Sorbonne Université

École doctorale de sciences mathématiques de Paris centre

MÉMOIRE

présenté pour l'obtention de

L'HABILITATION À DIRIGER DES RECHERCHES

Spécialité : Mathématiques

Some aspects of the geometric combinatorics of convex polytopes

par

Arnau Padrol

Soutenu le 10 décembre 2019 à Paris devant le jury composé de :

Frédéric Chapoton	CNRS Université de Strasbourg
Victor Chepoi	Aix-Marseille Université
Stéphane Gaubert	INRIA École Polytechnique
Xavier Goaoc	Université de Lorraine
Gil Kalai	Hebrew University of Jerusalem
Sophie Morier-Genoud	Sorbonne Université
Michel Pocchiola	Sorbonne Université
Michèle Vergne	CNRS Université de Paris

au vu des rapports de :

Xavier Goaoc	Université de Lorraine
Gil Kalai	Hebrew University of Jerusalem
Rekha R. Thomas	University of Washington

Arnau Padrol
Sorbonne Université
Institut de Mathématiques de Jussieu - Paris Rive Gauche (UMR 7586)
4 place Jussieu
75252 Paris Cedex 05
France
arnau.padrol@imj-prg.fr

Acknowledgements

I would like to start by thanking Xavier Goaoc, Gil Kalai and Rekha R. Thomas, who kindly accepted to review this manuscript. I am honored and very grateful for the time you devoted. I really appreciate that Gil Kalai and Xavier Goaoc additionally accepted to be part of the examining committee. I also want to express my sincere gratitude to the other members of the committee: Frédéric Chapoton, Victor Chepoi, Stéphane Gaubert, Sophie Morier-Genoud, Michel Pocchiola, and Michèle Vergne. It is an honor to have such a great jury, and it will be a pleasure to share this moment with you.

Many special thanks to the co-authors of the research articles on which this thesis is based: Karim A. Adiprasito, Philip Brinkmann, Cesar Ceballos, Hao Chen, Bernd Gonska, Francesco Grande, Jeffrey C. Lagarias, Yusheng Luo, Leonardo Martínez-Sandoval, Hiroyuki Miyata, Benjamin Nill, Yann Palu, Pavel Paták, Zuzana Patáková, Julian Pfeifle, Vincent Pilaud, Pierre-Guy Plamondon, Raman Sanyal, Camilo Sarmiento, Louis Theran, and Günter M. Ziegler.

Je tiens à remercier mes collègues de l'IMJ-PRG – enseignants, chercheurs, et équipe administrative – et en particulier les membres de l'Équipe Combinatoire et Optimisation, pour leur aide et encouragement.

Al Guillem, sense qui aquesta HDR s'hauria acabat molt abans, i a la Laura, sense qui no s'hauria pogut acabar.

Contents

CHAPTER 1

Introduction

The aim of this document is to give an overview of my work since the end of my PhD thesis in 2013 [00]. The topics presented cover different aspects of polytope theory, with an emphasis on questions related to the interaction between the geometric and combinatorial properties of convex polytopes. They concern the study of the influence of geometric constraints on the combinatorial structure, a combinatorial analysis of geometric constructions, and the search for geometric realizations of given combinatorial objects. They range from very classical subjects, such as inscribability, to very recent developments, like tropical polytopes and ν-Tamari lattices. The point of view is mainly that of combinatorial geometry, but during the path we will see many connections with other facets of the theory: linear optimization, computational geometry, combinatorial topology, metric geometry, enumerative combinatorics, tropical geometry, algebraic combinatorics…

A *polytope* is the convex hull of a finite set of points, and its combinatorial structure is given by its poset of faces, ordered by inclusion. The set of all geometric realizations of a fixed combinatorial isomorphism class is known as its *realization space*. Starting at 4-dimensional polytopes, realization spaces can be very complicated semialgebraic sets (topologically, algebraically, and algorithmically), a phenomenon revealed by the groundbreaking *Universality Theorems* of Mnëv [Mnë88] and Richter-Gebert [RG96]. Chapter 2 starts with a result from "The universality theorem for neighborly polytopes" [11], written in collaboration with Karim Adiprasito, that provides the missing final step for proving Mnëv's Universality Theorem for simplicial polytopes. Sections 2.2 and 2.3 present results from "A universality theorem for projectively unique polytopes and a conjecture of Shephard" [07], written also with Karim Adiprasito. It concerns *projectively unique polytopes*, the polytopes that have the most rigid realization space. We prove that every polytope described by algebraic coordinates is the face of a projectively unique polytope, and disprove a classical conjecture of Shephard [She74] by constructing a combinatorial type of 5-dimensional polytope that is not realizable as a subpolytope of any stacked polytope. The chapter ends with a part of "Extension complexity and realization spaces of hypersimplices" [13], coauthored with Francesco Grande and Raman Sanyal, concerning the realization spaces of hypersimplices.

Understanding which polytopes are *inscribable*, that is, have a realization with all the vertices on the sphere, is a classical subject that was first asked by Steiner in 1832 [Ste32]. Since then, the interest on inscribability of polytopes has soared, partially because of tight relations with Delaunay subdivisions and hyperbolic geometry. Inscribability is a very restrictive constraint, and in general it is hard to decide whether a polytope is inscribable. Nevertheless, Section 3.1 presents a construction for many combinatorially distinct inscribable polytopes, to the point that they give the current best lower bound for the number of combinatorial types of polytopes. It presents the point of view from "Neighborly inscribed polytopes and Delaunay triangulations" [08], joint work with Bernd Gonska, in which we prove the inscribability (in any smooth strictly convex body) of a family of neighborly polytopes first presented in "Many neighborly polytopes and oriented matroids" [01] (a publication from my PhD thesis). Section 3.2 describes results from "Universality theorems for inscribed polytopes and Delaunay triangulations" [02], written in collaboration with Karim Adiprasito and Louis Theran, that show that inscribed polytopes present Mnëv's universality phenomenon. The polars of inscribable polytopes are circumscribable, and in between there is a full spectrum of "scribability" notions. Some of them are studied in Section 3.3, which reports joint work with Hao Chen from "Scribability problems for polytopes" [12]. The open problems section presents some of the questions formulated in "Six topics on inscribable polytopes" [17], written in collaboration with Günter Ziegler.

The *Minkowski sum* is one of the most important geometric operations on polytopes. It is geometric in the sense that the combinatorial structure of $P+Q$ depends on the geometric realization of P and Q, not only their combinatorics. Chapter 4 presents results on Minkowski sums of polytopes and colorful point configurations from "Colorful simplicial depth, Minkowski sums, and generalized Gale transforms" [15], written with Karim Adiprasito, Philip Brinkmann, Pavel Paták, Zuzana Patáková, and Raman Sanyal. It consists of two very different parts. On the one hand, we present an upper bound on the *colorful simplicial depth* of colorful point configurations. The colorful simplicial depth is a measure of centrality of a point in $\mathbb{R}^d$ with respect to $d+1$ point configurations, introduced by Deza, Huang, Stephen, and Terlaky in [Dez+06] as a colorful generalization of Liu's simplicial depth [Liu90]. Our bound implies a conjectured upper bound from [Dez+06]. Furthermore, we introduce colorful Gale transforms and Minkowski transforms, a bridge between colorful configurations and Minkowski sums. Through them, our colorful upper bound yields a tight upper bound on the number of totally mixed facets of certain Minkowski sums of simplices, resolving a conjecture of Burton in the theory of normal surfaces [Bur03].

The *extension complexity* of a polytope is the minimal number of facets of a (usually higher dimensional) polytope that can be projected onto it. It is a geometric parameter, as it strongly depends on the geometric realization of the polytope. Introduced by the combinatorial optimization community because of its ties with the computational complexity of linear programming, it is also very relevant in many other areas for its tight relation with nonnegative factorizations of nonnegative matrices and with communication complexity [Yan91]. Polytopes for which the extension complexity is known are very rare, and a lot of effort is put on finding bounds for relevant families of polytopes. Section 5.1 presents "Polygons as sections of higher-dimensional polytopes" [06], written in collaboration with Julian Pfeifle. Our main result is an upper bound for the extension complexity of polygons, found independently by Shitov [Shi14a], that disproved a conjecture of Beasley and Laffey [BL09]. Section 5.2 contains the complete classification by extension complexity of all d-polytopes with up to $d+4$ vertices or facets, which form a super-exponentially large family of polytopes. This is a result from "Extension complexity of polytopes with few vertices or facets" [09]. For one particular case, the classification uses a combinatorial result from "Polytopes with few vertices and few facets" [10], related to Perles' Skeleton Theorem, that entails (surprisingly low) upper bounds on the number of polytopes with few vertices and facets. It is presented in Section 5.3. Section 5.4 gives lower bounds for the extension complexity of generic polytopes, also from "Extension complexity of polytopes with few vertices or facets" [09]. We end in Section 5.5 with results from "Extension complexity and realization spaces of hypersimplices" [13], written with Francesco Grande and Raman Sanyal, concerning the extension complexity of hypersimplices. They complement the results on realization spaces presented in Section 2.4.

The last chapter concerns triangulations of products of simplices. These are very important combinatorial objects, among other things, because of their correspondence with tropical (pseudo-)hyperplane arrangements [AD09; DS04]. In "Geometry of ν-Tamari lattices in types A and B" [16], joint work with Cesar Ceballos and Camilo Sarmiento, this correspondence is used to give a (tropical) geometric realization of the ν-Tamari lattice, presented in Section 6.1. This generalization of the Tamari lattice, whose study originated in the study of higher diagonal coinvariant spaces [BPR12], has been the object of a lot of attention lately, and the geometric realizability of some particular cases as a polyhedral subdivision of an associahedron was an open problem by F. Bergeron [Ber12]. As a by-product of our construction, a new ν-Tamari poset in type B is introduced, with its analogous geometric realization. This is the subject of Section 6.2. The central tool for our realizations is the *associahedral triangulation*. A close relative, the *Dyck path triangulation*, is key for Section 6.3, which reports results on extendability of partial triangulations of the product of two simplices from "Dyck path triangulations and extendability" [03], also joint work with Cesar Ceballos and Camilo Sarmiento.

Nearly all of my post-thesis journal publications are cited in the document in some way or an other, but with different level of detail. In particular, "Enumeration of neighborly polytopes and oriented matroids" [04], which is joint work with Hiroyuki Miyata, and in which we do a computer search to enumerate new cases of neighborly polytopes and neighborly oriented matroids, is only cited in relation to open problems in Sections 2.5 and 3.4. For most publications, one or two selected results are presented in separated sections. Relevant notions and definitions are introduced gradually, only when they are needed for the first time. Proofs are mostly omitted, or just informally outlined. However, most sections are complemented with a small appendix giving a glimpse to some of the technical tools used for the proofs. These are independent from the main text and can be safely skipped or left for a more detailed lecture. Nevertheless, they form the backbone connecting the subjects together. Indeed, some of the tools pop up being used in several very different contexts. Each chapter ends with some open problems and perspectives for future research.

Results from my PhD thesis [00] are mostly omitted. Besides some results not published elsewhere, it contains the papers "Many neighborly polytopes and oriented matroids" [01], which presents a new construction for a large family of neighborly polytopes that is mentioned in Section 3.1 as it is the starting point for [08], and "The degree of point configurations: Ehrhart theory, Tverberg points and almost neighborly polytopes" [05], written with Benjamin Nill, in which we study a combinatorial parameter of polytopes motivated from Ehrhart theory for lattice polytopes and related to neighborliness, the Generalized Lower Bound Theorem and Tverberg theory.

Similarly, results that have not been published in a journal yet, or only very recently, are barely discussed. This concerns particularly the article [14], that appeared after the writing of this manuscript started, and the preprints [18; 19; 20], that are completed and at different stages of the peer-review process. "Moser's shadow problem" [14], a collaboration with Jeffrey Lagarias and Yusheng Luo, provides complete answers to several variants of Moser's shadow problem [Mos66; Mos91] concerning the maximal shadows of 3-dimensional polyhedra. "The ν-Tamari lattice as the rotation lattice of ν-trees" [18], written with Cesar Ceballos and Camilo Sarmiento, gives new interpretations of the ν-Tamari lattice and uncovers its relation with several known combinatorial objects. Some of its open questions are briefly mentioned in Section 6.4. "Associahedra for finite type cluster algebras and minimal relations between g-vectors" [19], joint work with Yann Palu, Vincent Pilaud and Pierre-Guy Plamondon, revisits and expands a construction of the associahedron that recently appeared in the mathematical physics community [AH+18], and its extension to generalized associahedra arising from representation theory [BM+18]. It is cited in relation to open problems in Section 2.5. Finally, "The convex dimension of hypergraphs and the hypersimplicial Van Kampen-Flores Theorem" [20], written in collaboration with Leonardo Martínez-Sandoval, computes the convex dimension of complete uniform hypergraphs, first asked by Halman, Onn and Rothblum [HOR07], by means of the study of projections of hypersimplices.

Passages from the aforementioned publications are quoted and paraphrased all along the document. The figures are also reproduced from these articles. In particular, many figures from Chapter 6 were originally created by Camilo Sarmiento.

1.1. List of publications

This section contains my publication list[1], ordered by publication date, and omitting all conference proceedings. A schematic representation of the links between publications is given in Section 1.2.

PhD thesis.

[00] Arnau Padrol. "Neighborly and almost neighborly configurations, and their duals". Ph.D. Thesis. Advisor: Julian Pfeifle. Universitat Politècnica de Catalunya, Mar. 2013.

Journal articles.

[01] Arnau Padrol. "Many neighborly polytopes and oriented matroids". In: *Discrete Comput. Geom.* 50.4 (2013), pp. 865–902.

[02] Karim Adiprasito, Arnau Padrol, and Louis Theran. "Universality theorems for inscribed polytopes and Delaunay triangulations". In: *Discrete Comput. Geom.* 54.2 (2015), pp. 412–431.

[03] Cesar Ceballos, Arnau Padrol, and Camilo Sarmiento. "Dyck path triangulations and extendability". In: *J. Combin. Theory Ser. A* 131.0 (2015), pp. 187–208.

[04] Hiroyuki Miyata and Arnau Padrol. "Enumeration of neighborly polytopes and oriented matroids". In: *Exp. Math.* 24.4 (2015), pp. 489–505.

[05] Benjamin Nill and Arnau Padrol. "The degree of point configurations: Ehrhart theory, Tverberg points and almost neighborly polytopes". In: *European J. Combin.* 50 (2015). Combinatorial Geometries: Matroids, Oriented Matroids and Applications. Special Issue in Memory of Michel Las Vergnas, pp. 159–179.

[06] Arnau Padrol and Julian Pfeifle. "Polygons as sections of higher-dimensional polytopes". In: *Electron. J. Combin.* 22.1 (2015), Paper 1.24, 16 pp.

[07] Karim Adiprasito and Arnau Padrol. "A universality theorem for projectively unique polytopes and a conjecture of Shephard". In: *Israel J. Math.* 211.1 (2016), pp. 239–255.

[08] Bernd Gonska and Arnau Padrol. "Neighborly inscribed polytopes and Delaunay triangulations". In: *Adv. Geom.* 16.3 (2016), pp. 349–360.

[1] These reference numbers are used for my publications throughout the document. The remaining citations use alphanumberic style based on authors names and publication year.

[09] Arnau Padrol. “Extension complexity of polytopes with few vertices or facets”. In: *SIAM J. Discrete Math.* 30.4 (2016), pp. 2162–217.

[10] Arnau Padrol. “Polytopes with few vertices and few facets”. In: *J. Combin. Theory Ser. A* 142 (2016), pp. 177–180.

[11] Karim A. Adiprasito and Arnau Padrol. “The universality theorem for neighborly polytopes”. In: *Combinatorica* 37.2 (2017), pp. 129–136.

[12] Hao Chen and Arnau Padrol. “Scribability problems for polytopes”. In: *European J. Combin.* 64 (2017), pp. 1–26.

[13] Francesco Grande, Arnau Padrol, and Raman Sanyal. “Extension complexity and realization spaces of hypersimplices”. In: *Discrete Comput. Geom.* 59.3 (2018), pp. 621–642.

[14] Jeffrey C. Lagarias, Yusheng Luo, and Arnau Padrol. “Moser’s shadow problem”. In: *Enseign. Math.* 64.3-4 (2018), pp. 477–496.

[15] Karim A. Adiprasito, Philip Brinkmann, Arnau Padrol, Pavel Paták, Zuzana Patáková, and Raman Sanyal. “Colorful simplicial depth, Minkowski sums, and generalized Gale transforms”. In: *Int. Math. Res. Not. IMRN* 6 (2019), pp. 1894–1919.

[16] Cesar Ceballos, Arnau Padrol, and Camilo Sarmiento. “Geometry of ν-Tamari lattices in types A and B”. In: *Trans. Amer. Math. Soc.* 371.4 (2019), pp. 2575–2622.

Book chapters.

[17] Arnau Padrol and Günter M. Ziegler. “Six topics on inscribable polytopes”. In: *Advances in Discrete Differential Geometry.* Ed. by Alexander I. Bobenko. Berlin, Heidelberg: Springer, 2016, pp. 407–419.

Preprints.

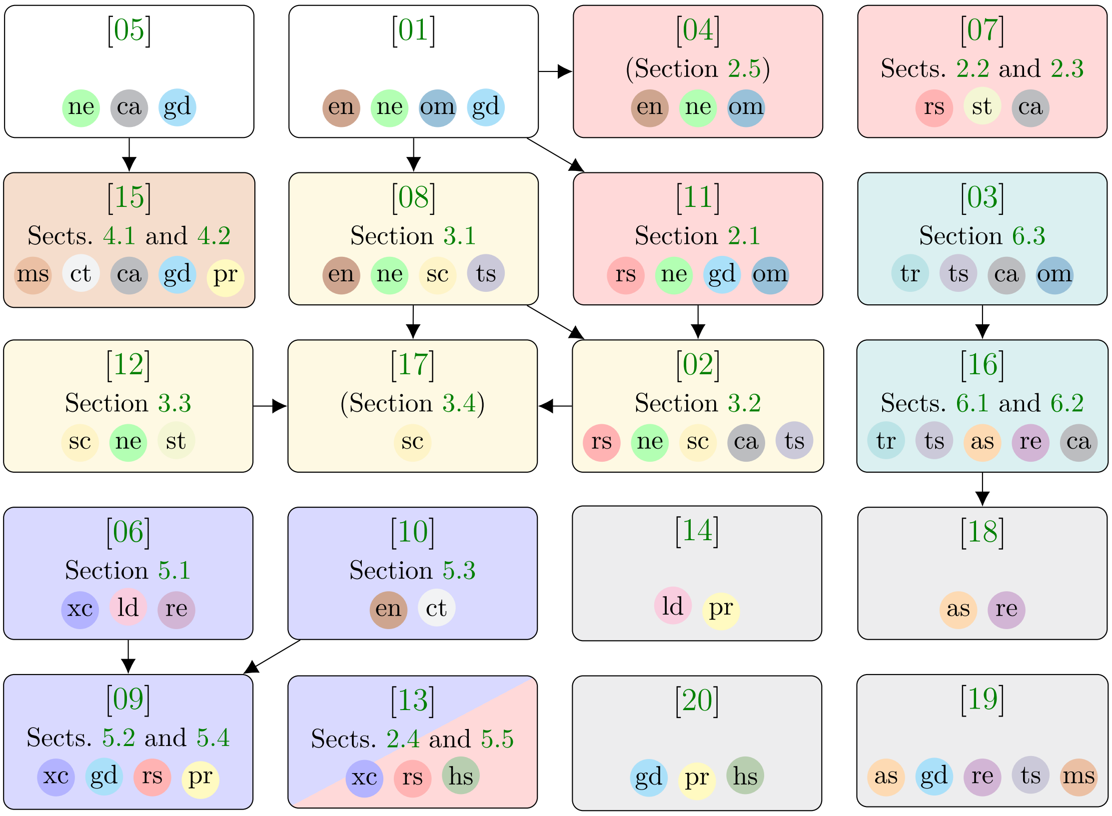
[18] Cesar Ceballos, Arnau Padrol, and Camilo Sarmiento. “The ν-Tamari lattice as the rotation lattice of ν-trees”. Preprint, 22 pp., arXiv: 1805.03566, submitted for publication. May 2018.

[19] Arnau Padrol, Yann Palu, Vincent Pilaud, and Pierre-Guy Plamondon. “Associahedra for finite type cluster algebras and minimal relations between g-vectors”. Preprint, 67 pp., arXiv: 1906.06861, submitted for publication. July 2019.

[20] Leonardo Martínez-Sandoval and Arnau Padrol. “The convex dimension of hypergraphs and the hypersimplicial Van Kampen-Flores Theorem”. Preprint, 18 pp., arXiv: 1909.01189, submitted for publication. Sept. 2019.

1.2. Publication scheme

The linear structure of this document makes it hard to display the logical dependencies within the sections, and the separation into the chapters is a poor representation of the distribution of topics. The following scheme attempts to present a better picture of the underlying connections in the publication list. Arrows represent logical dependence, understood in a loose sense: the explicit use or the generalization of a result, considering a question that naturally arises from the previous study, or using a concept introduced in the previous publication. There are also plenty of common themes treated in different subsets of publications. These key subjects are indicated by small tags.



Backgrounds:

PhD; Chapter 2; Chapter 3; Chapter 4; Chapter 5; Chapter 6; Preprint.

Tags:

rs: realization spaces; sc: (in)scribability; ms: Minkowski sums; xc: extension complexity; tr: tropical hyperplane arrangements; ts: triangulations and subdivisions; en: enumeration; om: oriented matroids; ct: combinatorial topology; gd: Gale duality; as: Tamari lattices and associahedra; re: Realizability; pr: Projections; ne: neighborly pol.; st: stacked pol.; ld: low-dimensional pol.; ca: Cayley and Lawrence pol.; hs: hypersimplices.

CHAPTER 2

Realization spaces and universality [07; 11; 13]

A *vector configuration* is a finite ordered collection $V = (v_1, \ldots, v_n)$ of vectors in $\mathbb{R}^d$, which we can identify with a matrix in $\mathbb{R}^{d\times n}$ that we will assume to be of full rank. Its *conical hull* is a *polyhedral cone* that is the image of the non-negative orthant $\mathbb{R}^n_{\geq 0}$ under the map $e_i \mapsto v_i$: $\operatorname{cone}(V) = \{\sum_{1\leq i\leq n} \lambda_i v_i \mid \lambda_i \geq 0\}$. The *face lattice* of a cone C, denoted by $\mathcal{F}(C)$, is the set of faces ordered by inclusion. Identifying each face with the indices of the vectors it contains, we interpret it as an induced poset of the Boolean lattice of subsets of $[n]$. This induces an equivalence relation which call the *face-lattice equivalence* of vector configurations. (Usually called just *labeled combinatorial equivalence* if we are at the level of polytopes/polyhedral cones.)

