\cdot q_{x+k}\}, \min\{1, d_c\cdot q_{x+k}\}, \min\{1, d_u\cdot q_{x+k}\}) \quad (10)$$

Thus the α-cuts of the substandard annuity fuzzy payment $\tilde{C}$ are given by

$$C_\alpha = [\underline{C}(\alpha), \overline{C}(\alpha)]$$
$$= \left[\frac{\Pi}{\sum_{t=1}^{\omega-x}(1+i)^{-t}\prod_{k=0}^{t-1}(1-\underline{q'_{x+k}}(\alpha))}, \frac{\Pi}{\sum_{t=1}^{\omega-x}(1+i)^{-t}\prod_{k=0}^{t-1}(1-\overline{q'_{x+k}}(\alpha))}\right] \quad (11)$$

Expression in (11) is analogous to those in Zhang, Mei, Lu, and Xiao (2011) that dealt with problems of project valuation and portfolio selection under fuzzy uncertainty. Note that in order to find the lower and upper bounds of (11), one only needs to apply (7) to two different scenarios: the lower and upper values of the the α-cuts of mortality probabilities, $\underline{q'_{x+k}}(\alpha)$ and $\overline{q'_{x+k}}(\alpha)$.

The FN $\tilde{C}$ is not a TFN but can be approximated by a TFN using the SA described in Section 2. It can be verified that $\tilde{C} \approx \tilde{C}' = (C_l, C_c, C_u)$ with:

$$C_l = \frac{\Pi}{\sum_{t=1}^{\omega-x}(1+i)^{-t}\prod_{k=0}^{t-1}(1-q'_{x+kl})},$$
$$C_c = \frac{\Pi}{\sum_{t=1}^{\omega-x}(1+i)^{-t}\prod_{k=0}^{t-1}(1-q'_{x+kc})},$$
$$C_u = \frac{\Pi}{\sum_{t=1}^{\omega-x}(1+i)^{-t}\prod_{k=0}^{t-1}(1-q'_{x+ku})} \quad (12)$$

So, the lower (upper) value of annuity payments C_l (C_u) is obtained by using the lower (upper) mortality probabilities q'_{x+kl} (q'_{x+ku}) in the annuity formula (7) whereas the most reliable payment, C_c, comes from the most reliable path of mortality probabilities q'_{x+kc}, $k = 1, 2, \cdots, \omega - x$.

3.1.1. Numerical application 1

Let us consider an insured aged $x = 75$, an interest rate $i = 2\%$ and a single premium $\Pi = 1000$ monetary units. To calculate some substandard annuity payments induced by different fuzzy mortality factors, we apply the TFN $\tilde{d}$ to adjust the standard mortality probabilities obtained from Human Mortality Database http://www.mortality.org/ (see also Wilmoth, Andreev, Jdanov, Glei, and Riffe, 2017) for the Spanish female population[2] in 2014. The fuzzy annuity payments are approximated by TFN $\tilde{C}'$, represented by its α-cuts, C'_α. The results are summarized in Table 2.

Alternatively, it is also possible to substitute the survival factor f in (8) by a triangular fuzzy survival factor $\tilde{f} = (f_l, f_c, f_u)$, with α-cuts $f_\alpha = [\underline{f}(\alpha), \overline{f}(\alpha)]$. Then the resulting annuity payment to be made by the insurer is also an FN, $\tilde{C}$. Now the fuzzy modified survival probability to price substandard annuities, ${}_t\tilde{p}'_x$, is defined as ${}_t\tilde{p}'_x = \tilde{f}\cdot {}_tp_x$. Under the assumption that $\tilde{f}$ is always less than 1

$${}_tp'_{x\alpha} = [\underline{{}_tp'_x}(\alpha), \overline{{}_tp'_x}(\alpha)]$$
$$= [\underline{f}(\alpha)\cdot {}_tp_x, \overline{f}(\alpha)\cdot {}_tp_x]$$
$$= [(f_l + (f_c - f_l)\alpha)\cdot {}_tp_x, (f_u - (f_u - f_c)\alpha)\cdot {}_tp_x] \quad (13)$$

So

Table 2
Substandard annuity payments represented by the α-CUTS, C'_α, of its approximation TFN $\tilde{C}'$.

	$\tilde{d} = (2.5, 3, 3.5)$		$\tilde{d} = (5, 6, 7)$		$\tilde{d} = (8, 10, 12)$	
α	$\underline{C}'(\alpha)$	$\overline{C}'(\alpha)$	$\underline{C}'(\alpha)$	$\overline{C}'(\alpha)$	$\underline{C}'(\alpha)$	$\overline{C}'(\alpha)$
1	140.11	140.11	210.37	210.37	305.58	305.58
0.9	138.87	141.31	208.05	212.69	300.74	310.67
0.8	137.64	142.52	205.74	215.02	295.90	315.75
0.7	136.40	143.72	203.43	217.34	291.06	320.83
0.6	135.16	144.92	201.11	219.67	286.21	325.91
0.5	133.93	146.13	198.80	222.00	281.37	331.00
0.4	132.69	147.33	196.48	224.32	276.53	336.08
0.3	131.46	148.53	194.17	226.65	271.69	341.16
0.2	130.22	149.73	191.86	228.97	266.85	346.24
0.1	128.98	150.94	189.54	231.30	262.00	351.33
0	127.75	152.14	187.23	233.62	257.16	356.41