A finer equivalence relation on vector configurations is given by oriented matroids. The *oriented matroid* of V is the equivalence class induced by the *chirotope* map $\chi^V\colon \binom{[n]}{d} \to \{+,-,0\}$ that sends a d-subset $\{i_1 < i_2 < \cdots < i_d\}$ of $[n]$ to the sign of their determinant $\operatorname{sign}\det(v_{i_1}, \ldots, v_{i_d})$, up to a global sign change. More generally, any non-zero alternating map that fulfills the Grassmann-Plücker relations is called a *chirotope*, see [Bjö+99] for details on this and other cryptomorphic axiomatizations of oriented matroids. Here, we will only consider oriented matroids arising from real vector configurations, *i.e.*, *realizable over the reals*. Note that, since the chirotope determines the face lattice, oriented matroids induce a finer stratification.

Of course, analogous definitions hold for the affine counterparts: *point configurations*, *convex hulls*, *polytopes*, *affine oriented matroids*, etc. They are inherited from the linear setting through the *homogenization map* $\hom : \mathbb{R}^d \to \mathbb{R}^{d+1}$ given by $\hom(p) = (p, 1)$. Linear properties of $\hom(A)$ correspond to affine properties of A. It is usually more comfortable to work and do the proofs in the linear setup, but to draw (and think about!) the affine counterparts, which are in one dimension less. This is the approach we will follow here, where sometimes the affine configuration and its linearization are used interchangeably.

These two equivalence relations stratify the set of full-rank $d\times n$ matrices. Since both are invariant under linear transformations, we can consider the quotient modulo the action of $\mathrm{GL}(\mathbb{R}^d)$ to obtain a stratification of the Grassmannian $\mathrm{Gr}_d(n)$. The strata contain the set of all realizations of a given oriented matroid or polyhedral cone.

- The *realization space of an oriented matroid* M (of rank d with n elements), that we denote $\mathcal{R}_{om}(M) \subset \mathbb{R}^{d \times n}$, is the set of vector configurations that realize M:

$$\mathcal{R}_{\text{om}}(M) = \{V \in \mathbb{R}^{d\times n} \mid \chi^V = M\} \Big/ \text{GL}(\mathbb{R}^d).$$

- The *realization space of a polytope/polyhedral cone/face lattice* P, that we denote $\mathcal{R}_{pol}(P) \subset \mathbb{R}^{d \times n}$, is the set of vector configurations whose face lattice is combinatorially equivalent to P:

$$\mathcal{R}_{\text{pol}}(P) = \{V \in \mathbb{R}^{d\times n} \mid \mathcal{F}(V) \cong P\} \Big/ \text{GL}(\mathbb{R}^d).$$

For a vector configuration V, we abbreviate $\mathcal{R}_{om}(V) := \mathcal{R}_{\text{om}}(\chi^V)$ and $\mathcal{R}_{pol}(V) := \mathcal{R}_{\text{pol}}(\mathcal{F}(V))$; and for a polytope P with vertex set A, $\mathcal{R}_{pol}(P) := \mathcal{R}_{\text{pol}}(\text{hom}(A))$.

A general principle in the theory of realization spaces for (semi-)algebraically defined objects is succinctly put in [Vak06]: "Unless there is some a priori reason otherwise, the deformation space may be as bad as possible."

Underlying a large number of these kinds of phenomena is a paradigmatic result of Mnëv. The Universality Theorem for oriented matroids states that for every primary basic semi-algebraic set there is an oriented matroid of rank 3 and a polytope whose realization spaces are stably equivalent to it (see [RG99] for an accessible presentation of this and related results). When the semi-algebraic sets are open, one can furthermore require the oriented matroids to be uniform and the polytopes to be simplicial. Section 2.1 presents the final step for the proof of the Universality Theorem for simplicial polytopes, and shows that it even holds for neighborly polytopes.

Projectively unique polytopes are those that have the smallest possible realization space: they have a unique realization up to projective transformation. Projectively unique polytopes are very rare. Only 11 projectively unique polytopes are known in dimension 4, a list that has been conjectured to be complete [McM76]. The first infinite family of projectively uniques in fixed dimension was only recently found, in dimension $d = 69$, answering an old question of Perles and Shephard [AZ15]. Nevertheless, they have the following universality property: every polytope described by algebraic coordinates is the face of a projectively unique polytope. This result from [07] is described in Section 2.2. A closely related result of Below is used in Section 2.3 to construct a combinatorial type of 5-dimensional polytope that is not realizable as a subpolytope of any stacked polytope. This disproves a classical conjecture in polytope theory, first formulated by Shephard in the seventies [She74].

Realization spaces of hypersimplices are discussed in Section 2.4. They will be later used to study the extension complexity of combinatorial hypersimplices in Section 5.5.

Another result concerning universality of realization spaces, in this case of inscribed polytopes, will be presented in Section 3.2.

The final section states some open problems and provides ideas for future research.

2.1. The universality theorem for (neighborly) simplicial polytopes

This section reports joint work with Karim Adiprasito from "The universality theorem for neighborly polytopes" [11].

The Universality Theorem was a fundamental breakthrough in the theory of oriented matroids and convex polytopes. It states that the realization spaces of oriented matroids and polytopes, i.e. the spaces of point (vector) configurations with fixed oriented matroid/face lattice, can be arbitrarily complex. It comes in four flavours:

THEOREM 2.1 (Universality Theorem [Mnë88]). *Let V be a primary basic semi-algebraic set defined over $\mathbb{Z}$, then*

- *(i) there is an* ***oriented matroid*** *of rank 3 whose realization space is stably equivalent to V, and*
- *(ii) there is a* ***polytope*** *whose realization space is stably equivalent to V;*

if moreover V is open, then

- *(iii) there is a* ***uniform oriented matroid*** *of rank 3 whose realization space is stably equivalent to V, and*
- *(iv) there is a* ***simplicial polytope*** *whose realization space is stably equivalent to V.*

Here, a *basic semialgebraic set* in $\mathbb{R}^d$ is the set of solutions to a finite number of rational polynomial equalities and inequalities; it is called *primary* if all the inequalities in its definition are strict. Realization spaces are primary basic semi-algebraic sets. A basic semialgebraic set $S \subset \mathbb{R}^d$ is a *stable projection* of a basic semialgebraic set $T \subset \mathbb{R}^{d+d'}$ if, for the projection $\pi : \mathbb{R}^{d+d'} \to \mathbb{R}^d$, we have that $\pi(T) = S$ and that for every $\mathbf{x} \in S$, the fiber $\pi^{-1}(\mathbf{x})$ is the relative interior of a non-empty polyhedron defined by equalities and strict inequalities that depend polynomially on $\mathbf{x}$. Two basic semialgebraic sets S and T are *rationally equivalent* if there is a homeomorphism $f : S \to T$ such that f and f^{-1} are rational functions. Two basic semialgebraic sets S and T are *stably equivalent* if they belong to the same equivalence class generated by stable projections and rational equivalences. See [RG96; RG99] for more detailed definitions of these concepts.

Mnëv announced this theorem in 1985 [Mnë85] and published a sketch of the proof in 1988 [Mnë88]. A more detailed reasoning can be found in his thesis [Mnë86] (in Russian). Shor [Sho91] simplified a key step in Mnëv's line of reasoning for part (i). It is also used for (iii), which is proved using constructible oriented matroids and a substitution technique from [Jag+89; Mnë88]. Moreover, part (i) of Theorem 2.1 was later elaborated upon by Richter-Gebert [RG95] and

Günzel [Gün96], who proved the stronger *Universal Partition Theorem* for oriented matroids.

The Universality Theorem for oriented matroids in particular entails a negative answer to Ringel's 1956 *isotopy problem*, which asked whether, given two point configurations A_0 and A_1 with the same oriented matroid (and orientation), is it always possible to find a continuous path of point configurations $\{A_t\}_{0\le t\le 1}$ with the same oriented matroid? (This weaker result also follows via examples from [Jag+89; RG96; Suv88; Tsu13; Ver88; Whi89].) Actually, the Universality Theorem shows that there are oriented matroids that have realization spaces with arbitrarily many connected components.

Another straightforward consequence of the Universality Theorem is that determining realizability of oriented matroids is polynomially equivalent to the *existential theory of the reals*, and in particular NP-hard [Mnë88; Sho91].

Using Lawrence extensions to rigidify the face lattices, it is easy to prove part (ii) from part (i) [Mnë88; RG99] (see also Section 2.2.B). Here, a face lattice is called *OM-rigid* if it uniquely determines the oriented matroid defined by its vertices (see Section 2.1.D). In principle, this polytope might be of a very high dimension. However, Theorem 2.1(ii) was generalized greatly by Richter-Gebert, who proved that already 4-dimensional polytopes exhibit universality [RGZ95; RG96].

For a proof of part (iv), in contrast, only Mnëv's original papers were available, apart of some preliminary results of Sturmfels [Stu88b] and Bokowski–Guedes de Oliveira [BO90]. Moreover, Mnëv's elaborations for this case in [Mnë86; Mnë88] are specially concise and, in our opinion, incomplete. Hence, we think part (iv) of Theorem 2.1, although widely believed to be true, should be considered open until our paper [11]. It is important to stress that, despite the wrong common belief, Lawrence extensions *cannot* be used to deduce the universality theorem for simplicial polytopes. We use a different approach to rigidify matroids, namely, one based on neighborly polytopes.

A polytope is *k-neighborly* if every subset of vertices of size at most k is the set of vertices of one of its faces, and simply *neighborly* if it is $\left\lfloor \frac{d}{2} \right\rfloor$-neighborly. A paradigmatic example is the *cyclic polytope* $C_d(n)$, the convex hull of n points on the *moment curve* in $\mathbb{R}^d$, $\{(t, t^2, \ldots, t^d) : t \in \mathbb{R}\}$. McMullen's Upper Bound Theorem [McM70], a milestone of modern combinatorial geometry, states that the number of i-dimensional faces of a d-polytope P with n vertices is maximized by simplicial neighborly polytopes, for all i.

We show that (even-dimensional) neighborly polytopes are universal. Since all neighborly polytopes of even dimension are simplicial, this provides a proof of Theorem 2.1(iv).

Theorem 2.2 ([11]). *Every open primary basic semialgebraic set defined over $\mathbb{Z}$ is stably equivalent to the realization space of some neighborly $(2n-4)$-dimensional polytope on $2n$ vertices.*

Note the contrast to the case of cyclic polytopes, who have trivial realization spaces [BS86, Example 5.1]. This holds more generally for all *Gale sewn* polytopes [01; She82], a large family of $n^{\lfloor \frac{d}{2} \rfloor n(1-o(1))}$ many neighborly polytopes whose construction is discussed in Section 3.1. This is a consequence of the fact that the construction is based on performing a sequence of lexicographic extensions starting on a configuration with trivial realization space (cf. Section 2.1.A).

Corollary 2.3. *The realization space of any even-dimensional neighborly d-polytope on n vertices constucted with the Gale sewing construction is contractible, and in fact an open, piecewise smooth $(d+1)\cdot(n-d-1)$-ball.*

Appendix 2.1. Tools for proving Theorem 2.2

2.1.A. Lexicographic extensions. Lexicographic extensions (see [Bjö+99, Section 7.2]) are a way to extend vector configurations with strong combinatorial control on the oriented matroid of the output.

For an oriented matroid realized by a configuration V, a *lexicographic extension* is realized by $V \cup w$, where $w = \sigma_1 v_1 + \varepsilon\sigma_2 v_2 + \cdots + \varepsilon^{k-1}\sigma_k v_k$, for some $v_i \in V$, $\sigma_i = \pm 1$, and $\varepsilon > 0$ small enough. Note that the oriented matroid of $V \cup w$ only depends on the the order of the v_i's and the signs σ_i.

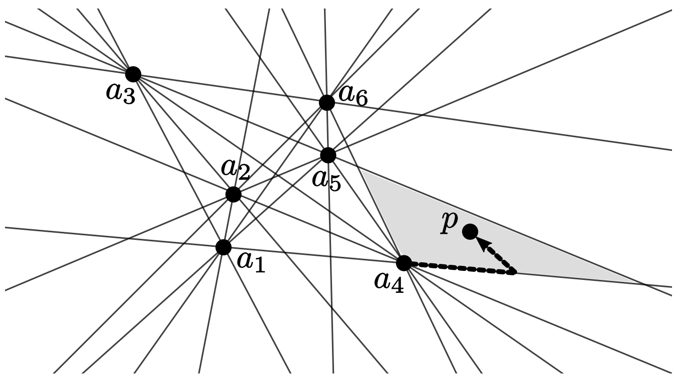


Figure 1. A lexicographic extension of a point configuration by $p = a_4 - \varepsilon a_1 + \varepsilon^2 a_6$.

Lexicographic extensions (see Section 2.1.A) are very useful for the study of realization spaces because they preserve them up to stable equivalence, see [Bjö+99, Lemma 8.2.1 and Proposition 8.2.2].

2.1.B. Gale (oriented matroid) duality. To each d-dimensional configuration of n vectors V one can associate a dual $(n-d)$-dimensional configuration V^*, called its *Gale dual* that realizes oriented matroid duality (circuits of V correspond to cocircuits of V^*, and vice versa). This is just an incarnation of the Grassmannian duality that sends a linear subspace to its orthogonal dual.

Our constructions work with the dual configurations. The important property for our purposes is that the realization space of a vector configuration and its Gale dual are stably equivalent [Bjö+99, Sec. 8.1].

2.1.C. Kortenkamp's construction. The next tool is a construction of Kortenkamp [Kor97, Thm. 1.2], who proved that every d-dimensional point configuration of at most $d+4$ points appears as a face figure of a neighborly polytope. This construction uses lexicographic extensions and works on the dual.

Using it, we can extend a rank 3 oriented matroid M whose realization space is stably equivalent to S from Theorem 2.1(iii) to the Gale dual of an even-dimensional neighborly oriented matroid by performing lexicographic extensions.

2.1.D. OM-rigidity. A polytope is called *OM-rigid* if its face lattice determines the oriented matroid spanned by its vertices (see [Zie95, Section 6.6]). Note that the realization space of an OM-rigid polytope coincides with the realization space of the oriented matroid of its vertices. The OM-rigidity of neighborly polytopes is the last ingredient for the proof of Theorem 2.2.

Lemma 2.4 ([She82, Thm. 2.10] and [Stu88a, Thm. 4.2])**.** *Every neighborly polytope of even dimension is OM-rigid.*

2.2. Faces of projectively unique polytopes

This section reports joint work with Karim Adiprasito from "A universality theorem for projectively unique polytopes and a conjecture of Shephard" [07].

A polytope $P \subset \mathbb{R}^d$ is *projectively unique* if any polytope $P' \subset \mathbb{R}^d$ combinatorially equivalent to P is *projectively equivalent* to P. In other words, P is projectively unique if for every polytope P' combinatorially equivalent to P, there exists a projective transformation of $\mathbb{R}^d$ that realizes the given combinatorial isomorphism from P to P'. These are the polytopes with the smallest possible realization space. Their rigid structure makes them a perfect tool to disprove statements of the kind "every polytope has a realization that...", as it suffices to find a projectively unique counterexample. For example, they were used by Perles to find the first polytope with no realization with rational coordinates [Grü03, Sec. 5.5, Thm. 4]. If the statement one wants to disprove concerns a property that is inherited by faces, then it suffices to find a projectively unique polytope that has the desired counterexample as a face.

By employing a technique developed by Adiprasito and Ziegler [AZ15], in [07] we prove the following (different kind of) universality theorem for projectively unique polytopes that characterizes its possible faces.

Theorem 2.5 ([07]). *For any algebraic polytope P, there exists a polytope $\widehat{P}$ that is projectively unique and that contains a face projectively equivalent to P.*

Here, a polytope is called *algebraic* if the coordinates of all of its vertices are real algebraic numbers. Theorem 2.5 would fail without the condition of P being algebraic: We cannot hope that every polytope is the face of a projectively unique polytope. Indeed, it is a consequence of the Tarski-Seidenberg Theorem [BM88; Lin71] that every combinatorial type of polytope has an algebraic realization. In particular, every projectively unique polytope, and every single one of its faces, must be projectively equivalent to an algebraic polytope. Hence, a d-dimensional polytope with $n \geq d+3$ vertices whose set of vertex coordinates consists of algebraically independent transcendental numbers is not a face of any projectively unique polytope.

A consequence of Theorem 2.5 is that for every finite field extension F over $\mathbb{Q}$, there exists a combinatorial type of polytope that is projectively unique, but not realizable in any vector space over F. This extends the famous result of Perles cited before.

Appendix 2.2. Tools for proving Theorem 2.5

2.2.A. Von Staudt constructions and functional arrangements. As a first step of our proof, we embed the algebraic vertices of our polytope inside a projectively unique point configuration (as an oriented matroid). This is done vertex by vertex.

Lemma 2.6. *Let ζ be any point in $\mathbb{R}^d_+$, $d \geq 3$, with algebraic coordinates. Then there is a projectively unique point configuration* $\mathrm{COOR}\,[\zeta]$ *containing ζ and* $\{0,1,2\}^d$.

The key idea to construct $\mathrm{COOR}\,[\zeta]$ is to realize the defining polynomials of the vertex coordinates in a *functional arrangement* (cf. [KM99, Def. 9.6]) that fixes them in any realization.

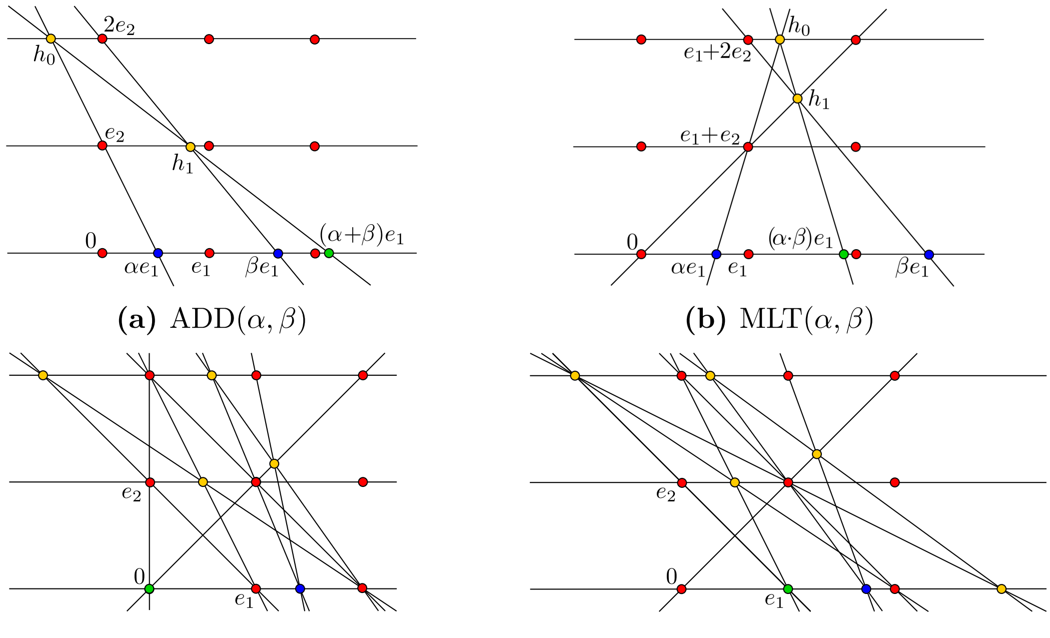
For a function $f : \mathbb{R}^k \mapsto \mathbb{R}$, a *functional arrangement* $\mathrm{FUNC}\,[f]\,(x)$ for f is a k-parameter family of point configurations in $\mathbb{R}^2$ that for all $x = (x_1, \dots, x_k) \in \mathbb{R}^k$, the functional arrangement $\mathrm{FUNC}\,[f]\,(x)$ contains the *output point* $f(x)e_1$ and the *input points* $x_i e_1$, for $1 \leq i \leq k$, and some fixed *frame points*; and that for any point configuration combinatorially equivalent to $\mathrm{FUNC}\,[f]\,(x)$ that fixes the frame and input points, the corresponding output point is fixed too. That is, a functional arrangement essentially computes a function by means of its point-line incidences alone.

The constuction of functional arrangements is based on the classical von Staudt constructions [Sta57] (compare also [RG11, Ch. 5], [RG96, Sec. 11.7] or [KM99, Sec. 5]); and composed of functional arrangements $\mathrm{ADD}(\alpha,\beta)$ and $\mathrm{MLT}(\alpha,\beta)$ for the functions $\mathrm{a}(\alpha,\beta)=\alpha+\beta$ and $\mathrm{m}(\alpha,\beta)=\alpha\cdot\beta$. Both functional configurations are shown below in Figure 2.

This construction is better aprehended with an example. Figure 2 shows a functional arrangement for $x\mapsto x^2-2=\mathrm{s}(\mathrm{m}(x,x),\mathrm{a}(1,1))$, evaluated at $\sqrt{2}$ and $\sqrt{3}$. Note that the functional arrangement can be written as combination of the functional arrangements for addition, subtraction and multiplication:

$$\begin{aligned}\mathrm{FUNC}\left[x^2-2\right](x) &= \mathrm{FUNC}\left[\mathrm{s}(\mathrm{m}(x_1,x_2),\mathrm{a}(x_3,x_4))\right](x,x,1,1)\\ &= \mathrm{SUB}(x^2,2)\cup\mathrm{MLT}(x,x)\cup\mathrm{ADD}(1,1),\end{aligned}$$

where $\mathrm{SUB}(\alpha,\beta)$ denotes the arrangement for $\mathrm{s}(\alpha,\beta)=\alpha-\beta$, constructed from ADD by switching input and output points.



(a) $\mathrm{ADD}(\alpha,\beta)$ **(b)** $\mathrm{MLT}(\alpha,\beta)$

(c) $\mathrm{FUNC}\left[x^2-2\right](\sqrt{2})$ **(d)** $\mathrm{FUNC}\left[x^2-2\right](\sqrt{3})$

Figure 2. Von Staudt constructions for addition, $\mathrm{ADD}(\alpha,\beta)$, and multiplication, $\mathrm{MLT}(\alpha,\beta)$. (The blue points of the configuration form the input, the red points show $\mathbb{Q}^2+\mathbf{1}$, the yellow points are auxiliary to the construction and the green points give the output.) And two evaluations of the functional arrangement $\mathrm{FUNC}\left[x^2-2\right]$.

2.2.B. Lawrence extensions. Lawrence extensions are a key ingredient for the proofs of the Universality Theorem for polytopes [Mnë88; RG96].

The *Lawrence extension* of a d-dimensional point configuration A on $a \in A$ is the $(d+1)$-dimensional point configuration

$$\Lambda(A, a) := (A \setminus a) \cup \overline{a} \cup \underline{a},$$

where A is embedded in the hyperplane $x_{d+1} = 0$ and the new points are $\underline{a} := (a, 1)$ and $\overline{a} := (a, 2)$. This operation does not change the realization space of the oriented matroid up to stable equivalence. Let $B \subseteq A$, then $\Lambda(A, B)$ is the point configuration obtained by Lawrence lifting the points of B one by one. The *Lawrence polytope* of a point configuration A is the polytope $\Lambda(A) = \mathrm{conv}(\Lambda(A, A))$.

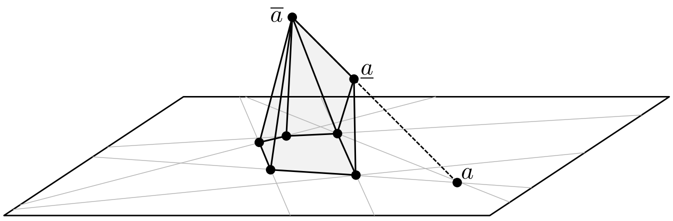


Figure 3. A Lawrence extension.

An key property of Lawrence polytopes is their OM-rigidity, which allows to translate between realization spaces of oriented matroids and polytopes.

Lemma 2.7 ([Zie95, Thms 6.26 and 6.27])**.** *For any point configuration A, $\Lambda(A, A)$ is the set of vertices of a OM-rigid polytope.*

2.2.C. Polytope–Point configurations. Performing a Lawrence extension on all the points of a projectively unique point configuration would yeld a projectively unique polytope, but it would not have the desired face.

The key is to work with the concept of polytope-point configurations (see also [AZ15, Sec. 5.1 & 5.2], [Grü03, Sec. 4.8 Ex. 30] or [RG96, Pt. I]), which is an equivalence class that interpolates between polytopes and oriented matroids.

A *polytope–point configuration* is a pair (P, R) where P is (the vertex set of) a polytope and R a point configuration such that $\mathrm{conv}(P) \cap R = \emptyset$. The combinatorial equivalence that one needs to consider is *Lawrence equivalence*, which is the coarsening of the oriented matroid data given by the covectors that are non-negative on P, *i.e.*, the hyperplanes that do not intersect the interior of $\mathrm{conv}(P)$. Observe that if $P = \emptyset$, then Lawrence equivalence coincides with oriented matroid equivalence on R; and if $R = \emptyset$, then we recover face lattice equivalence on P.

As it turns out, the face lattice of $\Lambda(P \cup R, R)$ encodes the Lawrence equivalence class of (P, R) (cf. [AZ15, Prp. 5.2], [RG96, Lem. 3.3.3 & 3.3.5]). Hence, if (P, R) is projectively unique (for Lawrence equivalence), then there is a projectively unique polytope that has P as a face. The missing piece for the proof of Theorem 2.5 is a construction by Adiprasito and Ziegler [AZ15, Lem. 5.8.] that (under certain conditions) lifts a convex subset of a projectively unique point configuration to a projectively unique polytope–point configuration that has this subset as a face.

2.3. A conjecture of Shephard

This section reports joint work with Karim Adiprasito from "A universality theorem for projectively unique polytopes and a conjecture of Shephard" [07].

Our original motivation to prove Theorem 2.5 was to disprove a conjecture of Shephard, who asked whether every polytope is a *subpolytope* of some stacked polytope. A *stacked polytope* is a connected sum of simplices (see Section 2.3.B). They are a very important family: By Barnette's Lower Bound Theorem [Bar71; Bar73], they have the minimum number of faces among all simplicial polytopes with the same number of vertices. The *subpolytopes* of a polytope P are the polytopes obtained as the convex hull of some subset of its vertices.

While Shephard disproved this statement in [She74], he conjectured it to be true in a combinatorial sense.

Conjecture 2.8 (Shephard [She74], Kalai [Kal04, p. 468], [Kal12]). *For every $d \geq 0$, every combinatorial type of d-dimensional polytope can be realized using subpolytopes of d-dimensional stacked polytopes.*

The conjecture was shown to be true for 3-dimensional polytopes by Kömhoff in 1980 [Köm80], but it remained open for dimensions $d > 3$. Despite the publicity by Kalai ([Kal04, p. 468], [Kal12]), no progress was done since.

Theorem 2.5 encourages to attempt a disproof of Shephard's conjecture by finding a projectively unique polytope that is not a subpolytope of any stacked polytope. Since any admissible projective transformation of a stacked polytope is a stacked polytope, no realization of the polytope provided this way is a subpolytope of any stacked polytope. This works. However, the method of Theorem 2.5 is highly ineffective: The counterexample to Shephard's conjecture it yields is of a very high dimension. In [07] we use a refined method, building on the same idea, to refute Shephard's conjecture.

Theorem 2.9 ([07]). *There exists a combinatorial type of 5-dimensional polytope that cannot be realized as a subpolytope of any stacked polytope.*

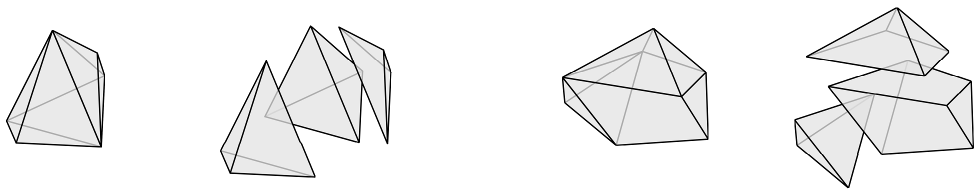
It remains open to decide whether every combinatorial type of 4-dimensional polytope can be realized as the subpolytope of some stacked polytope.

Appendix 2.3. Tools for proving Theorem 2.9

2.3.A. Below's stamp polytopes. For our counterexample, instead of Theorem 2.5, we use the following result of Below. Indeed, for projectivley fixing a face, we do not need the whole polytope to be projectively unique. This allows us to drastically reduce the dimension of the counterexamples.

Theorem 2.10 ([Bel02, Ch. 5], see also [Dob11, Thm. 4.1]). *Let P be an algebraic d-dimensional polytope. Then there is a polytope $\widehat{P}$ of dimension $d + 2$ that contains a face F that is projectively equivalent to P in every realization of $\widehat{P}$.*

2.3.B. Shephard's geometric counterexamples and [k-facet]-stacked polytopes. A polytope P is the *connected sum* of two polytopes Q and R if $P = Q \cup R$ and $F := Q \cap R$ is a common facet of Q and R whose boundary complex ∂F is still present in ∂P (cf. [RG96, Sec. 3.2]). A polytope is [*k-facet*]-*stacked* if it is the connected sum of polytopes with at most k facets each. With this, a [$(d+1)$-facet]-stacked d-dimensional polytope is simply a *stacked* polytope.



(a) [4-facet]-stacked **(b)** [5-facet]-stacked

Figure 4. A [4-facet]-stacked 3-polytope (i.e. a stacked 3-polytope) and a [5-facet]-stacked 3-polytope.

In [She74], Shephard proved that there are 3-dimensional polytopes that are not subpolytopes of stacked polytopes. His ideas can be used to prove a slightly stronger statement.

Lemma 2.11 ([07], based on [She74])**.** *For any subpolytope P of a [k-facet]-stacked d-dimensional polytope, $d \geq 3$, we have*

$$\mathrm{d_H}(P, \mathbb{B}_d) \geq 2^{-4k-10} \cdot 3^{-2},$$

where $\mathrm{d_H}(\cdot,\cdot)$ denotes the Hausdorff distance between compact convex subsets of $\mathbb{R}^d$, cf. [Sch93, Sec. 1.8], and $\mathbb{B}_d$ is the unit ball.

In particular, every d-polytope P that approximates $\mathbb{B}_d$ closely is not the subpolytope of any stacked polytope, and the same holds for any polytope projectively equivalent to P. We now only need to add a simple observation to Shephard's ideas to get all the pieces for the proof:

Lemma 2.12. *If P is a subpolytope of a [k-facet]-stacked polytope S, then any face σ of P is a subpolytope of a [k-facet]-stacked polytope as well.*

Now we can consider a 3-dimensional polytope P that is not a subpolytope of any [6-facet]-stacked polytope. Theorem 2.10 provides a polytope $\widehat{P}$ of dimension 5 that contains a face F that is projectively equivalent to P in every realization of $\widehat{P}$. Assume now that some polytope O combinatorially equivalent to $\widehat{P}$ is a subpolytope of some stacked polytope. The property of being a subpolytope of a [k-facet]-stacked polytope is inherited under taking faces, and hence any face of O is a subpolytope of some [6-facet]-stacked polytope. But the face of O corresponding to F is projectively equivalent to P, and hence not obtained by deleting vertices of a [6-facet]-stacked polytope. A contradiction.

2.4. Realization spaces of hypersimplices

This section reports joint work with Francesco Grande and Raman Sanyal from "Extension complexity and realization spaces of hypersimplices" [13].

As we have seen, realization spaces of polytopes are probably complicated objects, and computing them can be very hard. This is so even for the most elementary polytopes. In this section we illustrate this intrinsic difficulty with some results on realization spaces of hypersimplices that emerged in our study of the extension complexity of combinatorial hypersimplices in [13], which we relate in Section 5.5.

For $0 < k < n$, the (n, k)-*hypersimplex* $\Delta_{n,k}$ is the polytope whose vertices are the $\binom{n}{k}$ incidence vectors of k-subsets of $[n]$. Hypersimplices were first described (and named) in connection with moment polytopes of orbit closures in Grassmannians (see [Gel+87]) but, of course, are very natural polytopes that arise in very diverse contexts.

The *projective realization space* of combinatorial (n, k)-hypersimplices $\mathcal{R}_{n,k}$ is the quotient $\mathcal{R}_{\text{pol}}(\Delta_{n,k}) \big/ \mathbb{R}^{2n}_{>0}$ of the (conical) realization space modulo the action of scaling by positive scalars, and parametrizes the polytopes combinatorially isomorphic to $\Delta_{n,k}$ up to projective transformation. Despite the combinatorial simplicity of the definition of hypersimplices, their realization spaces can be surprisingly intricate.

For $k = 2, n - 2$, we are able to give a full description.

THEOREM 2.13 ([13]). *For $n \geq 4$, $\mathcal{R}_{n,2}$ is rationally equivalent to the interior of a $\binom{n-1}{2}$-dimensional cube. In particular, $\mathcal{R}_{n,2}$ is homeomorphic to an open ball and hence contractible.*

For $2 < k < n - 2$, the realization spaces are more involved and, in particular, deeply related to the algebraic variety of n-by-n matrices with vanishing principal k-minors that was studied by Wheeler [Whe15], and which is far from being completely understood. In [13] we give an upper bound of $\binom{n-1}{2}$ for the dimension of $\mathcal{R}_{n,k}$ when $2 \leq k \leq n - 2$. However, we can currently not exclude that $\mathcal{R}_{n,k}$ is disconnected and has components of different dimensions.

One of the first manifestations of the phenomenon of universality is that the realization of a facet of a high-dimensional polytope can not always be prescribed, a feature that we have seen and exploited in Sections 2.2 and 2.3. In contrast, the shape of any single facet of a 3-polytope can be prescribed [BG70]. In [13] we show that, despite their combinatorial simplicity, facets of hypersimplices cannot be prescribed in general. On the other hand, our description of $\mathcal{R}_{n,2}$ allows us to show that facets of $(n, 2)$-hypersimplices can be prescribed.

Theorem 2.14 ([13]). *Not every combinatorial (n, k)-hypersimplex is a facet of a combinatorial $(n+1, k+1)$-hypersimplex. In particular, there is a combinatorial $(6, 2)$-hypersimplex that is not a facet of any combinatorial $(7, 3)$-hypersimplex.*

In contrast, every combinatorial $(n, 2)$-hypersimplex is a facet of a combinatorial $(n + 1, 2)$-hypersimplex.

Appendix 2.4. Tools for proving Theorems 2.13 and 2.14

2.4.A. ***FG*-genericity.** Hypersimplices can be presented as the intersection of the 0/1-cube with the affine hyperplane $\sum x_i = k$. This presentation purports that

$$F_i := \Delta_{n,k} \cap \{x_i = 0\} \cong \Delta_{n-1,k}, \text{ and } G_i := \Delta_{n,k} \cap \{x_i = 1\} \cong \Delta_{n-1,k-1},$$

are disjoint facets for any $1 \leq i \leq n$. We call a combinatorial hypersimplex *F-generic* (resp. *G-generic*) if the hyperplanes supporting the F_i facets (resp. G_i facets) are not projectively concurrent, and we simply write *FG-generic* if it is both.

These notions turn out to play a crucial role. In particular, the following observation is instrumental in our study of $\mathcal{R}_{n,k}$, and also on our study of extension complexity in Section 5.5.

Lemma 2.15. *Every combinatorial (n, k)-hypersimplex is F-generic if $2k < n+2$, and G-generic if $2k > n-2$. In particular, every combinatorial (n, k)-hypersimplex is FG-generic for $n - 2 < 2k < n + 2$.*

2.5. Open problems and perspectives

The major open problem in this area is to prove the universality theorem for simplicial polytopes in fixed dimension. A universality conjecture for simplicial 4-polytopes is supported by the existence of simplicial 4-polytopes with disconnected realization space [BO90]. In view of Theorem 2.2, we can further conjecture that universality holds even when we restrict to neighborly polytopes.

Conjecture 2.16 ([11])**.** *Every open primary basic semialgebraic set defined over $\mathbb{Z}$ is stably equivalent to the realization space of some neighborly (and hence simplicial) 4-polytope.*

Note that in, particular, such a result would imply that it is ETR-hard to decide whether a given graph is the graph of a 4-polytope, as simple polytopes are determined by their graph [BML87], and realization spaces are preserved by polarity. This does not follow directly from Richter-Gebert's Universality Threorem for 4-polytopes, as in general polytopes are not determined by their graphs.

Mimicking the strategy to prove Theorem 2.2, one could try capture a uniform oriented matroid of rank 3 as the edge contraction of a neighborly 4 polytope. Vertex figures of neighborly 4-polytopes are always 3-dimensional stacked polytopes [AS73]. Despite the fact that realization spaces of stacked polytopes are trivial, their oriented matroids can be complicated: Notice that if M is any planar point configuration, then there exists a stacked 3-polytope P with a distinguished vertex v such that the contraction of v in P coincides with M.

This motivates the study of a question of Altshuler and Steinberg, who asked if every stacked 3-polytope is a vertex figure of a neighborly polytope [AS73, Problem 1]. The little computational evidence that is available supports this conjecture: Every stacked 3-polytope with at most 11 vertices is a vertex figure of a rank 5 neighborly oriented matroid. (This is included in [04], done in collaboration with Hiroyuki Miyata, where we enumerate new classes of neighborly polytopes and oriented matroids.) Bokowski and Shemer [BS87] generalized Altshuler and Steinberg's question to whether every $(m-1)$-stacked $(m-1)$-neighborly $(2m-1)$-polytope is a vertex figure of an m-neighborly $2m$-polytope. So far we know that every 2-stacked (2-)neighborly 5-polytope with up to 9 vertices is a vertex figure of a (3-)neighborly 6-polytope, by exhaustive search [04].

This is of course related to the conjecture, attributed to Perles and Sturmfels [Stu88b, Conj. 7.1], that every simplicial polytope appears as quotient (face contraction) of a neighborly polytope, and of which we used the resolution of the case of d-polytopes with up to $d+4$ vertices by Kortenkamp [Kor97]. Although this conjecture sounds similar to the conjecture of Shephard that we disproved in Section 2.3, they are of a very different nature.

Deciding whether the list of 11 projectively unique polytopes in dimension 4 is complete [McM76], or just constructing a low-dimensional infinite family, would be enormous progress.

One way to define realization spaces of oriented matroids and polytopes is to stratify the Grassmannian according to the combinatorial type of the orthogonal projection of the standard basis vectors onto ξ, for each $\xi \in \mathrm{Gr}_d(n)$. If instead of the whole Grassmannian one considers all d-subspaces containing a fixed $(d-1)$-subspace, then one can show that the polytope strata are actually contractible. This begs the question of understanding what's the first α such that the set of all k-subspaces containing a fixed $(k-\alpha)$-subspace presents universality. And in more generality to try to understand the polytope/oriented matroid strata in other Schubert varieties.

Actually, the structure they define is strongly related to the theory of *secondary fans and polytopes*, which encode the set of regular subdivisions of point and vector configurations [GKZ94] and can be constructed from the Gale transform [BFS90]. When one replaces the non-negative orthant by another cone, we recover the theory of generalized Gale transforms and *fiber fans and polytopes* that generalize secondary polytopes and fans [BS92]. See also Section 4.2.B.

The dual interpretation gives rise to McMullen's theory of *type cones* [McM73], which represent the set of polyhedra that have fixed facet normal vectors; that is, all polyhedra of the form $\{x \in \mathbb{R}^d \mid Ax \leq b\}$ for a fixed $m \times d$ matrix A and varying $b \in \mathbb{R}^m$, the strata being those b that give rise to the same combinatorial type.

Our recent preprint [19] concerns the study of type-cones of certain generalizations of the associahedron arising from representation theory. Our results extend a construction from [BM+18] motivated by realizations of the associahedron that arose from the mathematical physics community [AH+18]. Can this technique be applied to study other generalizations of the associahedron? In particular, to *quotientopes* [PS19], which are polytopes whose combinatorics encodes quotients of the weak order on the symmetric group?

CHAPTER 3

Inscribability and related notions [02; 08; 12]

The history of "*scribability*" problems goes back to at least 1832, when Steiner asked whether every 3-polytope is inscribable or circumscribable [Ste32]. A polytope is *inscribable* if it can be realized with all its vertices on a sphere, and *circumscribable* if it can be realized with all its facets tangent to a sphere. Steiner's problem remained open for nearly 100 years, until Steinitz showed that inscribability and circumscribability are dual through polarity, and presented a technique to construct infinitely many non-circumscribable 3-polytopes [Ste28]. A full characterization of inscribable 3-polytopes had to wait still more than 60 years, until Rivin gave one in terms of hyperbolic dihedral angles [Riv96] (see also [HRS92; Riv94; Riv96; Riv03]). Rivin's groundbreaking results allow to efficiently decide whether a 3-polytope is inscribable, whereas in higher dimensions the question of deciding inscribability is still wide open.

Inscribed polytopes have also been studied because, via stereographic projections, they are in correspondence with Delaunay subdivisions [Bro79]. These are central objects in computational geometry [Ede06]. Their applications include nearest-neighbors search, pattern matching, clustering and mesh generation.

The first topic we will consider is that of estimating the number of inscribable polytopes. Inscribability is a very restricting geometric constraint, and it is reasonable to expect that only very few combinatorial types are actually inscribable. For example, most 3-polytopes are known to be not inscribable [Smi91]. Surprisingly, the results in Section 3.1 go in the opposite direction. We present a construction for many combinatorially different inscribed d-polytopes that matches the current best lower bounds for the number of combinatorial types of d-polytopes (for $d \geq 4$). Even more, the constructed polytopes, which include cyclic polytopes, are inscribable in the boundary of any smooth strictly convex body, not only the sphere.

Once a polytope is known to be inscribable, it is natural to study its set of inscribed realizations. This is the subject of Section 3.2. It contains universality theorems for inscribed polytopes and Delaunay subdivisions the sets of inscribed realizations can be complicated in the sense of Mnëv. In particular, our results imply that the realizability problem for Delaunay triangulations is polynomially equivalent to the existential theory of the reals.

Other kinds of "scribability" problems are considered in Section 3.3. First, the classical k-scribability problem, studied by Steiner [Ste32] and Schulte [Sch87], that asks about the existence of d-polytopes that cannot be realized with all k-faces tangent to a sphere. Then a weaker version that only considers the affine hulls of k-faces. We finish the classification of the pairs d, k for which there is a d-polytope that is not weakly k-scribable. This problem was proposed by Grünbaum and Shephard [GS87], and almost solved by Schulte [Sch87]. Both versions generalize to (i, j)-scribability problems, which ask about the existence of d-polytopes that can not be realized with all their i-faces "avoiding" the sphere and all their j-faces "cutting" the sphere.

There are many challenging open questions in this area, some of which are related in Section 3.4 (see also [17]).

3.1. Many inscribable neighborly polytopes

This section reports joint work with Bernd Gonska from "Neighborly inscribed polytopes and Delaunay triangulations" [08].

Asking for the number of different combinatorial types of d-polytopes with n vertices or facets is a very natural question, so it comes as no surprise that this is a problem that has intrigued several generations of geometers (see the historical notes in [Grü03, Sec. 13.6]).

Steinitz's Theorem makes the enumeration of 3-polytopes a purely combinatorial problem. This allowed for enormous progress, and nowadays we have quite precise knowledge on their number and the distribution of many combinatorial parameters [RW82; BW88; BGR92].

Higher dimensional analogues of Steinitz's Theorem fail badly, as the Universality Theorem illustrates, and deciding realizability poses a major problem (see Section 2.1). This makes it very hard, if not infeasible, to obtain precise enumeration results for high dimensional polytopes. For high values of d the estimations for $\mathrm{p}_l(n, d)$, the number of combinatorial types of vertex-labeled d-polytopes with n vertices, are much weaker.

The first good lower bounds for $\mathrm{p}_l(n, d)$ arose from constructions of *neighborly* polytopes. In [She82], Shemer used a *sewing construction* to give a superexponential lower bound for the number of neighborly polytopes, of order $n^{c_d n(1+o(1))}$, where $c_d \to \frac{1}{2}$ when $d \to \infty$, improving a previous exponential lower bound by Barnette [Bar81]. (Here and below, the asymptotic notation $o(1)$ refers to fixed d and $n \to \infty$.)

A better lower bound for $\mathrm{p}_l(n, d)$ was found by Alon in 1986 [Alo86], who showed that $\mathrm{p}_l(n, d) \geq \left({(n-d)}/{d}\right)^{nd/4}$ for $n \geq 2d$. Nevertheless, the current best lower bound for the number of polytopes is actually also valid for the number of neighborly polytopes. This is one of the main results of my article [01], a publication containing some of the results of my PhD thesis.

Theorem 3.1 ([01]). *The number of combinatorial types of neighborly d-polytopes with n labeled vertices is at least $n^{\lfloor d/2 \rfloor n(1+o(1))}$ for fixed d and $n \to \infty$.*

The current best upper bounds for the number of vertex-labeled d-polytopes with n vertices, of order $({n}/{d})^{d^2 n(1+o(1))}$ when ${n}/{d} \to \infty$, are due to Alon [Alo86], who improved a similar bound for simplicial polytopes due to Goodman and Pollack [GP86]. Both results follow from bounds on the topological complexity of the real algebraic varieties defined by the chirotope map.