Note: In the case of using the standard mortality probabilities, the standard payment $C = 85.82$.

$$C_\alpha = \left[\frac{\Pi}{\sum_{t=1}^{\omega-x}(1+i)^{-t}(f_u - (f_u - f_c)\alpha)\ {}_tp_x}, \frac{\Pi}{\sum_{t=1}^{\omega-x}(1+i)^{-t}(f_l + (f_c - f_l)\alpha)\ {}_tp_x}\right] \quad (14)$$

Again $\tilde{C}$ is not a TFN and its approximation through the SA is $\tilde{C}' = (C_l, C_c, C_u)$, with

$$C_l = \frac{\Pi}{\sum_{t=1}^{\omega-x}(1+i)^{-t}\ {}_tp'_{xu}},$$
$$C_c = \frac{\Pi}{\sum_{t=1}^{\omega-x}(1+i)^{-t}\ {}_tp'_{xc}},$$
$$C_u = \frac{\Pi}{\sum_{t=1}^{\omega-x}(1+i)^{-t}\ {}_tp'_{xl}} \quad (15)$$

In either case, calculating the annuity payment by means of an FN allows the insurer to have a full range of values the payment can take depending on the considered variables (i.e. the mortality factor, survival factor, mortality probabilities or survival probabilities). This calculation also permits the ability to perform sensitivity analyses of the payment related to these variables.

It is worth pointing out that, although the interest rate is taken as a fixed value (as it is implicitly assumed to be determinable by using the returns of bonds and/or other tradable fixed income securities), fuzzy uncertainty on the interest rate can also be considered. The consideration of fuzzy interest rates in the actuarial literature was first suggested, as an alternative to stochastic discount rates, by Lemaire (1990). Subsequently, several papers used fuzzy interest rates (see, e.g., Ostaszewski, 1993; Betzuen, Jiménez, & Rivas, 1997; de Andrés-Sánchez & Terceño, 2003 and de Andrés-Sánchez & González-Vila, 2012, 2017).

3.2. Estimating fuzzy mortality factors using confidence intervals

To calculate the substandard annuity fuzzy payment $\tilde{C}$ (or $\tilde{C}'$), we need to fit the fuzzy mortality factor $\tilde{d}$ (or the value of credits and debits in (4)). In this Subsection, we will discuss three possible ways to obtain/estimate this fuzzy mortality factor.

Firstly, a fuzzy mortality factor can be quantified based on experts' opinions. For example, an expert may judge that disease X increases someone's probability of dying between age x and $x + 1$ by about 10 times (i.e., $\tilde{d} = \tilde{10}$) or that a certain bad habit decreases someone's life expectancy by around 20% (i.e., $\tilde{f} = \tilde{0.8}$ or $\tilde{\phi} = \tilde{0.2}$). Often imprecise or subjective quantitative predictions may come from a pool of experts, leading to a set of fuzzy quantifications. This set of fuzzy opinions can be aggregated simply by their arithmetic mean or other more sophisticated methods as it is done in Shapiro and Koissi (2015), that proposed using Fuzzy Analytic Hierarchy Process in a risk assessment

[2] While we use Spanish data for illustration in all examples in this paper, the proposed fuzzy approach provides a general framework for other mortality data.

process in an insurance context. For a wide review on this matter Mardani et al. (2018) can be consulted.

Fuzzy Regression can also be used to fit mortality models, in which a multiplier that represents heterogeneity in a risk class is introduced. Koissi and Shapiro (2006) and de Andrés-Sánchez and González-Vila Puchades (2019) proposed different fuzzy versions of the Lee-Carter model. These works can be extended to obtain fuzzy estimates of mortality (or survival) factors. Likewise, fuzzy literature provides a great deal of instruments to obtain an estimation expressed as an FN from imprecise information. A complete survey on applications of Fuzzy Regression in Actuarial Science can be found in de Andrés-Sánchez (2016).

Below we propose and develop a third way of estimating the fuzzy mortality factor. It is based on interpreting statistical confidence intervals estimates as FNs and on the use of the bootstrapping methodology. Following Sfiris and Papadopoulos (2014), we consider a standard $(1-\alpha)100\%$ statistical confidence interval as the observed α-cut of the FN, for some increasing values of $\varepsilon \le \alpha < 1$, where ε is an arbitrary probability near 0 (it is often chosen to be 0.001, 0.005 or 0.01). These confidence intervals are then placed on the top of each other to produce an FN $\tilde{d}$. Some practical applications of this procedure can be found in Ellina, Papaschinopoulos, and Papadopoulos (2019).