In this section we will study the enumeration problem for combinatorial types of polytopes verifying the geometric constraint of inscribability.

By McMullen's Upper Bound Theorem [McM70], the complexity of inscribed polytopes (and Delaunay triangulations) is bounded by that of neighborly polytopes. The existence of inscribed cyclic polytopes was already known to Carathéodory in 1911 [Car11]. While more inscribed realizations of the cyclic polytope have been found since [Gon13; GZ13; Sei87; Sei91], no other example of inscribable neighborly polytope was known. In [GZ13], Gonska and Ziegler stated:

> One can then proceed and try to characterize inscribability for some of these classes. This seems out of reach for neighborly polytopes, as according to Shemer [She82] there are huge numbers of combinatorial types, and no combinatorial classification in sight.

The main goal of this section is to present a result from [08] that shows that all *Gale sewn* polytopes (those constructed for the proof of Theorem 3.1, which strictly contain Shemer's *sewn polytopes* [She82]) are inscribable. Actually, we prove a much stronger result, and show that they are all K-inscribable for any smooth strictly convex body K (*i.e.*, that they admit a realization with all the vertices on ∂K).

THEOREM 3.2 ([08]). *For any smooth strictly convex body K, the number of combinatorial types of K-inscribable neighborly d-polytopes with n labeled vertices, and the number of $(d-1)$-dimensional K-Delaunay neighborly triangulations on $n-1$ labeled points, is at least $n^{\lfloor \frac{d}{2} \rfloor n(1+o(1))}$ for fixed d and $n \to \infty$.*

These are the first non-trivial lower bounds for the number of high-dimensional inscribable polytopes. Moreover, this is the first construction of arbitrarily large families of *universally inscribable* polytopes (*universal* referring to all smooth strictly convex bodies). In [08] we construct also a universally inscribable stacked polytope (see Section 2.3), which shows that the Lower Bound Theorem is also attained for simplicial polytopes in this family.

This concept is reminiscent of the renowned result of Schramm stating that every 3-polytope admits a realization with all the edges tangent to any smooth strictly convex body [Sch92]. In the converse direction, Ivanov proved that there exist *universally circumscribing* convex bodies $K \subset \mathbb{R}^d$ that fulfill that every d-polytope is K-inscribable [Iva12].

Appendix 3.1. Tools for proving Theorem 3.2

3.1.A. Many neighborly polytopes. The starting point is the construction from [01] that gives Theorem 3.1. It is an inductive construction that uses lexicographic extensions (see Section 2.1.A) in the dual space, producing Gale duals of neighborly polytopes instead (see Section 2.1.B).

The main insight from [01] is that every lexicographic extension can be followed by certain 'balancing' lexicographic extension in order to keep the property of being dual to neighborly. Note that every extension of the dual increases the dimension of the primal. Hence, starting with a trivial configuration, we do essentially dimension many extensions. Every two dimensions we have the choice of $v_1 \dots v_r$ (and of $\sigma_1, \dots, \sigma_r \in \{+, -\}$). Since $r = n - d$, and we are considering d fixed and $n \to \infty$, at each iteration we have roughly n^n choices, and this is repeated $\lfloor d/2 \rfloor$ times, which gives the $n^{\lfloor d/2 \rfloor n}$ from Theorem 3.1.

Of course, one still needs to make sure that the polytopes thus obtained are different. With mild restrictions on the choices at each step, one can be sure to obtain different oriented matroids. This is in principle not enough, as different oriented matroids could give rise to the same polytope. Fortunately, this is not the case for (even-dimensional) neighborly polytopes, which are OM-rigid (see Section 2.1.D).

3.1.B. K-Delaunay subdivisions. A *convex body* (a full-dimensional compact convex subset of $\mathbb{R}^d$) $K \subset \mathbb{R}^d$ is *strictly convex* at a boundary point $c \in \partial K$ if ∂K does not contain any segment through c, and it is *smooth* at c if c has a unique supporting hyperplane. K is called *strictly convex/smooth* if every point $c \in \partial K$ is strictly convex/smooth.

The stereographic projection s_c from a smooth strictly convex point $c \in \partial K$ maps $K \setminus c$ to $\mathbb{R}^{d-1}$, and defines a bijection between $\partial K \setminus c$ and $\mathbb{R}^{d-1}$. We define *K-spheres* and *K-balls* as the images $s_c(H \cap \partial K)$ and $s_c(H \cap K)$ for some hyperplane $H \subset \mathbb{R}^d \setminus c$. An example is sketched in Figure 1.

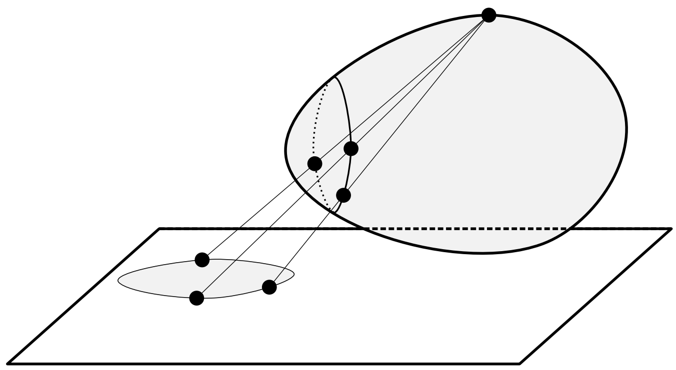


Figure 1. A stereographic projection of a smooth strictly convex body K, and a K-circumsphere.

The *K-Delaunay subdivision* $\mathcal{D}_K(A)$ of a point configuration $A \subset \mathbb{R}^{d-1}$ consists of all cells defined by the *empty K-sphere condition*: for $S \subseteq A$, $\operatorname{conv}(S)$ is a cell in $\mathcal{D}_K(A)$ if and only if there is a K-sphere that contains S and has all the remaining points of A outside the corresponding K-ball.

Note that the conditions defining the face structure of K-Delaunay triangulations are inherited from supporting hyperplanes of K-inscribed polytopes. From the combinatorial point of view, both concepts are almost interchangeable.

3.1.C. Lexicographic liftings. The construction from Section 3.1.A uses lexicographic extensions on the Gale dual. However, Gale duality is only defined up to linear transformation, and hence does not capture which polytopes are K-inscribed. We need an understanding of the effect of lexicographic extensions in the primal setting.

This is easier to visualize with affine point configurations. Performing a lexicographic lifting to the dual of $A = \{a_1, \dots, a_n\}$ can be interpreted as adding a new point a_0 at $(0, \dots, 0, +\infty)$ and then successively lifting the remaining points from a_n to a_1, in such a way that at each step we lift the point a_j arbitrarily high/low so that it is above/below all the hyperplanes spanned by the previous points. Such a lifting is called a *lexicographic lifting*.

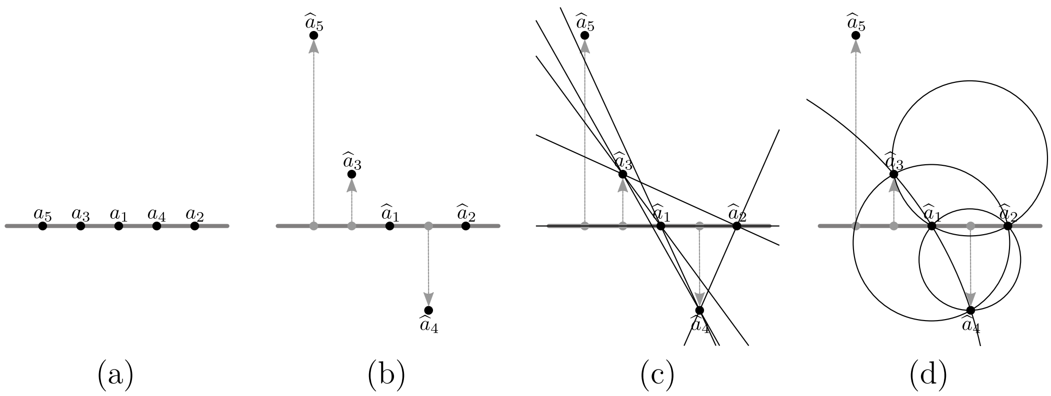


Figure 2. A point configuration $\{a_1, \dots, a_5\}$ and one of its K-Delaunay lexicographic liftings $\{\widehat{a}_0, \widehat{a}_1, \dots, \widehat{a}_5\}$ ($\widehat{a}_0$ is not visible since it is at $(0, +\infty)$), where K is the Euclidean ball.

If all the points of A are lifted up, then triangulation of A induced by the lower enveloppe of $\widehat{A}$ is the so-called *placing (or pushing) triangulation*. The key observation is that every lexicographic lifting can be performed in such a way that the placing triangulation is the K-Delaunay triangulation, by lifting the points outside the K-circumballs spanned by the previous points. This allows to control the combinatorial type of the inverse of the K-stereographic projection.

As it turns out, this is precisely the double lifting from Section 3.1.A, which sends k-neighborly configurations to $(k+1)$-neighborly configurations and constructs a family of polytopes with large combinatorial diversity.

3.2. Universality theorems for inscribed polytopes and Delaunay triangulations

This section reports joint work with Karim Adiprasito and Louis Theran from "Universality theorems for inscribed polytopes and Delaunay triangulations" [02].

The discovery of non-inscribable polytopes by Steinitz in 1928 [Ste28] naturally raised the question of how to decide whether a polytope is inscribable.

This is a question about realization spaces of Delaunay triangulations and inscribed polytopes, and deciding whether they are empty. The *realization space* of a Delaunay subdivision T, $\mathcal{R}_{del}(T)$, is a parametrization of the set of all configurations of n labeled points whose Delaunay subdivision has the combinatorial structure of T (as a labelled polytopal complex with vertex set $[n]$), modulo similarity. Analogously, $\mathcal{R}_{ins}(P)$, the realization space of an inscribed polytope P, is a parametrization of configurations of n points in the unit sphere whose convex hull has the same face lattice as P, up to Möbius transformation.

In dimension 3, this fundamental question was answered in the already mentionned series of breakthrough papers by Rivin [HRS92; Riv94; Riv96; Riv03] that connect 2-dimensional Delaunay subdivisions with metric properties of hyperbolic 3-dimensional polyhedra, and give a surprisingly sharp characterization of the inscribable types and their realization spaces.

Rivin's work in particular entailed that: (1) whether a (combinatorial) planar graph has a drawing as a Delaunay triangulation can be tested in polynomial time; (2) that the realization space of a planar Delaunay triangulation is homeomorphic to a polyhedron of so-called *angle structures*, and, in particular, connected.

In the language of polyhedra, (1) says that whether a graph is the 1-skeleton of an inscribable 3-polytope is efficiently checkable; and (2) says that the set of inscribed realizations is convex (and in particular contractible) in the parameterization by dihedral angles.

Our main results in [02] show that in arbitrarily high dimensions, there is again universality, and we cannot hope for Rivin-like characterizations. We have not found yet an appropriate notion of stable equivalence for this context. This forces us to separate the topological, algebraic and algorithmic statements:

Theorem 3.3 ([02]).

(i) For every primary basic semi-algebraic set there is an inscribed polytope (resp. a Delaunay subdivision) whose realization space is homotopy equivalent to it.

(ii) For every finite field extension $F/\mathbb{Q}$ of the rationals, there is an inscribed polytope (resp. a Delaunay subdivision) that cannot be realized with coordinates in F.

(iii) The problem of deciding if a poset is the face lattice of an inscribed simplicial polytope (resp. a Delaunay triangulation) is polynomially equivalent to the existential theory of the reals (ETR). *In particular, it is NP-hard.*

The polytopes used to prove items (i) and (ii) are based on Lawrence polytopes and far from being simplicial. Nevertheless, in the last point we can even ask the polytopes to be simplicial. This follows from a weak universality theorem for inscribed simplicial polytopes, in which we find polytopes whose realization space retracts onto the semi-algebraic set, instead of having homotopy equivalence as in the general case.

Appendix 3.2. Tools for proving Theorem 3.3

3.2.A. Inscribability of Lawrence polytopes. To go from realization spaces of oriented matroids to those of inscribed polytopes, we use Lawrence extensions (as the standard proof of the Universality Theorem for polytopes of unbounded dimension [Mnë88]), see Section 2.2.B. For our proof, we need to expand a proof of inscribability of Lawrence polytopes from [AZ15, Proposition 6.5.8] to make a statement about realization spaces. Namely, we have that for every planar point configuration A, $\mathcal{R}_{\text{om}}(A) \sim \mathcal{R}_{\text{ins}}(\Lambda(A))$, where $\sim$ denotes homotopy equivalence.

The proof is inductive, and considers realization spaces of point configurations in which a fixed subset lies on the boundary of the unit ball $\mathbb{B}_d$ and all the remaining points are outside. Figure 3 depicts the main idea for the inscribability of Lawrence extensions (and the fiber of the natural projection of realization spaces).

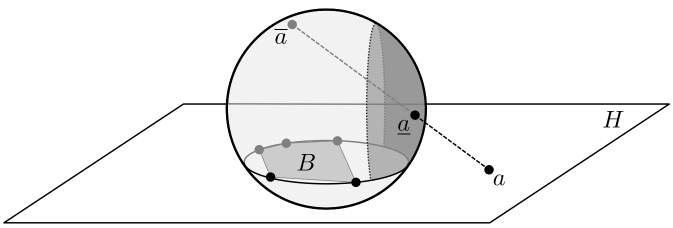


Figure 3. Inscribing a Lawrence extension.

Now Theorem 3.3(i) follows directly from the Universality Theorem 2.1. The algebraic universality (point (ii)) is deduced from the same construction.

3.2.B. The Universality Theorem for neighborly polytopes. To get universality for inscribed simplicial polytopes, we combine two tools that we have already presented. From any open primary basic semi-algebraic set S, we get from

Theorem 2.2 a neighborly polytope with a stably equivalent realization space. Performing the double lexicographic lifting from Section 3.1.C, we obtain an inscribed neighborly polytope whose realization space retracts to S.

3.3. Other scribability problems

This section reports joint work with Hao Chen from "Scribability problems for polytopes" [12].

A polytope is *k-scribable* if it has a realization with all its k-faces tangent to a sphere. This concept was studied by Schulte [Sch87], who constructed examples of d-polytopes that are not k-scribable for all the cases except for $k = 1$ and $d = 3$ and the trivial cases of $d \leq 2$. In fact, every 3-polytope has a realization with all its edges tangent to a sphere. This follows from Koebe–Andreev–Thurston's remarkable Circle Packing Theorem [Koe36; And71a; And71b; Thu79] (see [Che16; Zie07] for the relation of edge-scribed polytopes and circle packings). This was later greatly generalized by Schramm [Sch92], who showed that an edge-tangent realization exists even if the sphere is replaced by an arbitrary smooth strictly convex body.

Scribability problems expose the intricate interplay between combinatorial and geometric properties of convex polytopes and arise naturally from several seemingly unrelated contexts. However, our understanding on scribability problems is still quite limited. As Grünbaum and Shephard put it in 1987 [GS87]: "it is surprising that many simple and tangible questions concerning them remain unanswered." Very little progress has been made since then.

In [12] we study classical *k-scribability* for two important families of polytopes: *stacked polytopes* and *cyclic (and neighborly) polytopes*.

Stacked polytopes are the family that minimizes f-vectors among simplicial polytopes [Bar71; Bar73] (see Section 2.3). The *triakis tetrahedron* is a stacked polytope among the first and smallest examples of non-inscribable polytopes found by Steinitz [Ste28]. Recently, Gonska and Ziegler [GZ13] characterized inscribable stacked polytopes. On the other hand, Eppstein, Kuperberg and Ziegler [EKZ03] showed that stacked 4-polytopes are essentially not edge-scribable. In [12] we completely answer the k-scribability problem for stacked polytopes.

THEOREM 3.4 ([12]). *For any $d \geq 3$ and $0 \leq k \leq d - 3$, there are stacked d-polytopes that are not k-scribable. However, every stacked d-polytope is $(d-1)$-scribable (i.e. circumscribable) and $(d-2)$-scribable (i.e. ridge-scribable).*

On the other end of the face-numbers spectrum, we solve the k-scribability problem for cyclic polytopes.

THEOREM 3.5 ([12]). *For any $d > 3$ and $1 \leq k \leq d - 1$, a cyclic d-polytope with sufficiently many vertices is not k-scribable.*

Our results on k-scribability of general neighborly polytopes from [12] imply:

THEOREM 3.6 ([12]). *For any $d > 3$ and $1 \leq k \leq d-2$, there are f-vectors such that no d-polytope with those f-vectors are k-scribable.*

Note that Theorem 3.5 implies that, for $d \geq 4$, cyclic d-polytopes with sufficiently many vertices are not circumscribable. Our bound for the number of vertices that guarantee non-circumscribability has been greatly improved very recently. A team formed by Doolittle, Labbé, Lange, Sinn, Spreer, and Ziegler has shown that, for $d \geq 4$, cyclic d-polytopes with at least $d+4$ vertices are not circumscribable [Doo+19]. Their technique provides the first f-vector that is not the f-vector of an inscribable polytope.

Schulte [Sch87] also proposed a weak version of k-scribability, following an idea of Grünbaum and Shephard [GS87]. A d-polytope is *weakly k-scribable* if it can be realized with the affine hulls of all its k-faces tangent to a sphere. Schulte was able to construct examples of d-polytopes that are not weakly k-scribable for all $k < d-2$, and left open the cases $k = d-2$ and $k = d-1$. The main difficulty that prevented Schulte from settling these cases was that weak scribability does not behave well under polarity. For scribability problems, it is more natural to consider polytopes as pointed polyhedral cones in Lorentzian space. In this set-up, the definition of weak scribability is slightly weaker than in Euclidean space, but behaves well under polarity. This allows us to settle the cases left open by Schulte.

THEOREM 3.7 ([12]). *For any $d \geq 3$ and $0 \leq k \leq d-1$ with the exception of $(d,k) = (3,1)$, there are d-polytopes that are not weakly k-scribable (both in Euclidean and spherical spaces).*

We also propose the new concept of *(i,j)-scribability*. A polytope is *(i,j)-scribable* if it can be realized with all its i-faces "avoiding" the sphere and all its j-faces "cutting" the sphere. The definitions are designed to behave well under polarity, and to reduce to classical k-scribability when $i = j = k$. This makes (i,j)-scribability a very useful tool for studying classical k-scribability problems. They are also an interesting topic in their own right. Notably, $(0,1)$-scribed 3-polytopes have been studied as *hyperideal polyhedra* in hyperbolic space [BB02; Sch05].

One of our main results is the following theorem, which constructs examples of polytopes that are not (i,j)-scribable.

THEOREM 3.8 ([12]). *For $d > 3$ and $0 \leq i \leq j \leq d-1$, there are d-polytopes that are not strongly (i,j)-scribable for $j-i \leq d-2$ when d is even, or $j-i \leq d-3$ when d is odd.*

Appendix 3.3. Tools for proving Theorems 3.5 and 3.8

3.3.A. Spherical polytopes and Lorentzian space. The most natural and convenient framework for defining (i,j)-scribability is to consider spherical polytopes, which arise from pointed polyhedral cones in Lorentzian space. The main advantage is that, for spherical polytopes, polarity is always well-defined and well-behaved.

We work with the polarity induced by the Lorentzian scalar product. The *Lorentzian space* $\mathbb{L}^{1,d}$ is $\mathbb{R}^{d+1}$ endowed with the Lorentzian scalar product:

$$(x, y) := -x_0y_0 + x_1y_1 + \cdots + x_dy_d, \quad x, y \in \mathbb{R}^{d+1}.$$

The role of the unit sphere for the scribability problems is played by the *light cone*:

$$\mathcal{L} := \{x \in \mathbb{L}^{1,d} \mid (x, x) \leq 0,\ x_0 \geq 0\}.$$

The passage to d-dimensional spherical space $\mathbb{S}^d$ is done by intersecting with the unit d-sphere.

The use of spherical polytopes instead of Euclidean polytopes is specially relevant in the context of weak scribability. The definition is weakened, but acquires the desired properties with respect to polarity

3.3.B. k-ply systems and k-sets. Any point $x \notin \mathbb{B}_d$ can be associated with a closed spherical cap on $\mathbb{S}_{d-1}$, namely the set of points that are visible from x. A set of spherical caps is said to be a *k-ply system* if no point belongs to the interior of k caps. These systems were studied by Miller et al. [Mil+97], who proved the following Separation Theorem. Here, the *intersection graph* is the graph where every vertex represents a cap, and two caps form an edge if they intersect.

Proposition 3.9 (Sphere Separator Theorem [Mil+97])**.** *The intersection graph of a k-ply system consisting of n caps on a d-dimensional sphere can be separated into two disjoint parts, each of size at most $\frac{d+1}{d+2}n$, by removing $O(k^{1/d}n^{1-1/d})$ vertices.*

A subset S of cardinality k of a point configuration A is said to be a *k-set* if there is a hyperplane strictly separating S and $A \setminus S$. Under the above correspondence, $A \subset \mathbb{R}^d \setminus \mathbb{B}_d$ corresponds to a k-ply system on $\mathbb{S}_{d-1}$ if and only if the convex hull of every k-set intersects the sphere. In particular, the caps corresponding to x and y have disjoint interiors if and only if the segment $[x, y]$ intersects the ball.

The proof of Theorem 3.8 uses the fact that, for even $d \geq 4$ and when $k = k(d)$ is large enough, every k-set of a cyclic d polytope contains a facet. This implies that, for even $d \geq 4$, any cyclic polytope with enough vertices is not $(1, d-1)$-scribable. Indeed, in a $(1, d-1)$-scribed realization, every k-set intersects the sphere, and hence the spherical caps corresponding to the vertices form a k-ply system, and hence their intersection graph admits a small separator. But if the edges avoid the sphere, the graph is a complete graph, a contradiction.

In particular, for even $d \geq 4$, $C_d(n)$ is not (i, j)-scribable for $1 \leq i \leq j \leq d-1$ if n is large enough. Theorem 3.5 follows.

3.4. Open problems and perspectives

The combinatorial richness of the family of neighborly polytopes, as well as the results in Section 5.3 that show that there are few polytopes with few vertices and few facets, support the hypothesis that typical polytopes (drawn at random among all combinatorial types) have a large number of facets (which happens with some geometric models, e.g. [DT05]). It is even conceivable that the average number of facets has the same order of magnitude as the upper bound. One way to derive this kind of results would be to consider the enumeration of polytopes with respect to their number of vertices and facets. Good upper bounds for these could be combined with the existing lower bounds to show that most polytopes have many facets. Together with Eran Nevo, we even made the following daring conjecture (unpublished):

Conjecture 3.10. *The number of combinatorial types of (simplicial) d-polytopes with n vertices and m facets is $m^{n(1+o(1))}$.*

This would imply that the bounds from Theorem 3.1 are tight. The few cases that have been completely enumerated show that the polytopes that give the lower bound are only a small fraction from all neighborly polytopes [04], but it is hard to extrapolate whether this difference is asymptotically meaningful.