The fuzzy interpretation of probabilistic confidence intervals combined with the bootstrapping methodology allows insurers to use their information to fit the fuzzy multiplier $\tilde{d}$. According to Pitacco (2019), a heterogeneous population can be considered as a (finite) set of (more or less) homogeneous groups. Let us denote by N_x the number of people alive at age x and by D_x the number of people dying during the interval $(x, x+1)$ in a given homogeneous group. To obtain the confidence interval estimate for the modified mortality probability q_x' in this group (due to a disease or other risk factors) and the TFN estimate for the mortality factor d, the following procedure can be performed:

Step 1. Calculate the point estimate of the modified mortality probability by $\hat{q}_x' = \frac{D_x}{N_x}$.
Step 2. Resample B times of D_x with the bth sample $D_x^{(b)} \sim \text{Binomial}(N_x, \hat{q}_x')$, $b = 1, 2, \cdots B$.
Step 3. Obtain the estimate of q_x' by $\hat{q}_x'^{(b)} = \frac{D_x^{(b)}}{N_x}$, for $b = 1, 2, \cdots B$.
Step 4. The estimate of the mortality factor associated with people aged x for the bth sample is given by $\hat{d}_x^{(b)} = \frac{\hat{q}_x'^{(b)}}{q_x}$. Then the mortality factor estimate for all ages in this bth sample is obtained by the weighted average $\hat{d}^{(b)} = \frac{\sum_x N_x \hat{d}_x^{(b)}}{\sum_x N_x}$, for $b = 1, 2, \cdots B$.
Step 5. Let $\hat{F}_{\hat{d}}$ be the empirical cumulative distribution function of $\hat{d}$. Obtain the $(1-\alpha)100\%$ bootstrap confidence interval for $\hat{d}$ by $\left[\hat{F}_{\hat{d}}^{-1}\left(\frac{\alpha}{2}\right), \hat{F}_{\hat{d}}^{-1}\left(1-\frac{\alpha}{2}\right)\right]$. As pointed out by Sfiris and Papadopoulos (2014), this confidence interval is assimilated to the α-cut, d_α, such that $\underline{d}(\alpha) = \hat{F}_{\hat{d}}^{-1}\left(\frac{\alpha}{2}\right)$ and $\overline{d}(\alpha) = \hat{F}_{\hat{d}}^{-1}\left(1-\frac{\alpha}{2}\right)$.
Step 6. Following Sfiris and Papadopoulos (2014), the TFN approximation, $\tilde{d} = (d_l, d_c, d_u)$, of $\hat{d}$ is then obtained by setting $d_l = \hat{F}_{\hat{d}}^{-1}\left(\frac{\varepsilon}{2}\right)$ where ε is close enough to 0 (e.g. 0.01), $d_c = \hat{F}_{\hat{d}}^{-1}(0.5)$ and $d_u = \hat{F}_{\hat{d}}^{-1}\left(1-\frac{\varepsilon}{2}\right)$.

3.2.1. Numerical application 2

Let us consider a sample of people, aged between 60 and 89, who are affected by a well-known disease that implys a great increase in mortality. It is assumed that this increase in mortality is roughly $\tilde{d}$ (here $\tilde{d} = \widetilde{15}$) times of a standard mortality base table (without the disease). To estimate $\tilde{d}$ for a prospective policyholder aged $x = 60$ affected by this disease we use the standard mortality probabilities for ages 60 to

Table 3
Summary of the sampling data.

x	N_x	$N_x q_x$	D_x	x	N_x	$N_x q_x$	D_x
60	60	0	3	75	45	1	10
61	60	0	4	76	45	1	12
62	60	0	8	77	45	1	12
63	60	0	1	78	45	1	19
64	60	0	4	79	45	1	15
65	55	0	3	80	40	1	25
66	55	0	3	81	40	1	17
67	55	0	5	82	40	2	26
68	55	0	5	83	40	2	25
69	55	0	6	84	40	2	34
70	50	0	6	85	35	2	34
71	50	0	6	86	35	2	35
72	50	0	9	87	35	3	35
73	50	1	11	88	35	3	35
74	50	1	6	89	35	4	35

Notes:
The product $N_x q_x$ is rounded to the closest integer number.
Only ages $x = 60, 61, \cdots, 85$ are taken. Considering greater ages will lead us to miscalculate d. This is because only those ages should be taken into account and once $dq_x = 1$, greater ages should be disregarded.

Table 4
α-Cuts of fuzzy estimate of d.

α	$\underline{d}(\alpha)$	$\overline{d}(\alpha)$
1	14.96	14.96
0.9	14.81	15.11
0.8	14.67	15.26
0.7	14.52	15.40
0.6	14.37	15.56
0.5	14.21	15.73
0.4	14.03	15.92
0.3	13.82	16.17
0.2	13.57	16.47
0.1	13.15	16.89
0	12.25	17.93

Note: To fit the 0-cut, the (1–0.01)% confidence interval is used.

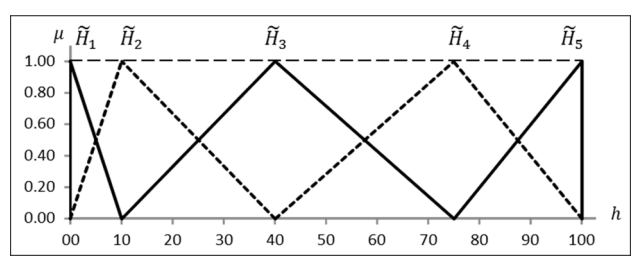

Fig. 1. Linguistic variable "Health" on a five-level scale.

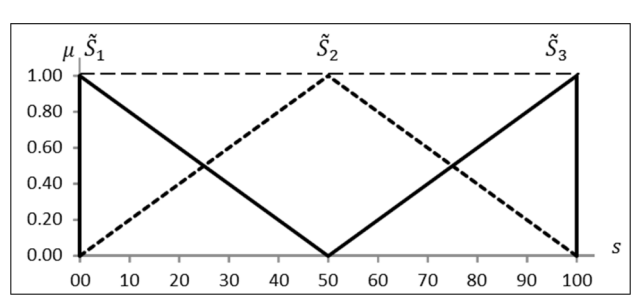

Fig. 2. Linguistic variable "Lifestyle" on a three-level scale.

Table 5
Fuzzy rules for determining "substandard annuity payment" based on H and S.

Health H	Lifestyle S		
	$\tilde{S}_1$	$\tilde{S}_2$	$\tilde{S}_3$
$\tilde{H}_1$	$\tilde{C}_5$	$\tilde{C}_5$	$\tilde{C}_4$
$\tilde{H}_2$	$\tilde{C}_4$	$\tilde{C}_4$	$\tilde{C}_3$
$\tilde{H}_3$	$\tilde{C}_3$	$\tilde{C}_3$	$\tilde{C}_2$
$\tilde{H}_4$	$\tilde{C}_2$	$\tilde{C}_2$	$\tilde{C}_1$
$\tilde{H}_5$	$\tilde{C}_2$	$\tilde{C}_1$	$\tilde{C}_1$

89 of Spanish female population in 2014 obtained from the Human Mortality Database (http://www.mortality.org/, Wilmoth et al., 2017). We perform steps 1 to 6 with $B = 5000$. The results are given in Tables 3 and 4.

In Table 3, the second column represents the number of people, aged x, subject to the disease. Column 3 shows the standard expected number of deaths between x and $x + 1$. The increased number of deaths, D_x, in the last column are generated by considering that $D_x \sim \text{Binomial}(N_x, 15q_x)$. So the effect of increasing 15 times of the standard mortality probabilities can be seen by comparing $N_x q_x$ and D_x.

Table 4 contains the α-cuts of $\tilde{d}$. From Step 6, the TFN approximation $\tilde{d} = (12.25, 14.96, 17.93)$ is obtained.

The estimation of the fuzzy mortality factor by considering different risk factors and their possible interaction may result in obtaining a set of n mortality factors $\tilde{d}_k$, $k = 1, 2, \cdots, n$, for each age x. In order to decide what mortality factor(s) correspond(s) to a prospective policyholder aged x and, from it (them), what substandard annuity payment should be paid to the policyholder, an FIS will be used. In the next Section, it is discussed how to use an FIS to determine substandard annuity payments.

4. Fuzzy inference systems to determine substandard annuity payments

Although several authors proposed applying FISs to underwrite or price life insurance contracts, in the existing literature, FISs have never been used to price substandard annuity payments. De Wit (1982) appeared to be the first research suggesting that FISs could be applied to insurance decision-making problems. He argued that a life insurance contract is not only based on its risk premium, estimated as close to its actual risk as possible, but also on a practical experience expressed by the company underwriting practice. However, De Wit (1982) only used a basic FIS to analyse the underwriting process of a life insurance contract. Lemaire (1990) extended the paper of De Wit (1982) and used an FIS in order to define the concept of "preferred policyholder" in life insurance contracts. While De Wit (1982) focused on technical and behavioural features of the underwriting process, Lemaire (1990) focused on some indicators of the policyholder's health. Horgby et al. (1997) applied an FIS for medical underwriting of life insurance prospective policyholders in the presence of diabetes mellitus. Also in a life insurance context, Horgby (1998) defined risk factors as fuzzy sets and showed that an insurer could use multiple prognostic factors that are imprecise and vague. He considered pricing life insurances according to both sharp criteria (such as gender and age) and a risk loading obtained by using fuzzy inference. In a general context related to a new product pricing, Haji and Assadi (2009) used an FIS for the representation and treatment of the uncertain knowledge and data. Shapiro and Koissi (2015) designed an FIS for the insurer's risk assessment. Finally, Torbati and Sayadi (2018) and Subartini et al. (2018) proposed using FISs to evaluate insurer's financial performance and insurance premiums for flood disaster policies, respectively.