In [17], we present many open problems on inscribed polytopes and related topics, none of which has been solved yet to the best of my knowledge. Some of them are inspired by the results presented here. In particular, the results from Section 3.1 beg the question, first asked in [08], of whether every neighborly polytope is inscribable. So far we do not know any counterexample. Moritz Firsching [Fir17] found inscribed realizations for: every neighborly 4-polytope with $n \leq 11$ vertices, every simplicial neighborly 5-polytope with $n \leq 10$ vertices, every neighborly 6-polytope with $n \leq 11$ vertices, and every simplicial neighborly 7-polytope with $n \leq 11$ vertices. These collections, enumerated in [04], include many neighborly polytopes not constructible with our methods. Even more, every simplicial 2-neighborly 6-polytope with $n \leq 10$ vertices is also inscribable. This lead Firsching to ask whether the even stronger statement that all 2-neighborly polytopes are inscribable might be true [Fir17]. Dillencourt and Smith proved that any 3 polytope with a "sufficiently rich" collection of Hamiltonian graphs is inscribable [DS96]. It would be interesting to know if similar conditions also hold in higher dimensions. Recall that the graph of a 2-neighborly polytope is complete, and hence has the richest possible structure of Hamiltonian subgraphs.

Another natural question is to find other universally inscribable polytopes. One observation of Karim Adiprasito shows that being inscribable on the sphere is not sufficient for being universally inscribable, see [08].

Concerning Section 3.2, the inscribed analogue to Richter-Gebert's result for 4-polytopes is still missing. Is there universality for inscribed polytopes in bounded dimension, say for inscribed 4-polytopes?

The proof of Theorem 3.3 strongly relies on the results of Mnëv. The strategy is to start with certain polytopes with intricate realization spaces, and then to show that their inscribed realization spaces are equally involved. In particular, it does not prove that it is hard to decide inscribability once we already know that the face lattice corresponds to a polytope. However, inscribability is itself a complex condition, and hence one can expect that it increases the complexity of the corresponding realization spaces. This could lead to a proof of universality that is intrinsic to inscribed polytopes, and hopefuly to advances in the previous question. A first step in this direction could be to find a polytope P such that $\mathcal{R}_{\text{ins}}(P)$ is disconnected while $\mathcal{R}_{\text{pol}}(P)$ is not.

The results from [12] leave open the question of whether every polytope has a realization where every vertex avoids the ball and every facet cuts the ball. For $d = 3$, the edge-scribed realization is also $(0, d-1)$-scribed. Every inscribable polytope has directly a $(0, d-1)$-scribed realization too. So, in particular, cyclic polytopes and many (all?) neighborly polytopes have such realizations. And since the property is self-polar, circumscribable polytopes also have such a realization. This includes stacked polytopes, which are always circumscribable. We suspect nevertheless that there exist polytopes that are not $(0, d-1)$-scribable.

CHAPTER 4

From colorful configurations to Minkowski sums [15]

This chapter presents results from [15] on two different topics concerning colorful point configurations and their relation to Minkowski sums.

The first topic is the colorful simplicial depth of colorful configurations. The Colorful Carathéodory Theorem was proved by Bárány in 1982 [Bár82], and it became an instant classic in discrete geometry. Inspired by it, Deza, Huang, Stephen, and Terlaky introduced in [Dez+06] a colorful generalization of Liu's simplicial depth [Liu90]: The *colorful simplicial depth* of a collection of $d+1$ finite sets of points in Euclidean d-space is the number of choices of a point from each set such that the origin is contained in their convex hull. Deza et al. conjectured upper and lower bounds for the special case where all the sets have size $d+1$. While the lower bound was later settled by Sarrabezolles [Sar15], the upper bound remained open.

Section 4.1 presents a tight upper bound on the colorful simplicial depth, proved in [15] using methods from combinatorial topology. It implies the upper bound conjectured in [Dez+06]. Our result yields also a tight upper bound on the number of totally mixed facets of certain Minkowski sums of simplices. The case of triangles resolves in the positive a completely independent conjecture of Burton in the theory of normal surfaces [Bur03].

These seemingly disparate conjectures (now turned into theorems) are connected by Colorful Gale transforms and Minkowski transforms, which are the subject of Section 4.2. The Gale transform is an incarnation of oriented matroid duality that has been extensively used in polytope theory, in particular to study polytopes with few vertices with respect to their dimension, and that we have already referred to in Section 2.1.B. One way to present it is in terms of projections, using the fact that every polytope is naturally associated to a projection of a simplex. This point of view can be generalized to arbitrary polytope projections, giving rise to *generalized Gale transforms*, first described by McMullen [McM79]. Colorful Gale transforms and Minkowski transforms are two alternative ways to apply (generalized) Gale transforms to study the combinatorics of Minkowski sums in terms of colorful point configurations.

4.1. An upper bound for Colorful simplicial depth

This section reports joint work with Karim Adiprasito, Philip Brinkmann, Pavel Paták, Zuzana Patáková, and Raman Sanyal from "Colorful simplicial depth, Minkowski sums, and generalized Gale transforms" [15].

Bárány's celebrated colorful generalization of Carathéodory's Theorem [Bár82] can be stated by saying that every centered colorful configuration contains a hitting colorful simplex. Here, a *colorful configuration* $C = (C_0, \ldots, C_d)$ is a collection of $d+1$ point configurations in $\mathbb{R}^d$; a subset $S \subseteq \bigcup_i C_i$ is a *colorful simplex* if $|S \cap C_i| \leq 1$ for all i; C is called *centered* if $0 \in \operatorname{relint}(\operatorname{conv}(C_i))$ for all $0 \leq i \leq d$; and S is *hitting* if $\dim S = |S| - 1 = d$ and $0 \in \operatorname{conv}(S)$.

Generalizing the simplicial depth introduced in [Liu90], Deza, Huang, Stephen and Terlaky [Dez+06] introduced the *colorful simplicial depth*, cs-depth(C), of a colorful configuration C as the number of hitting colorful simplices of C. They conjectured a lower bound of

$$1 + d^2 \;\leq\; \text{cs-depth}(C)$$

when $|C_i| = d+1$ and $\bigcap_i \operatorname{conv}(C_i)$ is of full-dimension d and contains the origin in its interior. The Colorful Carathéodory Theorem says that cs-depth$(C) \geq 1$. This initial lower bound was improved in a series of papers [BM07; Dez+06; DMS14; DSX11; ST08] culminating in the resolution of the conjectured (tight) lower bound by Sarrabezolles [Sar15].

In [Dez+06] also a conjectured upper bound was proposed.

Conjecture 4.1 ([Dez+06, Conj. 4.4.]). *Let $C = \{C_0, \ldots, C_d\}$ be a centered colorful configuration in $\mathbb{R}^d$ with $|C_i| = d+1$ for all $0 \leq i \leq d$ and $0 \in \operatorname{int} \bigcap_i \operatorname{conv}(C_i)$. Then*

$$\text{cs-depth}(C) \;\leq\; 1 + d^{d+1}.$$

In [15], we give a topological proof of this conjecture in the following stronger form.

Theorem 4.2 ([15]). *Let $C = \{C_0, \ldots, C_d\}$ be a centered colorful configuration in relative general position in $\mathbb{R}^d$ with $|C_i| \geq 2$ for all $0 \leq i \leq d$. Then*

$$\text{cs-depth}(C) \;\leq\; 1 + \prod_{i=0}^{d} (|C_i| - 1).$$

And this bound is tight.

Here, a colorful configuration C in $\mathbb{R}^d$ is called in *relative general position* if no colorful simplex S of C of dimension $d-1$ contains the origin in its convex hull. This is a natural assumption to avoid trivial pathologic examples.

The tightness of the bound is easily seen with this example.

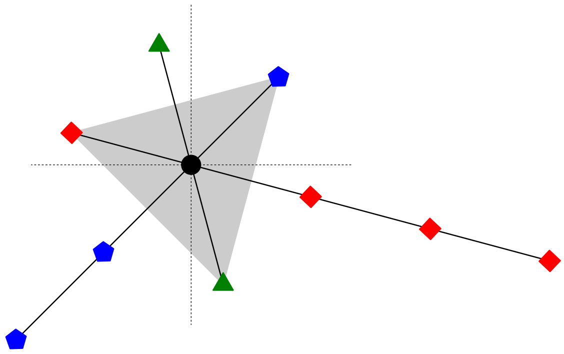


Figure 1. Configuration from Example 4.3 for $(n_0, n_1, n_2) = (2, 3, 4)$.

Example 4.3. Let $v_0, \ldots, v_d \in \mathbb{R}^d$ be the vertices of a simplex containing the origin in its interior and for $n_0, \ldots, n_d \geq 2$ define

$$C_i \; := \; \{v_i, -v_i, -2v_i, \ldots, -(n_i - 1)v_i\}$$

for $0 \leq i \leq d$. Then $C = \{C_0, \ldots, C_d\}$ is a centered colorful configuration with exactly $1 + (n_0 - 1) \cdots (n_d - 1)$ hitting simplices.

Recall that the *Minkowski sum* of $P_0, \ldots, P_s \subset \mathbb{R}^d$ is $P_0 + \cdots + P_s \; = \; \{p_0 + \cdots + p_s : p_i \in P_i\}$. This operation is key to many deep results in many areas, notably convex geometry [Sch93] and computational commutative algebra (eg. [GS93]). The combinatorial complexity of Minkowski sums of polytopes has been subject of several studies [AS16; MPP11; RS12]. Using Gale transforms and Cayley embeddings, we introduce *colorful Gale transforms* associated to a collection $P_0, \ldots, P_s$ that, similar to ordinary Gale transforms, capture the facial structure of Minkowski sums in the combinatorics of colorful configurations (see Section 4.2). Under this correspondence, Theorem 4.2 implies:

Corollary 4.4. *For $d_0, \ldots, d_s \geq 1$ and $D = d_0 + \cdots + d_s - s$, let $P_i \subset \mathbb{R}^D$ be d_i-dimensional simplices whose Minkowski sum is of full dimension D, for $0 \leq i \leq s$. Then the number of totally mixed facets of $P_0 + \cdots + P_s$ is at most*

$$1 + d_0 d_1 \cdots d_s.$$

Here, a face $F = F_0 + \cdots + F_s$ of the Minkowski sum $P = P_0 + \cdots + P_s$ is called *totally mixed* if each $F_i \subset P_i$ is a facet of P_i.

As it turns out, this also solves an at first sight unrelated conjecture of Benjamin Burton [Bur03, Conj. 5.5.14] about the complexity of projective edge weight solution spaces in *normal surface* theory. Edge weights are coordinates used to represent normal surfaces in a one-vertex triangulation with n tetrahedra as vectors in $\mathbb{R}^{n+1}$ instead of the standard triangle and quadrilateral coordinates in $\mathbb{R}^{7n}$. Burton's conjecture has a formulation in terms of certain *balanced fans* that can be interpreted as normal fans of triangles embedded in $\mathbb{R}^d$, and asks for the maximal number of totally mixed faces of a Minkowski sum of $d - 1$ triangles embedded in $\mathbb{R}^d$. His conjectured bound of $1 + 2^{d-1}$ is exactly the outcome of our Corollary 4.4 with these parameters. See [Bur03, Ch. 5] for the original formulation, the topological connections, and implications.

Appendix 4.1. Tools for proving Theorem 4.2

4.1.A. Betti numbers of avoiding complexes. In [15] we introduce the notion of the *avoiding complex* associated to a colorful point configuration $C = (C_0, \dots, C_d)$. It is the simplicial complex $\mathcal{A}(C)$ on the vertex set $C_0 \cup \dots \cup C_d$ consisting of the colorful simplices that do not contain the origin in their convex hull. That is, $S \in \mathcal{A}(C)$ if and only if $|S \cap C_i| \leq 1$ for all $0 \leq i \leq d$ and $0 \notin \operatorname{conv}(S)$.

Using the Euler-Poincaré formula and computing the reduced Euler characteristic $\tilde{\chi}(\mathcal{A})$ in two different ways, one can see that:

Lemma 4.5. *Let $C = (C_0, \dots, C_d)$ be a centered colorful configuration in relative general position in $\mathbb{R}^d$ with $n_i = |C_i| \geq 2$ for $0 \leq i \leq d$. Then*

$$\text{cs-depth}(C) \;\leq\; \prod_{i=0}^{d}(n_i - 1) + \tilde{\beta}_{d-1}(\mathcal{A}),$$

where $\tilde{\beta}_k(\mathcal{S}) = \dim_{\mathbb{Z}_2} \tilde{\mathrm{H}}_k(\mathcal{S})$ denotes the k-th reduced Betti number *of $\mathcal{S}$.*

Hence, to prove Theorem 4.2 it suffices to show that the $(d-1)$-Betti number of the avoiding complex is always constant equal to one independently of the configuration. To prove this, we introduce the notion of *flips* between colorful configurations, and we prove our result by 'flipping' any configuration to the configuration of Example 4.3 whose avoiding complex is homotopy equivalent to a $(d-1)$-sphere.

4.2. Colorful Gale transforms and Minkowski transforms

This section reports joint work with Karim Adiprasito, Philip Brinkmann, Pavel Paták, Zuzana Patáková, and Raman Sanyal from "Colorful simplicial depth, Minkowski sums, and generalized Gale transforms" [15].

In this section, we present colorful Gale transforms and Minkowski transforms, two techniques that allow to translate between results on colorful configurations and results on Minkowski sums. In particular, they allow to derive Corollary 4.4 from Theorem 4.2, and hence to solve Burton's conjecture. Even if both techniques coincide for the case of simplices, they can be of independent interest in future applications beyond this scope, as they present different features.

The usual *Gale transform* assigns a configuration $G = (g_1, \dots, g_n)$ of n vectors in $\mathbb{R}^{n-d-1}$ to every (full-dimensional) configuration $A = (a_1, \dots, a_n)$ of n points in $\mathbb{R}^d$. The configuration G encodes the (dual) oriented matroid of A (see Section 2.1.B), and in particular its face lattice, but can be of a much lower dimension.

Because of this, Gale duality has been a very powerful tool in the study of polytopes with few vertices (see [Mat02, Section 5.6] for a very accessible treatment in terms of matrices and to [Zie95, Lecture 5] for its relation to oriented matroids).

We may adapt the notion of Gale transforms to Minkowski sums by way of Cayley embeddings (see Section 4.2.A), and what we recover is a colorful configuration that encodes the combinatorics of the Minkowski sum and all its subsums:

THEOREM 4.6 ([15]). *Let $A = (A_0, \dots, A_s)$ be a colorful point configuration in $\mathbb{R}^d$ with full-dimensional Minkowski sum. Then there is a centered colorful point configuration $G = (G_0, \dots, G_s)$ in $\mathbb{R}^{n-d-s-1}$, where $n = \sum_{0 \le i \le s} n_i$ and $n_i = |A_i| = |G_i|$, called the* colorful Gale transform *of A, with the following property:*

For $\emptyset \neq J_i \subset [n_i]$, let $B_i = \{a_j \in A_i \mid j \in J_i\} \subset A_i$ and $H_i = \{g_j \in G_i \mid j \notin J_i\} \subset G_i$ for $0 \le i \le s$. Then $\sum_{0 \le i \le s} B_i$ is a face of $\sum_{0 \le i \le s} A_i$ if and only if

$$0 \in \operatorname{relint} \operatorname{conv} \bigcup_{0 \le i \le s} H_i.$$

More generally, for $I \subseteq [s]$, we have that $\sum_{i \in I} B_i$ is a face of $\sum_{i \in I} A_i$ if and only if

$$0 \in \operatorname{relint} \operatorname{conv} \left(\bigcup_{i \in I} H_i \cup \bigcup_{i \notin I} G_i \right).$$

Example 4.7. Figure 2 depicts the Minkowski sums $T + T_i$ with $i = 1, 2, 3$ for the following triangles in $\mathbb{R}^3$:

$$T = \operatorname{conv}\{(2,0,0),(0,1,0),(0,-1,0)\}, \quad T_1 = \operatorname{conv}\{(2,0,0),(0,0,1),(0,0,-1)\},$$
$$T_2 = \operatorname{conv}\{(0,2,0),(0,0,1),(0,0,-1)\}, \quad T_3 = \operatorname{conv}\{(-2,0,0),(0,0,1),(0,0,-1)\}.$$

The figure also shows the one-dimensional colorful Gale transforms associated to each of these Minkowski sums. We invite the reader to verify how the condition of Theorem 4.6 allows to recover the full face structure of $T + T_i$ from its colorful Gale transform.

Minkowski transforms are a strongly related alternative way to associate a colorful configuration to a Minkowski sum. This is by way of *generalized* Gale transforms that were first described by McMullen [McM79] (see Section 4.2.B). This construction works at the level of polytopes, their facets, and their vertex sets.

THEOREM 4.8. *Let $P_0, \dots, P_s \subset \mathbb{R}^d$ be polytopes with full-dimensional Minkowski sum, where P_i is d_i-dimensional and has m_i facets. Then there is a centered colorful point configuration $M = (M_0, \dots, M_s)$ in $\mathbb{R}^{d_0 + \dots + d_s - d}$, with $|M_i| = m_i$, called the* Minkowski transform *of $P = P_0 + \dots + P_s$, with the following property:*

For faces $F_i \subseteq P_i$, $0 \le i \le s$, we have that $\sum_{0 \le i \le s} F_i$ is a proper face of $\sum_{0 \le i \le s} P_i$ if and only if

$$0 \in \operatorname{relint} \operatorname{conv} \bigcup_{0 \le i \le s} N_i.$$

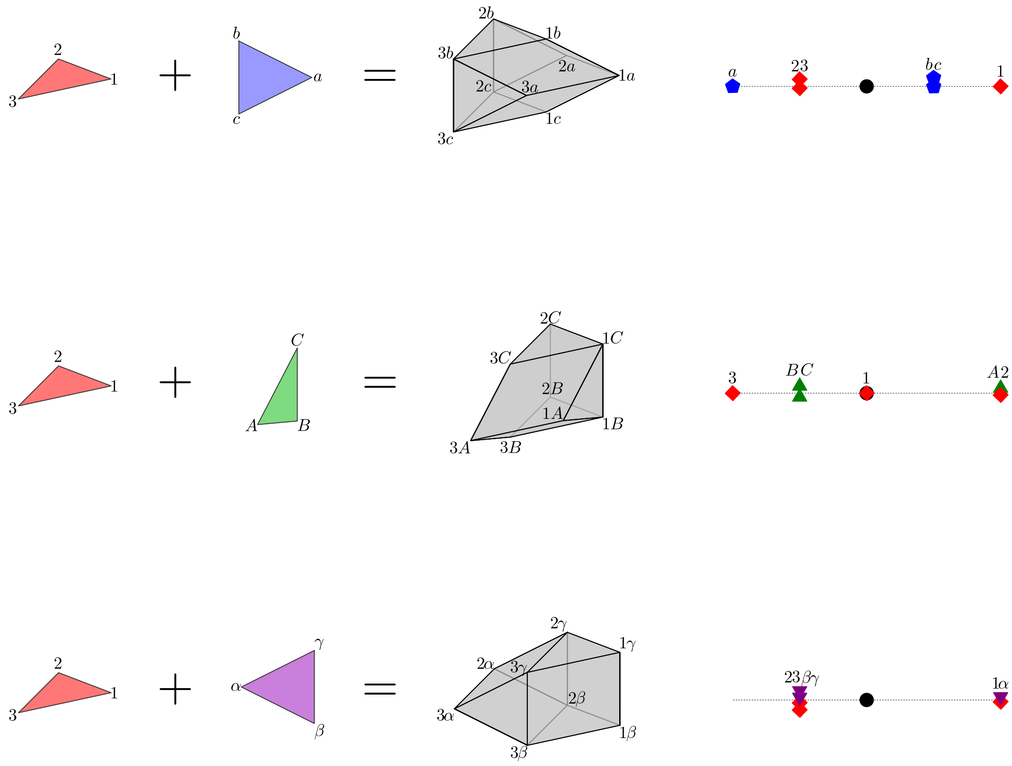


Figure 2. Three different Minkowski sums of two triangles in $\mathbb{R}^3$, and their corresponding colorful Gale transforms in $\mathbb{R}^1$.

where $N_i = \{m_j \in M_i \,|\, j \in I(F_i)\}$, *and* $I(F_i) \subseteq [m_i]$ *indexes the facets of* P_i *that contain* F_i.

Note that Minkowski transforms can be of much lower dimension than colorful Gale transforms. Moreover, the number of elements of the Minkowski transform is indexed by facets, whereas colorful Gale transforms are indexed by vertices. Hence, for polytopes with much fewer facets than vertices, this approach can reduce the complexity considerably.

One reason for this drop in complexity is that the colorful Gale transform also contains information about subsums $\sum_{i \in I} P_i$ for any $I \subseteq [s]$, and in particular, of each of the individual summands. This is not true any more for the Minkowski transform M.

However, for any collection of simplices $P_0, \ldots, P_s \subset \mathbb{R}^d$, the Minkowski transform and the colorful Gale transform coincide up to a choice of coordinates.

Appendix 4.2. Tools for proving Theorems 4.6 and 4.8

4.2.A. Cayley embeddings. Let $A = (A_0, \ldots, A_s)$ be a collection of point configurations in $\mathbb{R}^d$. Its *Cayley embedding* is the configuration $\mathrm{Cay}(A)$ in $\mathbb{R}^{s+d}$ consisting of the points (e_k, a_j) for $a_j \in A_k$, where $e_0 := 0$ and $e_1, \ldots, e_s$ is the standard basis of $\mathbb{R}^s$. Let $b = \frac{1}{s+1}(e_0 + \cdots + e_s)$ be the barycenter of $e_0, \ldots, e_s$ and consider the affine subspace $\Lambda = \{(x, y) \in \mathbb{R}^s \times \mathbb{R}^d : x = b\}$. Then it is straightforward to check that

$$\mathrm{conv}(\mathrm{Cay}(A)) \cap \Lambda \;\cong\; \mathrm{conv}\,(A_0 + \cdots A_s)\,.$$

In particular, this induces a bijection between faces of $\mathrm{Cay}(A)$ and the Minkowski sum $\sum_{0 \le i \le s} A_i$. Cayley embeddings have many favorable properties, in particular in relation to triangulations and mixed subdivisions; see [HRS00].