In the following, we will show how to determine the payments for a substandard annuity with a fixed single premium by using an FIS. Pricing substandard annuities requires collecting policyholders' health and lifestyle information. Collection of these data is usually made through questionnaires (example of such questionnaire is the Retirement Health & Lifestyle (2020), https://www.retirementhealthform.co.uk/) and/or interviews with the insured

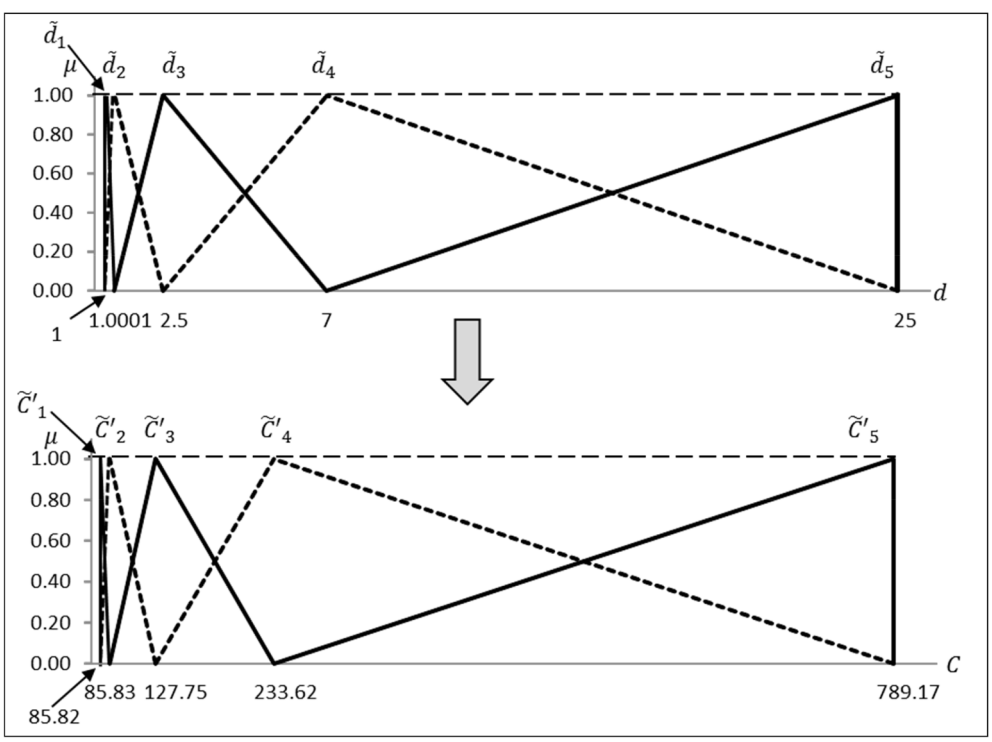

Fig. 3. "Substandard Annuity Payment" induced by "Mortality Factor".