The *colorful Gale transform* of $A = (A_0, \ldots, A_s)$ is the Gale transform of $\mathrm{Cay}(A)$. This centered point configuration encodes the combinatorics of the Cayley embedding, and hence of the Minkowski sum $A_0 + \cdots + A_s$ and its subsums.

4.2.B. Generalized Gale transforms. McMullen's generalized Gale transforms [McM79] are a very powerful tool to study polytopes under projections; see for example [SZ10; RS12]. They generalize ordinary Gale transforms, and also the so-called zonal and central transforms, that is, Gale transforms tailor made for zonotopes [McM71] and centrally symmetric polytopes [MS68].

Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope containing the origin in its interior, and consider its presentation

$$P \;=\; \{x \in \mathbb{R}^d : \ell(x) \le 1 \text{ for all } i = 1, \ldots, m\}.$$

for linear forms $\ell_1, \ldots, \ell_m : \mathbb{R}^d \to \mathbb{R}$. We are interested in the facial structure of $\pi(P)$, where $\pi : \mathbb{R}^d \to \mathbb{R}^e$ is a linear projection with $e \le d$.

The adjoint of π is an injection $\pi^* : (\mathbb{R}^e)^* \hookrightarrow (\mathbb{R}^d)^*$. Let $L \subseteq (\mathbb{R}^d)^*$ be its image. Finally, let $\phi : (\mathbb{R}^d)^* \twoheadrightarrow (\mathbb{R}^d)^*/L \cong \mathbb{R}^{d-e}$ be the canonical projection. We define the *P-transform* of π as the point configuration $G = \{g_1, \ldots, g_m\}$ given by $g_i = \phi(\ell_i)$ for $i = 1, \ldots, m$. This, in a strong way, depends on the *geometry* of P.

The following projection lemma has been discovered in different contexts. See [SZ10; Zie04] for strengthenings.

Lemma 4.9. *Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope. For $\pi : \mathbb{R}^d \to \mathbb{R}^e$, let G be the associated P-transform. For a proper face $F \subset P$ the following are equivalent*

(i) *$F' = \pi(F)$ is a proper face of $\pi(P)$ and $\pi^{-1}(F') \cap P = F$,*
(ii) *$0 \in \mathrm{relint}\,\mathrm{conv}\{g_i \,|\, i \in I\}$, where $I = \{i \in [m] \,|\, \ell_i(x) = 1$ for all $x \in F\}$.*

The ordinary Gale transforms are exactly Δ_n-transforms for the standard projection from a simplex Δ_n to the polytope; and *central transforms* and *zonal transforms* P-transforms for projections of crosspolytopes and cubes, respectively.

Of course, we may also fix the projection $\pi : \mathbb{R}^d \to \mathbb{R}^e$ and vary the polytope $P \subset \mathbb{R}^d$. For fixed s and d, the *Minkowski projection* is the linear map $\mu : (\mathbb{R}^d)^{s+1} \to \mathbb{R}^d$ given by $(x_0, \ldots, x_s) \mapsto x_0 + \cdots + x_s$. For polytopes $P_0, \ldots, P_s \subset \mathbb{R}^d$ one has $\mu(P_0 \times \cdots \times P_s) = P_0 + \cdots + P_s$.

The *Minkowski transform* of $P_0, \ldots, P_s$ is then the $(P_0 \times \cdots \times P_s)$-transform for the Minkowski projection μ.

4.3. Open problems and perspectives

The assumption of centeredness is very natural in our framework, as colorful Gale transforms and Minkowski transforms are always centered. However, the conclusion of the Colorful Carathéodory Theorem has been shown to hold with diverse weaker assumptions [Aro+09; Bár82; HPT08; MD13]. This prompts the question of under which conditions other than centeredness does the upper bound of Theorem 4.2 hold. In particular, what are other conditions that guarantee that $\tilde{\beta}_{d-1}(\mathcal{A}) = 1$?

Lower bounds for the colorful simplicial depth can be seen as quantitative versions of the Colorful Carathéodory Theorem. The bound $1+d^2 \leq \text{cs-depth}(C)$ by Sarrabezolles [Sar15] only concerns the case considered in [Dez+06] in which all color classes have $d+1$ points. However, our setup in Section 4.1 works more generally with color classes that have an arbitrary number of points. What is the appropriate general lower bound?

Concerning the second part of the chapter, we expect colorful Gale transforms and Minkowski transforms to become a useful tool in the study of Minkowski sums of polytopes with few vertices or facets. Moreover, the generalized Gale duality setup used for their definition (and the relation between both concepts) invites to develop Gale transforms for projections followed by sections, or viceversa. Many existing theories could be unified within this new framework, which could be particularly useful in the study of non-negative factorizations and extended formulations, the subject of Chapter 5 (see Section 5.6 for more details).

CHAPTER 5

Extension complexity bounds and a structural result on polytopes with few vertices and facets [06; 09; 10; 13]

The *extension complexity* of a polytope P, denoted by $\mathrm{xc}(P)$, is the minimal number of facets of a polytope $\widehat{P}$, called an *extended formulation*, that can be linearly projected onto P. This terminology is motivated by applications in combinatorial optimization, as polytopes with small extension complexity correspond to optimization problems that have efficient formulations as linear programs. This is because the computational complexity of the simplex algorithm is intimately tied to the number of linear inequalities and hence it can be advantageous to optimize over $\widehat{P}$ instead of P. Many well-known problems are naturally associated with polytopes whose number of facets is exponential, but admit extended formulations of polynomial size. This has had both theoretical and practical applications (see the surveys [CCZ10; Kai11]).

The study of extension complexity is strongly related to nonnegative matrix factorizations. The *nonnegative rank* of a nonnegative $n \times m$ matrix M, denoted $\mathrm{rank}_+(M)$, is the minimal number r such that there exist $n \times r$ and $r \times m$ nonnegative matrices R and S such that $M = RS$. A seminal result of Yannakakis [Yan91] states that the extension complexity of a polytope coincides with the nonnegative rank of its *slack matrix* (the matrix whose entries are the evaluations of the facet-defining functionals on the vertices). General nonnegative matrix factorizations have a related geometric interpretation (cf. [GG12]), and several applications in diverse disciplines such as linear algebra [Ber73; BL09], statistics [CR93; KRS15], and data analysis [Ber+07; LS99]. Yannakakis also discovered deep connections with communication complexity theory [Yan91].

Extended formulations are also a very interesting and intriguing topic from the point of view of combinatorial polytope theory. Many applications show that it is often advantageous to treat polytopes as affine shadows of higher dimensional polytopes. For example, this observation is already key in the proof of the Minkowski-Weyl Theorem via Fourier-Motzkin Elimination [Zie95, Lec. 1]. It is at the heart of Gale duality, which later gave rise to McMullen's theory of transforms, diagrams and representations [McM71; McM79; MS68]. It is related to the introduction of mixed subdivisions of Minkowski sums [HRS00; HS95], and later to Billera and Sturmfels' fiber polytopes [BS92]. More recently, projection techniques have been fruitfully used for polytope constructions [JZ00; MPP11; SZ10; Zie04].

Despite being the subject of extensive research, the extension complexity is a geometric parameter of polytopes still very far from being well understood, and many basic questions are still unresolved (see the reports [Bea+13; Kla+15] for the latest results and many open problems). In particular, there are very few polytopes for which the exact extension complexity is known. Examples are cubes, Birkhoff polytopes and bipartite matching polytopes [Fio+13]. Exponential lower bounds for important classes of polytopes, obtained recentlty in [Fio+15; Rot13; Rot14], solved problems asked more than 30 years ago and attracted renewed interest in the field.

Actually, determining extension complexity is a very challenging problem already for $d = 2$, and even the possible range of values of the extension complexity of an n-gon is still unknown. This is the subject of Section 5.1, where an upper bound for the extension complexity of n-gons is presented. This result, found independently by Shitov [Shi14a], disproved a conjecture of Beasley and Laffey [BL09].

Section 5.2 presents the complete classification of d-polytopes with at most $d+4$ vertices according to their extension complexity. My main motivation for this project was to provide examples of high-dimensional polytopes for which the explicit determination of the extension complexity is still treatable, in order to obtain a ground set for testing open problems and looking for examples and counterexamples. The first of the goals is largely fulfilled, as this is a super-exponentially large family. However, in view of the final classification, it is not clear that this family will be a rich source of interesting examples and counterexamples. Complete understanding of the extension complexity of the next natural family, d-polytopes with $d + 5$ vertices, seems out of reach right now, specially if we take into account that this was a highly non-trivial problem already for $d = 2$, as we will see in Section 5.1.

A special case of this classification concerns d-polytopes with $d+4$ vertices and at most $d + 3$ facets. It turns out that there are only finitely many (eight) non-pyramidal such polytopes. This observation is generalized in Section 5.3, where it is proven that the number of combinatorial types of d-polytopes with $d + 1 + \alpha$ vertices and $d + 1 + \beta$ facets is bounded by a constant independent of d. This follows from a structural result on polytopes with few vertices and facets related to Perles' Skeleton Theorem.

Section 5.4 gives a lower bound for the extension complexity of generic realizations of combinatorial types of polytopes. It implies that generic simplicial/simple d-polytopes with $d + 1 + \alpha$ vertices/facets have extension complexity at least $2\sqrt{d(d + \alpha)} - d + 1$, which shows that for all $d > (\frac{\alpha-1}{2})^2$ there are d-polytopes with $d + 1 + \alpha$ vertices or facets and extension complexity $d + 1 + \alpha$.

Finally, in Section 5.5 we explicitly determine the extension complexity of all hypersimplices as well as of certain classes of combinatorial hypersimplices.

The chapter ends with some open problems and perpectives.

5.1. The extension complexity of polygons

This section reports joint work with Julian Pfeifle from "Polygons as sections of higher-dimensional polytopes" [06].

Obviously, every n-gon has extension complexity at most n, and for those with $n \leq 5$ it is indeed exactly n. It is not hard to check that hexagons can have complexity 5 or 6 (cf. [GPT13, Example 3.4]). Our first result in [06] shows that it is easy to decide which is the exact value.

Proposition 5.1 ([06]). *For a hexagon P, the following are equivalent:*

(i) $\mathrm{xc}(P) = 5$,
(ii) $P = \pi(Q)$ *for* $Q \cong \Delta_1 \times \Delta_2$, *and*
(iii) P *is* Desarguian*; that is, the lines* $p_0 \wedge p_1$, $p_5 \wedge p_2$ *and* $p_3 \wedge p_4$ *are (projectively) concurrent for some cyclic labeling of its vertices* $p_0, \ldots, p_5$.

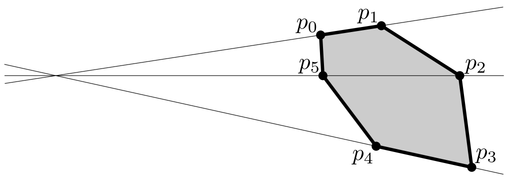


Figure 1. A Desarguian hexagon.

Then we turn our attention to heptagons. We show that every heptagon is a section of a 3-polytope with no more than 6 vertices, and a projection of a 3-polytope with no more than 6 facets. Our proof reveals the geometry behind a result found independently by Shitov in [Shi14a].

THEOREM 5.2. *Every heptagon has extension complexity* 6.

In general, the minimal extension complexity of an n-gon is $\theta(\log n)$, which is attained by regular n-gons [BTN01; FRT12]. On the other hand, there exist n-gons whose extension complexity is at least $2\sqrt{2n-2}-1$ (see Section 5.4). As a consequence of Theorem 5.2 we automatically get upper bounds for the complexity of arbitrary n-gons, and for the nonnegative rank of rank 3 matrices.

THEOREM 5.3 ([06]). *The nonnegative rank of any nonnegative $n \times m$ matrix of rank 3 is at most $\left\lceil \frac{6}{7}\min(n,m) \right\rceil$. In particular, $\mathrm{xc}(P) \leq \left\lceil \frac{6n}{7} \right\rceil$ for every n-gon P with $n \geq 7$.*

This disproved a conjecture of Beasley and Laffey (originally stated in [BL09, Conjecture 3.2] in a more general setting), who asked if for any $n \geq 3$ there is an $n \times n$ nonnegative matrix M of rank 3 with $\mathrm{rank}_+(M) = n$. Subsequently, Shitov improved Theorem 5.3 and announced a sublinear upper bound for the intersection/extension complexity of n-gons [Shi14b]. In higher dimensions, no non-trivial upper bound for extension complexity in terms of the number of vertices/facets is known yet.

Appendix 5.1. Tools for proving Theorem 5.2

5.1.A. Stretchability of pseudoline arrangements. After a combinatorial and geometric analysis, we reduce the problem of determining the extension complexity of heptagons to a problem about stretchability of pseudoline arrangements; that is, about oriented matroid realizability. In short, we prove that the arrangement from Figure 2 is not realizable by straight lines, which allows us to conclude Theorem 5.2.

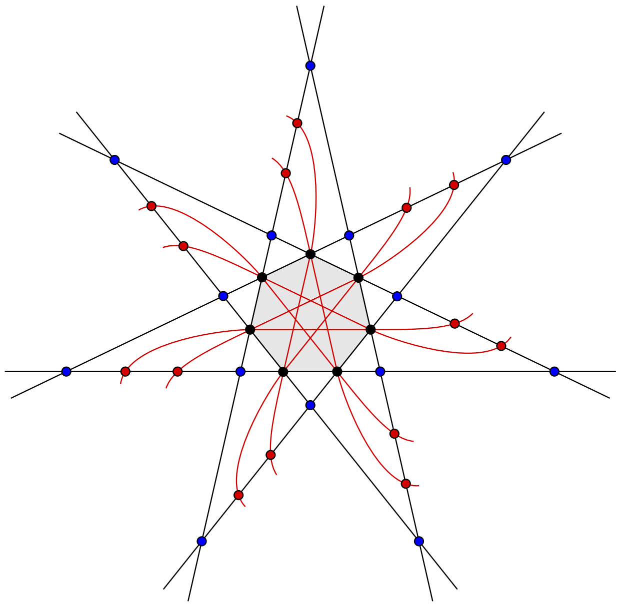


Figure 2. A non-stretchable pseudo-line arrangement.

5.2. The extension complexity of polytopes with few vertices or facets

This section reports work from "Extension complexity of polytopes with few vertices or facets" [09].

In this section, we vastly generalize Proposition 5.1 to determine the extension complexity of all d-polytopes with up to $d + 4$ vertices. Of course, since the extension complexity is preserved by polar duality, this is equivalent to studying the extension complexity of d-polytopes with up to $d + 4$ facets.

Many properties shared by d-polytopes with at most $d+3$ vertices start failing for d-polytopes with $d + 4$ vertices (in a similar way as properties of polytopes of dimension at most 3 usually start failing for 4-polytopes). This led Sturmfels to call d-polytopes with $d + 4$ vertices the "threshold for counterexamples" [Stu88b]. As we have seen in previous sections, this concerns the combinatorial diversity (there are super-exponentially many combinatorial types of d-polytopes with $d+4$

vertices) and the realization spaces (d-polytopes with $d+4$ vertices present Mnëv's universality).

One is tempted to ask whether d-polytopes with $d+4$ vertices are also the threshold of counterexamples for extension complexity. This would have been of particular interest in the context of the long-term open question that asked whether the rational and real nonnegative rank always coincide [BL09; CR93], and that has been recently disproved by two independent teams [Chi+17; Shi16b]. Shitov has even proved a much stronger result, a universality theorem for nonnegative factorizations [Shi16a].

Unfortunately, the answer to this question is negative. They all have complexity $d+4$ except for some sporadic instances that can be constructed via some elementary operations from a finite collection of polytopes. In particular, it is easy to compute the extension complexity of every d-polytope with $d+4$ vertices (or facets, by duality).

THEOREM 5.4. *Let P be a d-polytope with $d+4$ vertices, then*

- *(1)* $\mathrm{xc}(P) = d+2$ *if and only if P has $d+2$ facets.*
- *(2)* $\mathrm{xc}(P) = d+3$ *if and only if:*
  - *(2.1) P has $d+3$ facets, or*
  - *(2.2) $P = \pi(Q)$, where $Q \cong \mathrm{pyr}_{d-2}(\Delta_1 \times \Delta_2)$ for some affine projection π. In this case, either*
    - *(2.2.1) $P = \mathrm{pyr}_k(Q)$ where Q is a Desarguian hexagon, or*
    - *(2.2.2) P has a subset of 6 vertices forming a triangular prism.*
- *(3)* $\mathrm{xc}(P) = d+4$ *otherwise.*

Here, Δ_d denotes a d-dimensional simplex; $P \oplus Q$ and $P \times Q$ represent, respectively, the *direct sum* and the *Cartesian product* of the polytopes P and Q; and $\mathrm{pyr}_k(P)$ denotes the *k-fold pyramid* over P. See [HRGZ04, Sec. 15.1.3] for the corresponding definitions.

More precisely, for a d-polytope P with $d+4$ vertices:

- (1) $\mathrm{xc}(P) = d+2$ if and only if $P \cong \mathrm{pyr}_{d-4}(\Delta_1 \times \Delta_3)$;
- (2) $\mathrm{xc}(P) = d+3$ if and only if:
  - (2.1) P is an iterated pyramid over one of the 8 sporadic non-pyramidal d-polytopes with $d+4$ vertices and $d+3$ facets, or
  - (2.2) $P = \pi(Q)$, where $Q \cong \mathrm{pyr}_{d-2}(\Delta_1 \times \Delta_2)$ for some affine projection π. In this case, either
    - (2.2.1) $P = \mathrm{pyr}_k(Q)$ where Q is a Desarguian hexagon, or
    - (2.2.2) P has a subset of 6 vertices forming a triangular prism. Which means that either
      - (2.2.2.1) $P \cong \mathrm{pyr}_k((\Delta_1 \times \Delta_2) \star (\Delta_n \oplus \Delta_m))$ (where $n+m+k = d-4$), or

(2.2.2.2) P can be obtained from $\Delta_1 \times \Delta_2$ via the operations of *pyramid*, *one-point-suspension* or *Lawrence extension* on an extra (projective) point.

(3) $\mathrm{xc}(P) = d + 4$ otherwise.

Here, $P \star Q$ denotes the *join* of the polytopes P and Q (see [HRGZ04, Sec. 15.1.3]). Lawrence extensions were defined in Section 2.2.B. See [DLRS10, Sec. 4.2.5] for the definition of *one-point-suspensions*.

In the families corresponding to the cases (1), (2.1) and (2.2.2.1), the extension complexity is completely determined by the combinatorial type. Deciding the extension complexity in the cases (2.2.1) and (2.2.2.2) amounts to checking whether certain lines are concurrent.

The finitude of the family (2.1) is the subject of Section 5.3. For fixed d, families (2.2.2.1) and (2.2.2.2) have a size quadratic in d. Hence, out of the super-exponentially many combinatorial types of d-polytopes with $d + 4$ vertices, there are only $\theta(d^2)$ that have realizations with extension complexity smaller than $d+4$.

The proof makes extensive use of generalized Gale diagrams (see Section 4.2.B).

5.3. There are few polytopes with few vertices and facets

This section reports work from "Polytopes with few vertices and few facets" [10].

In this section we take a break from extension complexity and show that there are few (combinatorial types of) polytopes that have both few vertices and few facets. This result, combined with a computer assisted search, lead to the statement from Section 5.2 claiming that there are only 8 sporadic non-pyramidal d-polytopes with $d + 4$ vertices and $d + 3$ facets.

THEOREM 5.5 ([10]). *For each pair of nonnegative integers α and β there is a constant $K(\alpha, \beta)$, independent from d, such that the number of combinatorial types of d-polytopes with no more than $d + 1 + \alpha$ vertices and no more than $d + 1 + \beta$ facets is bounded above by $K(\alpha, \beta)$.*

This might come as a surprise, considering that the number of d-polytopes with $d+3$ vertices is exponential in d [Fus06], and the number of d-polytopes with $d + 4$ vertices is already super-exponential in d (this follows from the full version of Theorem 3.1 from [01]). Of course, the same numbers apply for polytopes with few facets, by polarity.

Theorem 5.5 is a direct consequence of the following structural result.

Proposition 5.6 ([10])**.** *For each pair of nonnegative integers α and β there is a constant $D(\alpha, \beta)$ such that every d-polytope with no more than $d + 1 + \alpha$ vertices and no more than $d + 1 + \beta$ facets is a join of a simplex and an at most $D(\alpha, \beta)$-dimensional polytope.*

Equivalently, every d-polytope with $d > D(\alpha, \beta)$ either is a pyramid, has more than $d + 1 + \alpha$ vertices or has more than $d + 1 + \beta$ facets.

Indeed, this proposition shows that for every d the number of combinatorial types of d-polytopes with no more than $d+1+\alpha$ vertices and no more than $d+1+\beta$ facets is bounded above by those in dimension $D(\alpha,\beta)$. Since the vertex-facet incidences determine the combinatorial type, we get the following crude estimate for $K(\alpha,\beta)$:

$$K(\alpha,\beta) < 2^{(D(\alpha,\beta)+\alpha+1)(D(\alpha,\beta)+\beta+1)}.$$

Appendix 5.3. Tools for proving Proposition 5.6

5.3.A. Unneighborly polytopes. The proof is based on a result of Marcus on minimal positively 2-spanning configurations [Mar81; Mar84], which via Gale duality provides lower bounds on the number of vertices of what Wotzlaw and Ziegler call *unneighborly polytopes* [WZ11]. A polytope P is *unneighborly* if for every vertex v of P there is some vertex w such that (v,w) does not form an edge of the graph of P.

THEOREM 5.7 ([Mar81]). *If P is an unneighborly d-polytope with $d+\alpha+1$ vertices, then*

$$d \leq \begin{cases} 3\alpha - 1 & \textit{if } \alpha \leq 5, \\ \binom{\alpha}{2} + 4 & \textit{if } \alpha \geq 5. \end{cases}$$

As Wotzlaw and Ziegler point out in [WZ11], this upper bound is actually tight up to a constant factor. A slightly worse, but still quadratic, upper bound can also be deduced from [Wot09, Theorem 7.2.1]. This is a quantitative version of *Perles' Skeleton Theorem*, a remarkable result first proved by Perles (unpublished, ca. 1970), reported by Kalai [Kal94] and elaborated upon by Wotzlaw [Wot09, Part II].

A quadratic upper bound for $D(\alpha,\beta)$ is given in [10]. The proof is inductive along the following lines: If P is a d-polytope with $d+1+\alpha$ vertices and $d+1+\beta$ facets and d is large enough, then by Marcus's result P has a neighborly vertex v connected to all the other vertices of P by an edge. If P is not a pyramid with apex v, then the vertex figure P/v has $(d-1)+1+\alpha$ vertices and strictly less than $(d-1)+1+\beta$ facets. We deduce by induction that P/v is a pyramid, which implies that P is also a pyramid because v is a neighborly vertex.

5.4. Lower bounds for the extension complexity of generic polytopes

This section reports work from "Extension complexity of polytopes with few vertices or facets" [09].

In view of Theorem 5.2, one is tempted to ask whether there is an α such that every d-polytope with $d+1+\alpha$ vertices has extension complexity at most $d+\alpha$, as this would similarly provide upper bounds for the extension complexity of d-polytopes in terms of their number of vertices. (It is still an open problem whether for each n there exist d-polytopes with n vertices and extension complexity n; for which we only know the answer, in the negative, for the case $d \leq 2$.)

Unfortunately, such an α does not exist. In this section we provide a lower bound for the extension complexity of generic polytopes in terms of the dimension of their realization space. It entails that for every $d, \alpha, \beta \geq 0$,

- a generic (simplicial) d-polytope P with $d + 1 + \alpha$ vertices has extension complexity
$$\mathrm{xc}(P) \geq 2\sqrt{d(d+\alpha)} - d + 1;$$
- a generic (simple) d-polytope P with $d + 1 + \beta$ facets has extension complexity
$$\mathrm{xc}(P) \geq 2\sqrt{d(d+\beta)} - d + 1.$$

Here, *generic* has to be understood in terms of the vertex coordinates and the inequality description, respectively.

In particular, when $d > (\frac{\alpha-1}{2})^2$ there are d-polytopes with $d + 1 + \alpha$ vertices (or facets) with extension complexity $d + 1 + \alpha$.

Observe also that the bounds above, when specialized to $d = 2$, give a lower bound of $2\sqrt{2n-2} - 1$ for the extension complexity of a generic (even rational) n-gon. This bound is tight for $n \leq 15$ [Van+15], but also for general n up to a multiplicative constant, as the *admissible n-gons* of Shitov show [Shi14b]. This order of magnitude was already attained by the previous best lower bound of $\sqrt{2n}$ for the extension complexity of generic n-gons, by Fiorini, Rothvoß and Tiwary [FRT12]. However, their approach did not extend directly to the rational case, where they got a lower bound of order $\Omega(\sqrt{n/\log n})$.

The precise statement of the lower bound uses *unreduced realization spaces* of polytopes. This is the set $\tilde{\mathcal{R}}(P)$ of realizations of P, parametrized by the vertex coordinates or by the facet defining inequalities, without taking the quotient by any transformation group. We use $\tilde{\mathcal{R}}_{\mathrm{xc}\leq K}(P)$ to denote the subset containing those instances with extension complexity at most K.

THEOREM 5.8 ([09]). *Let P be a polytope whose unreduced realization space has dimension r, then $\tilde{\mathcal{R}}(P) \setminus \tilde{\mathcal{R}}_{\mathrm{xc}\leq K}(P)$ is a full-dimensional dense semi-algebraic subset of $\tilde{\mathcal{R}}(P)$ for every*
$$K < 2\sqrt{r-d} - d + 1.$$
In particular:

(1) *For every $Q \in \tilde{\mathcal{R}}(P)$ there is some polytope $Q' \in \tilde{\mathcal{R}}(P)$ arbitrarily close to P in Hausdorff distance such that*
$$\mathrm{xc}(Q') \geq 2\sqrt{r-d} - d + 1.$$
(2) *If R is drawn randomly from a continuous probability distribution on $\tilde{\mathcal{R}}(P)$, then almost surely*
$$\mathrm{xc}(R) \geq 2\sqrt{r-d} - d + 1.$$

To recover the statements above, observe that the realization space of a simplicial d-polytope with n vertices always has dimension dn; and the realization space of a simple d-polytope with m facets always has dimension dm. Moreover, polytopes with rational coordinates are dense in these realization spaces, and hence one can also impose the approximating polytope Q' to be rational.

Appendix 5.4. Tools for proving Theorem 5.8

5.4.A. Bounds for real algebraic geometry. Theorem 5.8 follows from an estimation of the dimension of $\tilde{\mathcal{R}}_{\mathrm{xc}\le K}(P)$, obtained by counting degrees of freedom. Both $\tilde{\mathcal{R}}(P)$ and $\tilde{\mathcal{R}}_{\mathrm{xc}\le K}(P)$ are semi-algebraic sets, and $\tilde{\mathcal{R}}_{\mathrm{xc}\le K}(P)$ is the union of the $\tilde{\mathcal{R}}_{N,D}(P)$ that contain all realizations of P that arise as projections of D-polytopes with N facets, for all $N \le K$ and $d \le D \le N-1$. Fixing the projection $\pi : \mathbb{R}^D \to \mathbb{R}^d$, and setting $\mathcal{Q}_{N,D}(P)$ the set of D-polytopes Q with N facets such that $\pi(Q) = P$, one can easily construct a continuous semi-algebraic surjective map $\phi : \mathcal{Q}_{N,D}(P) \to \tilde{\mathcal{R}}_{N,D}(P)$. Our bounds for $\dim \tilde{\mathcal{R}}_{\mathrm{xc}\le K}(P)$ come from estimating the dimensions of $\mathcal{Q}_{N,D}(P)$ and the fibers of ϕ, with classical dimension bounds from real algebraic geometry related to semialgebraic triviality of continuous semialgebraic mappings (see [BCR98, Thm. 9.3.2] and [Cos00, Cor. 4.2]).

5.5. The extension complexity of hypersimplices

This section reports joint work with Francesco Grande and Raman Sanyal from "Extension complexity and realization spaces of hypersimplices" [13].

In this section we explicitly determine the extension complexity of the family of hypersimplices. Recall that for $0 < k < n$, the (n, k)-*hypersimplex* is the convex polytope

$$\Delta_{n,k} \;=\; \operatorname{conv}\{x \in \{0,1\}^n : x_1 + \cdots + x_n = k\}\,.$$

They are prominent objects in combinatorial optimization, appearing in connection with packing problems, and also in matroid theory as $\Delta_{n,k}$ is the matroid base polytope of the uniform matroid $U_{n,k}$. This marks hypersimplices as polytopes of considerable interest and naturally prompts the question as to their extension complexity.

Note that $\Delta_{n,k}$ is affinely isomorphic to $\Delta_{n,n-k}$. The hypersimplex $\Delta_{n,1} = \Delta_{n-1}$ is the standard simplex of dimension $n-1$ and $\mathrm{xc}(\Delta_{n-1}) = n$. Our first result concerns the extension complexity of the *proper* hypersimplices, that is, the hypersimplices $\Delta_{n,k}$ with $2 \le k \le n-2$.

THEOREM 5.9 ([13]). *The hypersimplex $\Delta_{4,2}$ has extension complexity 6, the hypersimplices $\Delta_{5,2} \cong \Delta_{5,3}$ have extension complexity 9. For any $n \ge 6$ and $2 \le k \le n-2$, we have $\mathrm{xc}(\Delta_{n,k}) = 2n$.*

As we have seen, the extension complexity is not an invariant of the combinatorial type (see for example Proposition 5.1). On the other hand, the extension complexity of any polytope combinatorially isomorphic to the n-dimensional cube is always $2n$. The close connection to simplices and cubes and Theorem 5.9 raises the question whether all *combinatorial* (n, k)-hypersimplices are *extension maximal* (have extension complexity equal to their number of facets) when $n \geq 6$ and $2 \leq k \leq n-2$.

Our understanding of $\mathcal{R}_{n,2}$ (see Section 2.4) allows us to show in [13] that generic combinatorial $(5, 2)$-hypersimplices have extension complexity 10, even if the standard realization verifies $\mathrm{xc}(\Delta_{5,2}) = 9$. For $n \geq 6$, we still do not know of any realization of a (n, k)-hypersimplex of extension complexity less than $2n$, but we do not think that every combinatorial (n, k)-hypersimplex with $n \geq 6$ and $2 \leq k \leq n$ is extension maximal. All we can say is that the locus $E_{n,k} \subseteq \mathcal{R}_{n,k}$ of extension maximal (n, k)-hypersimplices is open and non-empty for $n \geq 6$ and $2 \leq k \leq n-2$.

Our best bound for the extension complexity of generic hypersimplices is:

THEOREM 5.10 ([13]). *If P is a combinatorial (n, k)-hypersimplex with $n \geq 6$ and $2 \leq k \leq \lceil \frac{n}{2} \rceil$, then*

$$\mathrm{xc}(P) \;\geq\; \begin{cases} n + 2k + 1 & \text{if } k < \left\lfloor \frac{n}{2} \right\rfloor, \\ 2n & \text{otherwise.} \end{cases}$$

Appendix 5.5. Tools for proving Theorems 5.9 and 5.10

5.5.A. Rectangle covering numbers. The rectangle covering number, introduced in [Fio+13], is a very elegant combinatorial approach to lower bounds on the nonnegative rank of a polytope. A *rectangle* of a nonnegative $m \times n$ matrix S_P is an index set $R = I \times J$ with $I \subseteq [m]$, $J \subseteq [n]$ such that $(S_P)_{ij} > 0$ for all $(i, j) \in R$. The *rectangle covering number* $\mathrm{rc}(S_P)$ is the smallest number of rectangles $R_1, \ldots, R_s$ such that $(S_P)_{ij} > 0$ if and only if $(i, j) \in \bigcup_t R_t$. As explained in [Fio+13, Section 2.4], we have $\mathrm{rc}(S_P) \;\leq\; \mathrm{xc}(P)$, if S_P is the slack matrix of a polytope P.

In [13], we use the rectangle covering number to compute the extension complexity of "small" hypersimplices (with $n \leq 6$). This is done computationally, reducing it to a satisfiability problem and using a SAT solver.

The extension complexity of "large" hypersimplices (with $n \geq 7$) is then solved using an inductive argument that combines the combinatorics and geometry of $\Delta_{n,k}$.

5.5.B. FG-genericity. The geometric part of our proof extends to all FG-generic hypersimplices (see Section 2.4.A for the definition), showing that they are extension maximal. Unfortunately, FG-genericity is not a property met by all hypersimplices, which is confirmed by the existence of a non-FG-generic realization of $\Delta_{6,2}$. On the other hand, Lemma 2.15 shows that that hypersimplices with $n \geq 6$

and $\lfloor \frac{n}{2} \rfloor \leq k \leq \lceil \frac{n}{2} \rceil$ are FG-generic. This, together with an inductive argument, gives Theorem 5.10.

5.6. Open problems and perspectives

Extension complexity is a very active field, and our knowledge has greatly advanced during the past years. Many longstanding open questions have been resolved, and many new challenging problems have arisen. Some of these are reported in [Bea+13; Kla+15].

Despite being far from the applications in optimization, our knowledge on the extension complexity of polygons is a very good indicator of the difficulty of the topic, and serves as a testing field for new questions. Shitov's sublinear upper bound on the extension complexity of n-gons is of order $n(\ln\ln\ln\ln\ln\ln n)^{-1/2}$ [Shi14b]. This bound is unlikely to be tight, and in any case, it is very far from $2\sqrt{2n-2}-1$, the current best lower bound for the worst-case extension complexity obtained using generic polygons in Section 5.4. It is even conceivable that this lower bound is optimal, as it happens when $n \leq 15$ [Van+15]. Finding non-trivial upper bounds for the extension complexity of d-polytopes with n vertices for $d > 2$ would be an exciting development, for its possible consequences in terms of linear optimization. A starting step would be to study cyclic polytopes, as the higher dimensional version of the Erdös-Szekeres Theorem states that every large enough point configuration in general position contains a cyclic polytope with n vertices [Suk14].

Planar configurations also form a building block for many higher dimensional constructions. For example, they are the starting point in the recent proof of the Universality Theorem for nonnegative factorizations by Shitov [Shi16a]. It states that the set of nonnegative factorizations of a nonnegative matrix can be rationally equivalent to any bounded semialgebraic set. A very tempting strengthening of this result would be a universality theorem for extended formulations, which would in particular entail that it is algorithmically hard to compute the extension complexity of a polytope. There is no straightforward way to adapt Shitov's construction to get a result on extended formulations, and hence the proof would probably need to start from scratch. A related result, the universality theorem for nested polytopes, has been recently proved [DHM19].

The classification of Section 5.2 exploits generalized Gale duality (see Section 4.2.B) in an essential way. Together with the constructions of Section 4.2, they hint that the generalized Gale transform of a projection of a polytope P should be considered simultaneously with the Gale diagram of its polar polytope P°. This amounts to extending Gale duality to the context of polytopes obtained by combining first a section and then a projection, or vice versa. The study of extended formulations is, in its core, the study of such constructions. Even more, this can be generalized to arbitrary nonnegative factorizations if one considers arbitrary polytope pairs fulfilling $Q \subseteq P^\circ$

Motivated by its geometric interpretation, nonnegative matrix factorization has been also generalized to the so-called cone factorizations [GPT13], in which the non-negative orthant is replaced by other convex cones. When this is the cone of symmetric positive semidefinite matrices, one obtains psd-factorizations, which are of particular interest for their consequences in semidefinite programming. In this set-up one tries to approximate polytopes by spectrahedral shadows, which are projections of sections of the psd cone. The framework of Gale transforms for projections and sections seems to fit perfectly in this context.

Our original motivation for studying the nonnegative rank of hypersimplices in [13] comes from matroid theory [Oxl11]. The (n, k)-hypersimplex is the matroid base polytope of the uniform matroid $U_{n,k}$. In [GS17], Grande and Sanyal studied 2-*level matroids*, which are those that can be constructed from uniform matroids by taking direct sums or 2-sums. They exhibit extremal behavior with respect to various geometric and algebraic measures of complexity. In particular, it is shown that M is 2-level if and only if its matroid polytope P_M has minimal psd-rank, $\dim P_M + 1$. Our starting point was the natural question whether the class of 2-level matroids also exhibits an extremal behavior with respect to the nonnegative rank. To extend Theorem 5.9 to all 2-level matroids, it would be necessary to understand the effect of taking direct and 2-sums on the nonnegative rank. The direct sum of matroids translates into the Cartesian product of matroid polytopes. The following conjecture, first asked by François Glineur during a Dagstuhl seminar in 2013 [Bea+13], remains surprisingly elusive.

Conjecture 5.11. *The extension complexity is additive with respect to Cartesian products, that is,*

$$\mathrm{xc}(P_1 \times P_2) \;=\; \mathrm{xc}(P_1) + \mathrm{xc}(P_2),$$

for polytopes P_1 and P_2.

We show in [13] that the conjecture holds whenever one of the factors is a simplex, which has been later improved to the case when one of the factors is a pyramid [TWZ17].

Finally, concerning Section 5.3, [10] gives the upper bound $D(\alpha, \beta) \leq \binom{\alpha}{2} + \beta + 3$ when $\alpha \geq 5$. The lack of symmetry suggests that it might not be optimal. Which is the optimal value for $D(\alpha, \beta)$? As far as we know, it could be linear in both α and β, like in our current best examples. These are based on the join of n squares, which is $(3n - 1)$-dimensional and has $4n$ vertices and facets. Our proof method cannot provide a linear bound because, despite Marcus' original conjecture, the maximal dimension of an unneighborly polytope is quadratic in α (see [Man74; WZ11]). However, this does not take into account the number of facets. In fact, the unneighborly polytopes with few vertices from [Man74] and [WZ11] have many facets and do not improve the join of squares. So it is conceivable that a different approach might yield better bounds.

CHAPTER 6

Triangulations of products of simplices and tropical oriented matroids [03; 16]

The Cartesian product of two simplices is the convex polytope:

$$\Delta_m \times \Delta_{\overline{n}} := \operatorname{conv}\left\{(e_i, e_{\overline{j}}) \colon 0 \le i \le m,\ \overline{0} \le \overline{j}\ \le \overline{n}\right\} \subset \mathbb{R}^{n+m+2},$$

where e_i and $e_{\overline{j}}$ denote the standard basis vectors of $\mathbb{R}^{m+1}$ and $\mathbb{R}^{n+1}$, respectively; and overlined indices are used to distinguish the labels of the two factors.

Their triangulations are very interesting intricate objects [DLRS10, Sec. 6.2] that have been extensively studied with various purposes. They are a key ingredient for understanding triangulations of products of polytopes [DL96; Hai91; OS03; San00]. Via the Cayley trick, they are in bijection with fine mixed subdivisions of a dilated simplex $m\Delta_{n-1}$ [San05], which provides the relation to tropical (pseudo-)hyperplane arrangements and tropical oriented matroids [AD09; DS04]. Moreover, they have also attracted interest in algebraic geometry and commutative algebra [BB98; CHT07; GKZ94; Stu96], and in Schubert calculus [AB07].

This chapter revolves around two sibling triangulations of $\Delta_n \times \Delta_{\overline{n}}$ with a deep combinatorial structure: the *associahedral/cyclohedral triangulation* and the *Dyck path triangulation*.

The associahedral triangulation is introduced in Section 6.1. It serves as the starting point to construct a polyhedral realization of Préville-Ratelle and Viennot's ν-Tamari lattice [PRV17]. In particular, we use it in [16] to obtain geometric realizations of m-Tamari lattices as polyhedral subdivisions of associahedra induced by an arrangement of tropical hyperplanes, giving a positive answer to an open question of F. Bergeron [Ber12]. This reveals a simplicial complex structure underlying the ν-Tamari lattice, which generalizes the classical associahedron, whose combinatorics is governed by the so-called $(I, \overline{J})$*-trees*.

The associahedral triangulation does not cover all of $\Delta_n \times \Delta_{\overline{n}}$, only a subpolytope. However, it is amenable to a cyclic symmetry, giving rise to a full triangulation of $\Delta_n \times \Delta_{\overline{n}}$. This is the *cyclohedral triangulation*, presented in Section 6.2. Its name refers to the *cyclohedron*, the *generalized associahedron* associated to the B_n root system [CFZ02; FZ03b]. It naturally provides type B analogues of our constructions. Notably, it gives rise to a partial order that generalizes the type B Tamari lattice, introduced independently by Thomas [Tho06] and Reading [Rea06], along with its corresponding tropical geometric realization.

With the right presentation, the associahedral triangulation can be considered a "non-crossing" combinatorial object. It has a "non-nesting" analogue, also amenable to the same cyclic action, resulting in the *Dyck path triangulation*. Its study is motivated by extendability problems of partial triangulations of products of two simplices. The main result of Section 6.3 states that whenever $m \geq k > n$, any triangulation of the product of the k-skeleton of Δ_m with $\Delta_{\overline{n}}$ extends to a unique triangulation of $\Delta_m \times \Delta_{\overline{n}}$. The Dyck path triangulation is used to show that the bound $k > n$ is optimal. These results can be interpreted in the language of tropical oriented matroids, providing analogues to classical results in oriented matroid theory.

The last section of this chapter is devoted to open questions and conjectures.

6.1. Tropical ν-Associahedra

This section reports joint work with Cesar Ceballos and Camilo Sarmiento from "Geometry of ν-Tamari lattices in types A and B" [16].

The ν-*Tamari* lattice is a partial order on the set of lattice paths weakly above a given path ν that generalizes the Dyck/ballot-path formulation of the classical Tamari lattice [MHPS12; Tam51]. It has been recently introduced by Préville-Ratelle and Viennot [PRV17] as a further generalization of the m-*Tamari lattice* on Fuss-Catalan paths, which was first considered by F. Bergeron and Préville-Ratelle in connection to the combinatorics of higher diagonal coinvariant spaces [BPR12]. These lattices have attracted considerable attention in other areas such as representation theory and Hopf algebras [BMCPR13; NT14; Nov14], and remarkable enumerative, algebraic, combinatorial, and geometric properties have been discovered [Ber12; BMFPR11; Cha05; FPR17].

One of the striking characteristics of the Tamari lattice is that its Hasse diagram can be realized as the edge graph of a polytope, the *associahedron*. The realization problem of this "mythical polytope" [Hai84] was explicitly posed by Stasheff in 1963 [Sta63], who constructed it as a cellular ball. After its first constructions as a polytope [Hai84; Lee89], many systematic construction methods emerged, with different remarkable geometric and combinatorial properties [CSZ15; CFZ02; FZ03b; GZK90; GZK91; HL07; HLT11; JK10; Lod04; Pos09; RSS03; SS93].

It is natural to ask if ν-Tamari lattices admit similar constructions. This question was posed by Bergeron, who in [Ber12, Figures 4 and 6] presented geometric realizations of a few small m-Tamari lattices as the edge graph of a subdivision of an associahedron and asked if such realizations exist in general. In [16] we provide a positive answer to this question (see Figure 1 for examples of such geometric realizations) by means of tropical geometry (see Section 6.1.B).

THEOREM 6.1 ([16]). *Let ν be a lattice path from $(0,0)$ to (a,b). The Hasse diagram of the ν-Tamari lattice* Tam_ν *can be realized geometrically as the edge*

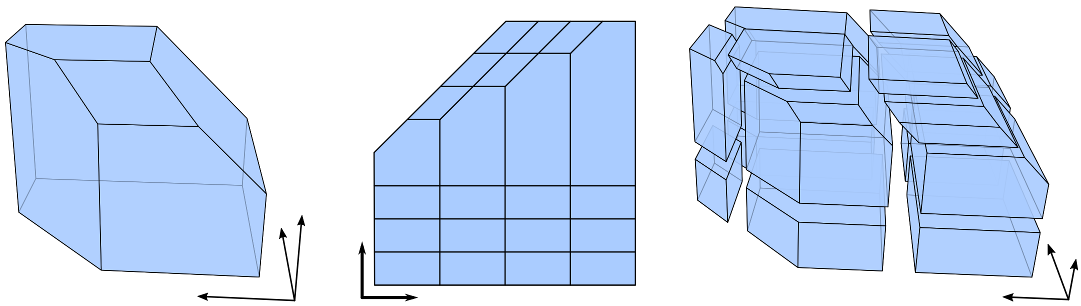


Figure 1. Geometric realizations of m-Tamari lattices by cutting classical associahedra with tropical hyperplanes.

graph of a polyhedral complex induced by an arrangement of tropical hyperplanes (in $\mathbb{TP}^a \cong \mathbb{R}^a$ and in $\mathbb{TP}^b \cong \mathbb{R}^b$).

The associated polyhedral complex is called the *ν-associahedron* Asso_ν; and in the Fuss-Catalan case we refer to it as the *m-associahedron*. Although it is not always a subdivision of a classical associahedron (it can even be non-pure in some cases), this holds for the m-associahedron. In more generality, if ν does not contain two (non-initial) consecutive north steps, then the ν-associahedron is a regular subdivision of a classical associahedron into Cartesian products of associahedra. Moreover, the edges of the ν-associahedron can be oriented by a linear functional to give rise to the ν-Tamari lattice, mimicking a property of the classical associahedron.