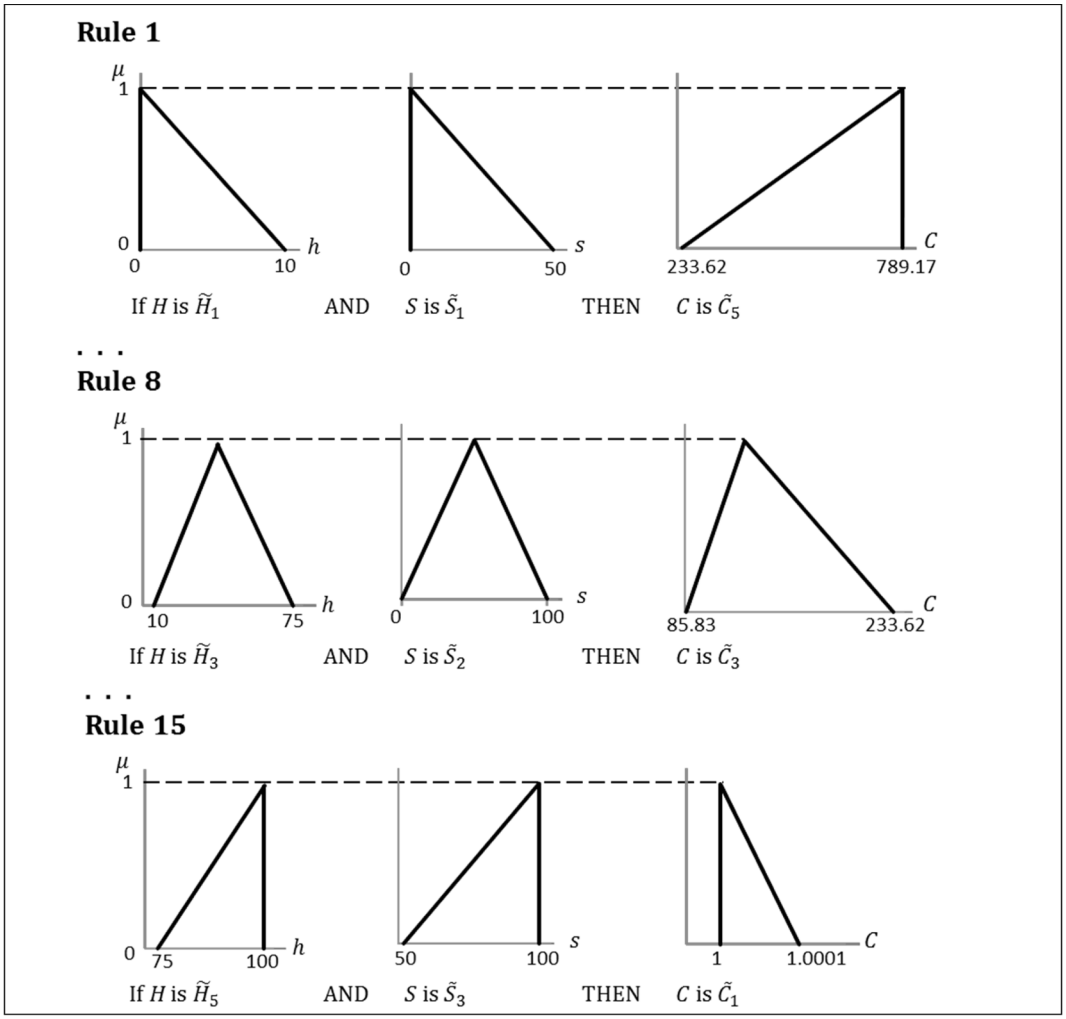

Fig. 4. Fuzzy rules of the FIS.

Table 6
Crisp outputs for the substandard annuity payments.

Health punctuation h	*AND* and *OR* connectives: Min-max			*AND* and *OR* connectives: probabilistic *t-norm* and *t-conorm*		
	Lifestyle punctuation			Lifestyle punctuation		
	$s = 10$	$s = 50$	$s = 90$	$s = 10$	$s = 50$	$s = 90$
0	650.28	650.28	406.89	650.28	650.28	406.89
5	606.82	498.16	317.25	498.16	498.16	295.54
10	589.43	346.04	184.20	346.04	346.04	184.20
20	413.85	278.60	161.34	278.60	278.60	158.06
30	293.24	211.17	150.05	211.17	211.17	131.93
40	143.74	143.74	105.79	143.74	143.74	105.79
50	136.96	130.18	100.94	130.18	130.18	100.69
60	136.96	116.63	98.80	116.63	116.63	95.58
70	136.36	103.08	95.05	103.08	103.08	90.47
80	121.91	94.21	87.32	95.89	94.21	87.50
90	108.36	90.02	87.32	95.05	90.02	86.66
100	94.21	85.82	85.82	94.21	85.82	85.82

parties' doctors (or general practitioners if in the UK). A questionnaire may include age, gender, nationality, marital status, postcode, height, weight, waist measurement, intake of alcoholic drinks, blood pressure, level of cholesterol, past or current diseases such as diabetes, cancer, stroke, etc. As stated in Gatzert et al. (2012) and Ridsdale (2012), many pricing structures of substandard annuities use rule-based expert systems that determine the mortality factor by using such underwriting data. These systems are based on statistical data and ordinary sets, which provide information about the excess of mortality associated with several diseases and lifestyles. However, as argued by Woo (2013) and Charrington (2013), these pricing structures are still narrow on mortality risk, algorithms to price substandard annuities should be refined and other potential risk factors (such as gym membership, food shopping trends, cognitive and social functioning, well-being and online computer gaming hours) should also be taken into consideration when pricing.

Thus, more complete expert inference systems can be designed by incorporating vague information, described by linguistic variables such as "*high* level of cholesterol", "*normal* cognitive functioning", "*very high* blood pressure", "*small* number of computer gaming hours", etc. Such imprecise but important information can be incorporated, when pricing substandard annuities, using an FIS. So we will design such an FIS to determine payments for a substandard annuity based on some vague information relevant to the annuity contract[3]. A Mamdani FIS (Mamdani, 1974) to model the fuzzy outputs by means of linguistic variables is adopted, but a Takagi-Sugeno FIS (Takagi & Sugeno, 1985) can also be applied in a similar way with a set of crisp functions of input linguistic variables that can be fitted by using conventional or neural regression techniques.

[3] While the development here can be used as a general FIS framework for pricing substandard annuities, companies or practitioners may wish to design an FIS that best suits their specific practice and environment.

An FIS can be developed with a set of $n(n \geq 1)$ linguistic variables $\{V_1, \cdots, V_n\}$ as inputs. These inputs may embed some risk factors of heterogeneous nature (such as health status, socio-economic and marital status, habits etc.). For each of the linguistic variables V_k, $k = 1, 2, \cdots, n$, we define a reference set as $[V_{k_{\min}}, V_{k_{\max}}]$. We then divide the reference set into J levels (i.e. J fuzzy sets) $\tilde{V}_{kj}$, $j = 1, 2, \cdots, J$, which are assumed to be TFNs[4], as $V_{k_{\min}} = V_{k_1} < V_{k_2} < V_{k_3} < \cdots < V_{k_{J-1}} < V_{k_J} = V_{k_{\max}}$ and $\{\tilde{V}_{k_1} = (V_{k_1}, V_{k_1}, V_{k_2})$; $\tilde{V}_{kj} = (V_{kj-1}, V_{kj}, V_{kj+1}), j = 2, 3, \cdots, J-1$; $\tilde{V}_{kJ} = (V_{kJ-1}, V_{kJ}, V_{kJ})\}$ such that $\sum_j \mu_{\tilde{V}_{kj}}(v_k) = 1$, for a given crisp value v_k of V_k.

In our Madamni FIS, the mortality factor[5] d is also modelled as a linguistic variable. The reference set of d is denoted by $[d_{\min}, d_{\max}]$ with $d_{\min} = 1$ implying no payment enhancement and $d_{\max} \leq \frac{1}{q_x}$. Now d is divided into P TFNs, $\tilde{d}_p$, $p = 1, 2, \cdots, P$, as[6] $d_{\min} = d_1 = 1 < d_2 = 1 + \delta < d_3 < \cdots < d_p < \cdots < d_{P-1} < d_P = d_{\max}$ and $\{\tilde{d}_1 = (1, 1, d_2); \tilde{d}_p = (d_{p-1}, d_p, d_{p+1}), \quad p = 2, 3, \cdots, P-1$; $\tilde{d}_P = (d_{P-1}, d_P, d_P)\}$, such that, $\sum_p \mu_{\tilde{d}_p}(d) = 1$, for a given crisp value of d.

Inputs are linked with the outputs via some fuzzy rules that may be given by experts or may be generated empirically as it is done in Tan, Shum, Chao, Vijayakumar, and Yang (2019). For example, we can apply the following fuzzy rules to determine the output variable "mortality factor":

If " V_1" is $\tilde{V}_{1_j}$ AND " V_2" is $\tilde{V}_{2_j}$ AND … " V_n" is $\tilde{V}_{n_j}$ THEN the "mortality factor " is $\tilde{d}_p$.

As seen in Section 3.1, a fuzzy mortality factor induces a fuzzy substandard annuity payment. So, the above fuzzy rules are equivalent to the following ones:

If " V_1" is $\tilde{V}_{1_j}$ AND " V_2" is $\tilde{V}_{2_j}$ AND … " V_n" is $\tilde{V}_{n_j}$ THEN the "substandard annuity payment" is $\tilde{C}_p$, where $\tilde{C}_p$ is obtained from $\tilde{d}_p$ by using (9) and (11).

According to Telford et al. (2011), there are three main types of substandard annuities: Lifestyle annuities, Impaired life annuities and Immediate needs annuities. *Lifestyle annuities* take into account policyholders' habits and minor medical problems that involve a small enhancement of the payment. Typical risk factors to be considered are policyholders' postal code of residence, whether they smoke or not, body mass index, etc. *Impaired life annuities*, which represent a more substantial payment improvement, are designed for people with significant health problems such as cancer, diabetes or chronic asthma. *Immediate needs annuities* are typically designed for elderly people, in a severe dependence situation, who need immediate attention. This type of substandard annuities generates the greatest payment improvement. Based on these three types of substandard annuities, we can divide the factors considered for pricing such annuities into two generic variables: lifestyle and health variables.

We now develop an FIS to price substandard annuities that, by considering the input variables "Lifestyle" and "Health", can be used to obtain the payments of any substandard annuity type described in Telford et al. (2011). The first variable embeds habits such as, regular exercise, practice of risky sports, alcohol intake, smoking, marital status, etc. The second variable may take into account, among other variables: blood pressure, level of cholesterol or past and present severe diseases (e.g. cancer, heart attack, stroke, etc.). We assume that "Health" (H) and "Lifestyle" (S) of a prospective policyholder can be measured as a numerical value in a given reference set [0, 100]. Some widely-used questionnaires (e.g. Behavioral Risk Factor Surveillance, 2018 or Health Promoting Lifestyle Profile, 2018) can convert a set of questions related to the respondent's health status and/or lifestyle into numerical values[7]. Thus, after the completion of some questionnaires and with the knowledge from experts, numerical values can be obtained corresponding to each policyholder's health and lifestyle.

Suppose now that H is granulated in 5 levels $\tilde{H}_i$, $i = 1, 2, \cdots, 5$, by dividing the reference set [0, 100] (from the worst to the best health status) as $0 = h_1 < h_2 < h_3 < h_4 < h_5 = 100$ and $\{\tilde{H}_1 = (h_1, h_1, h_2)$; $\tilde{H}_i = (h_{i-1}, h_i, h_{i+1}), i = 2, 3, 4$; $\tilde{H}_5 = (h_4, h_5, h_5)\}$. Fig. 1 shows a five-level linguistic variable for "Health" with $\{\tilde{H}_1$: Very bad health, $\tilde{H}_2$: Bad health, $\tilde{H}_3$: Normal health, $\tilde{H}_4$: Good health and $\tilde{H}_5$: Very good health$\}$.

Similarly, S is divided into 3 linguistic labels, $\tilde{S}_j$, $j = 1, 2, 3$. Fig. 2 shows a three-level linguistic variable for "Lifestyle" with $\{\tilde{S}_1$: Bad lifestyle, $\tilde{S}_2$: Normal lifestyle and $\tilde{S}_3$: Good lifestyle$\}$.

And for any value h and s in the interval [0, 100], it is satisfied that

$$\sum_i \mu_{\tilde{H}_i}(h) = \sum_j \mu_{\tilde{S}_j}(s) = 1$$

The output of the FIS is the "Substandard Annuity Payment" (C) also defined as a linguistic variable. The FIS links the inputs and output via a set of fuzzy rules as:

If "Health" is $\tilde{H}_i$ AND "Lifestyle" is $\tilde{S}_j$, then the "Substandard Annuity Payment" is $\tilde{C}_p$.

To illustrate how the FIS works, we consider 15 fuzzy rules as shown in Table 5[8].

4.1. Numerical application 3

Consider an individual aged $x = 75$, a mortality multiplier d with five possible linguistic values: {Very Low mortality, Low mortality, Medium mortality, High mortality, Extremely High mortality}.

We thus divide the mortality factor d into 5 TFNs, $\tilde{d}_p$, $= 1, 2, \cdots, 5$, with the reference set [1, 25] being divided into $d_{min} = d_1 = 1 < d_2 = 1.0001 < d_3 = 2.5 < d_4 = 7 < d_5 = 25$. So, the linguistic variable "Mortality Factor" is labelled as

$\{\tilde{d}_1 = (1, 1, 1.0001); \tilde{d}_2 = (1, 1.0001, 2.5); \tilde{d}_3 = (1.0001, 2.5, 7);$

$\tilde{d}_4 = (2.5, 7, 25); \tilde{d}_5 = (7, 25, 25)\}$

We can now obtain the linguistic variable "Substandard Annuity Payment" associated to the variable "Mortality Factor". We use standard mortality rates of Spanish female population in 2014 obtained from the Human Mortality Database (http://www.mortality.org/, Wilmoth et al., 2017). Further, a net single premium $\Pi = 1000$ monetary units and the interest rate $i = 0.02$ are assumed.

Each of the 5 linguistic labels $\tilde{C}_p$, $p = 1, \cdots, 5$, of the "Substandard Annuity Payment", as in Table 5, corresponds to each $\tilde{d}_p$, $p = 1, \cdots, 5$, through Eqs. (9) and (11). We then obtain the approximations $\tilde{C}'_p$, $p = 1, 2, \cdots, 5$ as depicted in Fig. 3.

Further, fuzzy rules in Table 5 for this numerical application are presented in Fig. 4.

Finally, in order to determine the substandard annuity payment for a given value of "Health" and "Lifestyle", a defuzzification method has to be applied. Table 6 shows the crisp output of the FIS (i.e. the crisp value of the substandard annuity payment), obtained by using the

[4] They can also be trapezoidal, Gaussian or semi-exponential FN. Further, as done by Dalkilic et al. (2009), input membership functions may be built up by applying fuzzy clustering techniques to insurer's database.

[5] An FIS with the survival factor f, if life expectancy is considered instead of mortality, can be developed in a similar way.

[6] Note that d_1 corresponds to the mortality probability of the base table while d_2 refers to the mortality factor allowing for the minimum enhancement (i.e. $\delta \to 0$).

[7] The Behavioral Risk Factor Surveillance System, available at https://www.cdc.gov/brfss/, collects data about US residents regarding their health-related risk behaviours, chronic health conditions and use of preventive services. The Health Promoting Lifestyle Profile II, available at https://www.unmc.edu/nursing/faculty/health-promoting-lifestyle-profile-II.html, focuses on lifestyle and habits questions.

[8] Different number of fuzzy rules can also be considered depending on the actual problem.

minimum and the probabilistic *t-norm* for the connector *AND* in the fuzzy rules, the maximum and the probabilistic *t-conorm* for the connector *OR* and the centre of gravity defuzzifying method.

5. Conclusions and further extensions

Pricing substandard annuities and determining their payments are complex processes that require the blend of both crisp and fuzzy information. Our paper demonstrates that actuarial valuation with fuzzy parameters and FISs are suitable tools to price substandard annuities and determine their payments.

In this paper, we introduce fuzzy mortality factors to capture the imprecise or vague nature of prospective annuitants' health status and lifestyle of substandard annuities. The fuzzy mortality factors modify mortality probabilities in the base table and lead us to determine substandard annuities' payments with the mortality (or survival) probabilities given by FNs. We then discussed how to estimate the fuzzy mortality factors with special attention to the fuzzy interpretation of statistical confidence intervals as it is proposed in Sfiris and Papadopoulos (2014). Using other fuzzy set tools to fit mortality factors (e.g. fuzzy multiple criteria decision making methods or fuzzy regression) can be considered for further research.

Section 4 studies how to apply an FIS to obtain substandard annuity payments based on imprecise or vague annuitant's information such as health status and lifestyle. The use of such a fuzzy expert system allows introducing linguistic variables, fuzzy information and fuzzy rules, with the additional advantage that the determination of the crisp annuity payment will always be performed in a consistent manner. FIS methods are flexible enough to allow introducing data from a wide database to build up input membership functions that infer outputs. The use of FIS to price substandard annuities may still be subject to further developments. Other types of membership functions instead of triangular shapes or other FIS architectures, such as the one developed by Takagi and Sugeno (1985), can also be considered. Likewise, assessing alternative methods to build up the linguistic labels of input and output variables, such as fuzzy clustering or genetic algorithms, are also interesting topics for further developments.

Even if certain risk factors cannot be used when pricing standard annuities due to country legislations (e.g. race, ethnicity, religion or sex), they may be considered when setting reserves, forecasting life expectancies, or establishing valuation processes for a more efficient mortality and longevity risks management. In our opinion, the application of FISs in such circumstances is a promising field for further extensions.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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