Our starting point is a triangulation of a subpolytope $\mathcal{U}_n$ of the Cartesian product of simplices $\Delta_n \times \Delta_{\overline{n}}$, which we call the *associahedral triangulation* $\mathfrak{A}_n$, that is flag, regular and, as a simplicial complex, isomorphic to the join of a simplex with the boundary of a simplicial $(n-1)$-associahedron.

The fact that $\mathfrak{A}_n$ is embedded in the product of two simplices has several advantages. First, for each lattice path ν there is a pair $I, \overline{J} \subseteq [n], [\overline{n}]$ such that the restriction of $\mathfrak{A}_n$ to its face $\Delta_I \times \Delta_{\overline{J}}$ induces a triangulation $\mathfrak{A}_{I,\overline{J}}$ dual to Tam_ν. Moreover, there is a correspondence between regular triangulations of $\Delta_m \times \Delta_{\overline{n}}$ and tropical hyperplane arrangements, conceived by Develin and Sturmfels in [DS04] and further developed in [AD09; FR15]. We exploit it to get the desired polyhedral realizations.

The triangulation $\mathfrak{A}_{I,\overline{J}}$ provides a full simplicial complex structure supported on ν-paths, the *ν-Tamari complex*, which shares several properties with the classical simplicial associahedron and provides definitions for their Fuss-Catalan and rational-Catalan extensions. For example, the ℓth entry of its h-vector is the number of ν-Dyck paths with exactly ℓ valleys, generalizing the classical Narayana numbers for classical Dyck paths. In the Fuss-Catalan case, these numbers were considered in [Ath05; AT06; FR05; Tza06]; and in the rational Catalan case, they appear in the work of Armstrong, Rhoades, and Williams [ARW13], who introduced a simplicial complex called the rational associahedron, different from ours, whose h-vector entries are given by the corresponding ν-Narayana numbers too. It would be interesting to understand the relation between the ν-Tamari complex and the rational associahedron.

Our construction from [16] has been subsequently extended by Pilaud to the Cambrian setting [Pil20].

Appendix 6.1. Tools for proving Theorem 6.1

6.1.A. $(I,\overline{J})$-trees and the $(I,\overline{J})$-associahedral triangulation. Consider two copies of the natural numbers $\mathbb{N}$ and $\overline{\mathbb{N}}$, and regard $\mathbb{N}\sqcup\overline{\mathbb{N}}$ as the totally ordered set with covering relations $i \prec \overline{i} \prec i+1$. Let I and $\overline{J}$ be nonempty finite subsets of $\mathbb{N}$ and $\overline{\mathbb{N}}$, respectively, such that $\min(I \sqcup \overline{J}) \in I$ and $\max(I \sqcup \overline{J}) \in \overline{J}$. An $(I,\overline{J})$-*forest* is a subgraph of $K_{I,\overline{J}}$, the complete bipartite graph with node set $I \sqcup \overline{J}$, that is

(1) **Increasing:** each arc $(i,\overline{j})$ fulfills $i \prec \overline{j}$ (i.e. $i \leq j$); and
(2) **Non-crossing:** it does not contain two arcs $(i,\overline{j})$ and $(i',\overline{j}')$ satisfying $i \prec i' \prec \overline{j} \prec \overline{j}'$ (i.e. $i < i' \leq j < j'$).

An $(I,\overline{J})$-*tree* is a maximal $(I,\overline{J})$-forest.

Lemma 6.2. *The set of $(I,\overline{J})$-forests indexes the simplices of a flag regular triangulation of*

$$\mathcal{U}_{I,\overline{J}} := \operatorname{conv}\left\{(e_i, e_{\overline{j}}) \ : \ i \in I, \overline{j} \in \overline{J} \text{ and } i \prec \overline{j}\right\},$$

called the $(I,\overline{J})$-associahedral triangulation $\mathfrak{A}_{I,\overline{J}}$.

When $(I,\overline{J}) = ([n],[\overline{n}])$, one recovers a triangulation that is dual to the classical associahedron. Relatives of this triangulation have been found independently several times, under various guises, in a number of different contexts [GGP97; SP02; Més11; RSS03; JK10]. Every $(I,\overline{J})$-associahedral triangulation arises as the intersection of certain $([n],[\overline{n}])$-associahedral triangulation with a supporting hyperplane.

The set of $(I,\overline{J})$-trees can be turned into a partial order (which is actually a lattice) by considering a cover relation induced by flips (akin to the description of the Tamari lattice as a rotation lattice). This lattice is isomorphic to the ν-Tamari lattice for certain $\nu(I,\overline{J})$ (and for every ν, there are some $(I,\overline{J})$ such that $\nu = \nu(I,\overline{J})$). An example is depicted in Figure 2.

6.1.B. Tropical hyperplane arrangements. Tropical geometry refers to geometry in the *tropical semiring* $(\mathbb{R}\cup\infty,\oplus,\odot)$ where the tropical addition $\oplus$ and tropical multiplication $\odot$ are defined by $a \oplus b = \min(a,b)$ and $a \odot b = a+b$

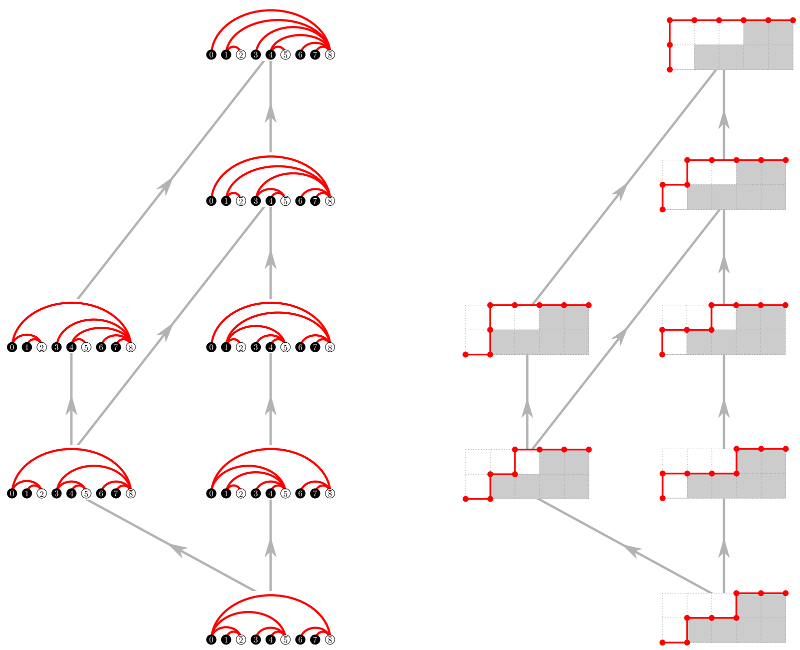


Figure 2. The $(I, \overline{J})$-Tamari lattice for $I = \{0, 1, 3, 4, 6, 7\}$ and $\overline{J} = \{\overline{2}, \overline{5}, \overline{8}\}$, and the corresponding representation in terms of $\nu(I, \overline{J})$-paths.

(see [MS15] for an introduction to the subject). The *tropical projective space* is

$$\mathbb{TP}^d = \left((\mathbb{R} \cup \infty)^{d+1} \setminus (\infty, \infty, \ldots, \infty)\right) / \mathbb{R}(1, 1, \ldots, 1),$$

and a *tropical hyperplane* is the "tropical vanishing locus" of a tropical linear equation $\bigoplus a_i \oplus x_i$, where the tropical vanishing locus is the set of points where the minimum $\min(a_i + x_i)$ is attained at least twice. Each tropical hyperplane subdivides the space as the normal fan of a simplex.

Combinatorial types of arrangements of (possibly degenerate) tropical hyperplanes in tropical projective space are in correspondence with regular subdivisions of subpolytopes of products of simplices [DS04; AD09; FR15]. *Regular subdivisions* are those obtained by considering the lower envelope of a lift of the vertices into an extra dimension (see [DLRS10] for details). The heights of the lift determine the coefficients of the equations of the tropical hyperplanes of an arrangement that induces a polyhedral decomposition of $\mathbb{TP}^d$ whose poset of bounded cells is anti-isomorphic to the poset of interior cells of the triangulation (see [DS04]).

6.2. ν-Tamari posets and ν-Associahedra in type B

This section reports joint work with Cesar Ceballos and Camilo Sarmiento from "Geometry of ν-Tamari lattices in types A and B" [16].

There are several connections between associahedra and Coxeter groups. The *generalized associahedra* are a family of simple polytopes that encode the mutation graphs of cluster algebras of finite types [FZ02; FZ03a; FZ03b], and for which various realizations have been found [CFZ02; HLT11; PS15; RS09; Ste12]. For the A_n root system, one obtains a classical n-dimensional associahedron. The generalized associahedron corresponding to B_n is the n-dimensional *cyclohedron*, a polytope that had appeared first in the work of Bott and Taubes [BT94], and was later realized as a convex polytope by Markl [Mar99] and Simion [Rod03].

The associahedral triangulation $\mathfrak{A}_n$ admits a cyclic action whose orbit gives a flag regular triangulation of $\Delta_n \times \Delta_{\overline{n}}$ that, combinatorially, is the join of a simplex with the boundary complex of a simplicial n-cyclohedron. For this reason we call it the *cyclohedral triangulation* $\mathfrak{C}_n$, and see it as a type B analogue of $\mathfrak{A}_n$.

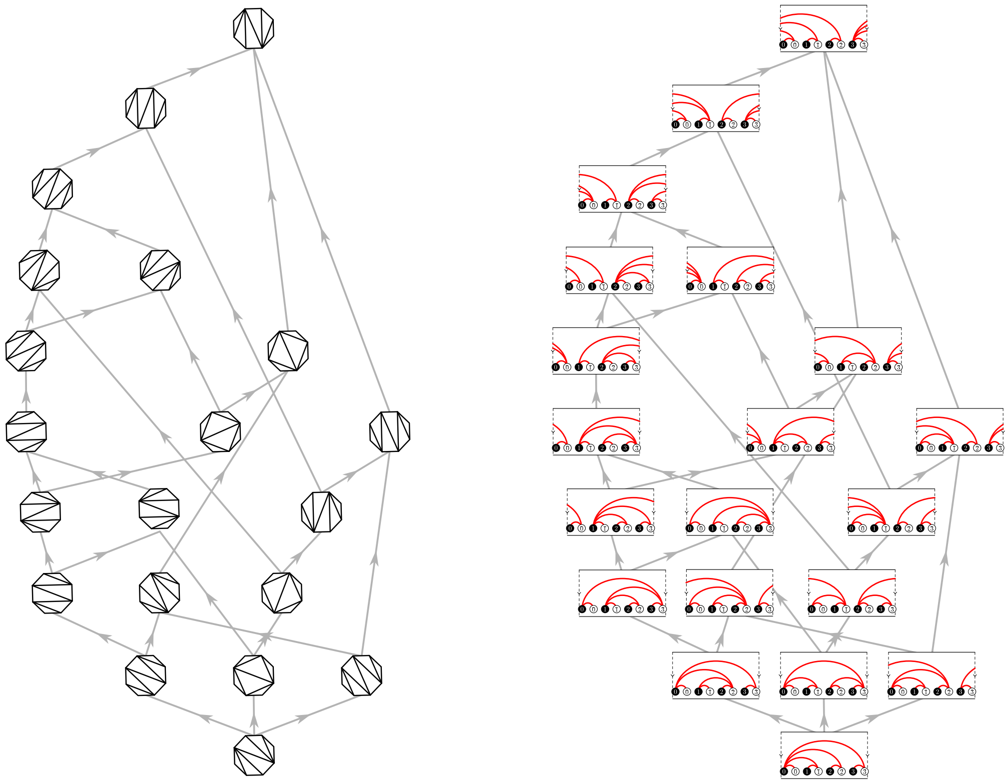


Figure 3. The Hasse diagram of the type B_n Tamari lattice for $n = 3$ from [Tho06, Figure 5] on the left, and the Hasse diagram of the cyclic $([3], [\overline{3}])$-Tamari poset on the right.

Maximal simplices of the restriction of $\mathfrak{C}_n$ to $\Delta_I \times \Delta_{\overline{J}}$ are indexed by *cyclic* $(I, \overline{J})$*-trees*. These are *cyclically non-crossing* subgraphs $K_{I,\overline{J}}$ that can be drawn on the surface of a cylinder to make the parallelism with $(I, \overline{J})$-trees more evident. By analogy with $(I, \overline{J})$-trees, they can be naturally given the structure of a poset that we call the *cyclic* $(I, \overline{J})$*-Tamari poset*. This new poset is a generalization of the type B Tamari lattice, independently discovered by Thomas [Tho06] and

Reading [Rea06], and whose Hasse diagram can be realized geometrically as the edge graph of the cyclohedron. The $n = 3$ case is shown in Figure 3.

The same techniques used for Theorem 6.1 in type A can be used to get tropical realizations of these posets.

THEOREM 6.3 ([16]). *The Hasse diagram of cyclic $(I, \overline{J})$-Tamari poset can be realized geometrically as the edge graph of a polyhedral complex induced by an arrangement of tropical hyperplanes.*

Figures 4 and 5 display $(I, \overline{J})$-cyclohedra corresponding to the first few Fuss-Catalan cases in dimensions two and three. Note that they are polyhedral subdivisions of classical cyclohedra into Cartesian products of associahedra and at most one cyclohedron. The support of the $(I, \overline{J})$-cyclohedron is convex whenever $(I, \overline{J})$ does not have two cyclically consecutive elements of $\overline{J}$ (when $|I| \geq 2$ and $|\overline{J}| \geq 3$), and in this case the convex hull is a classical cyclohedron.

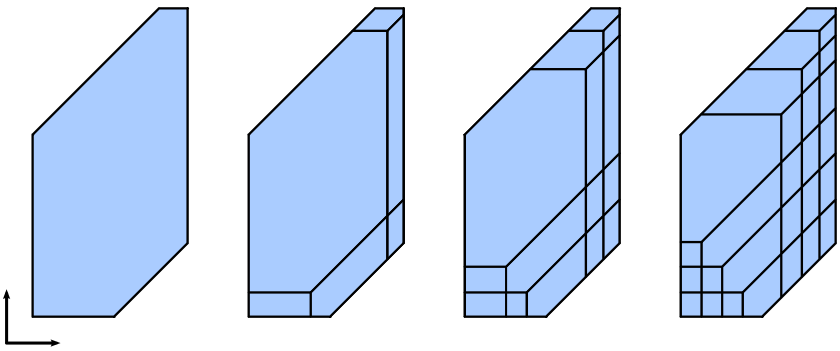


Figure 4. Some Fuss-Catalan $(I, \overline{J})$-cyclohedra in dimension two.

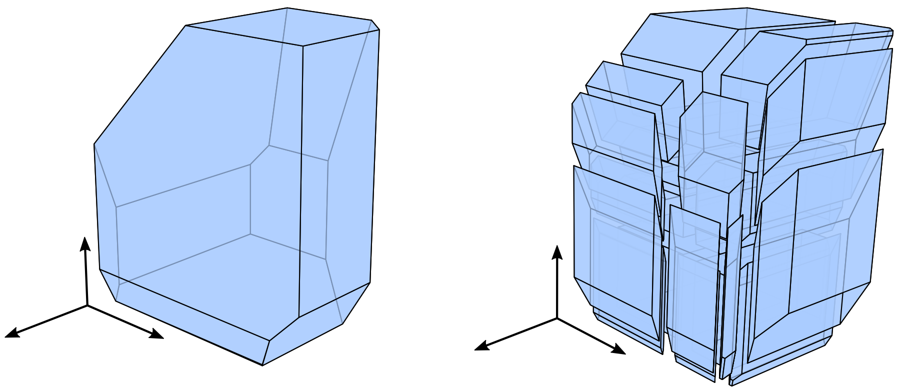


Figure 5. Some Fuss-Catalan $(I, \overline{J})$-cyclohedra in dimension three.

6.3. Dyck path triangulations and extendability

This section reports joint work with Cesar Ceballos and Camilo Sarmiento from "Dyck path triangulations and extendability" [03].

In this section, we introduce another nice family of triangulations of $\Delta_n \times \Delta_{\overline{n}}$. Our presentation describes the associahedral triangulation $\mathfrak{A}_n$ as "non-crossing". Its "non-nesting" counterpart is the restriction of the staircase triangulation of $\Delta_n \times \Delta_{\overline{n}}$ to $\mathcal{U}_n$, and its maximal simplices are described in terms of Dyck paths. Relatives of this triangulation have also appeared under different guises alongside $\mathfrak{A}_n$ in [GGP97; PPS10; SSW17; SP02; Sta86]. We can apply the same cyclic action that constructed the cyclohedral triangulation $\mathfrak{C}_n$ from the associahedral triangulation $\mathfrak{A}_n$ to obtain a full triangulation of $\Delta_n \times \Delta_{\overline{n}}$ called the *Dyck path triangulation* $\mathfrak{D}_{n+1}$.

The study of the Dyck path triangulation is motivated by extendability problems of partial triangulations of $\Delta_m \times \Delta_{\overline{n}}$. The *k-skeleton* of Δ_m, which we denote by $\Delta_m^{(k)}$, is the polyhedral complex of all faces of Δ_m of dimension less than or equal to k. A *partial triangulation* of $\Delta_m \times \Delta_{\overline{n}}$ is a triangulation of the polyhedral complex $\Delta_m^{(k)} \times \Delta_{\overline{n}}$. Such a triangulation is said to be *extendable* if it is equal to the restriction of a triangulation of $\Delta_m \times \Delta_{\overline{n}}$ to $\Delta_m^{(k)} \times \Delta_{\overline{n}}$. The question of extendability of triangulations of $\Delta_m^{(k)} \times \Delta_{\overline{n}}$ was first systematically considered for $k = 1$ by Ardila and Ceballos in [AC13], who completely characterized the extendable triangulations of $\Delta_2^{(1)} \times \Delta_{\overline{n}}$. There, in an attempt to prove the *Spread Out Simplices Conjecture* of Ardila and Billey [AB07, Conjecture 7.1], the authors formulated the *Acyclic System Conjecture* [AC13, Conjecture 5.7], which concerned a sufficient condition for the extendability of triangulations of $\Delta_m^{(1)} \times \Delta_{\overline{n}}$. Shortly after, however, the Acyclic System Conjecture was disproved by Santos [San13]. These results motivate the search for necessary and sufficient conditions for extendability.

Our first contribution in [03] is the following extendability theorem.

THEOREM 6.4 ([03]). *Let m, n, k be nonnegative integers such that $m \geq k > n$. Every triangulation of $\Delta_m^{(k)} \times \Delta_{\overline{n}}$ extends to a unique triangulation of $\Delta_m \times \Delta_{\overline{n}}$.*

In considering whether the bound $k > n$ in Theorem 6.4 is optimal, we are led to the Dyck path triangulation of $\Delta_n \times \Delta_{\overline{n}}$. This triangulation is the main tool to explicitly construct a family of partial triangulations that shows that the assertion of Theorem 6.4 does not generally hold when $m > k = n$.

THEOREM 6.5 ([03]). *For every positive n there is a non-extendable triangulation of $\partial(\Delta_{n+1}) \times \Delta_{\overline{n}}$.*

Apart from providing a characterization of extendable triangulations of $\Delta_m^{(k)} \times \Delta_{\overline{n}}$, these results admit additional interpretations that render them of broader interest.

In particular, Theorems 6.4 and 6.5 naturally translate into the language of tropical oriented matroids (which we abbreviate as TOMs). This concept was introduced by Ardila and Develin as an analogue of classical oriented matroids for the tropical semiring [AD09]. The combinatorics of an arrangement of m tropical pseudohyperplanes in the tropical space $\mathbb{T}^{n-1}$ is captured by its TOM. The *Topological Representation Theorem* establishes a correspondence between TOMs (with parameters (m, n)) and subdivisions of $\Delta_m \times \Delta_{\overline{n}}$ [AD09; Hor12; OY11]. More concretely, triangulations of $\Delta_m \times \Delta_{\overline{n}}$ correspond to generic TOMs and triangulations of $\Delta_m^{(k)} \times \Delta_{\overline{n}}$ correspond to compatible collections of generic subarrangements of k pseudohyperplanes. In this context, our results imply the following statement.

Corollary 6.6 ([03]). *The TOM of any generic arrangement of tropical pseudohyperplanes in $\mathbb{T}^{n-1}$ is completely determined by the TOMs of its subarrangements of n pseudohyperplanes.*

If $\mathcal{M}$ is a TOM of an arrangement whose pseudohyperplanes have labels in $[m]$, denote by $\mathcal{M}|_S$ the TOM of the subarrangement corresponding to the hyperplanes with labels in $S \subseteq [m]$. Theorem 6.4 can be read as follows.

Corollary 6.7 ([03]). *For each $S \in \binom{[m]}{n+1}$, let $\mathcal{M}_S$ be the TOM of a generic arrangement of $n+1$ pseudohyperplanes in $\mathbb{T}^{n-1}$ with labels in S. If the TOMs in this collection are compatible in their intersections, then there exists a unique arrangement of m pseudohyperplanes in $\mathbb{T}^{n-1}$ whose TOM $\mathcal{M}$ fulfills $\mathcal{M}|_S = \mathcal{M}_S$.*

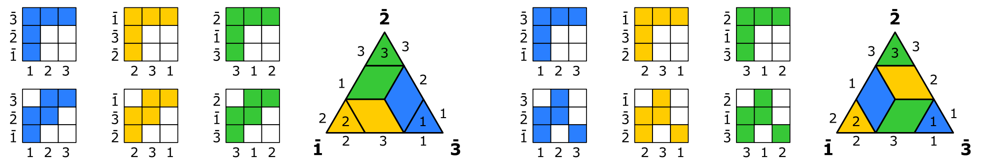
These corollaries should be compared with analogous results in classical oriented matroid theory: every oriented matroid of rank $n-1$ is completely determined by its submatroids with n elements and every compatible collection of submatroids with $n+1$ elements can be completed to a full oriented matroid (cf. [Bjö+99, Corollaries 3.6.3 and 3.6.4]).

Appendix 6.3. Tools for proving Theorem 6.4

6.3.A. Matching ensemble representation. Identifying the vertices of $\Delta_m \times \Delta_{\overline{n}}$ with the edges of $K_{n,\overline{m}}$, every triangulation of $\Delta_m \times \Delta_{\overline{n}}$ gives rise to a collection of perfect matchings on all subgraphs of $K_{n,\overline{m}}$ induced by subsets $I \subset [m]$ and $\overline{J} \subset [\overline{n}]$ of the same cardinality. Roughly, it collects the information of what subset of every circuit of $\Delta_m \times \Delta_{\overline{n}}$ appears as a simplex of the triangulation.

Suho Oh and Hwanchul Yoo [OY13] found a concise characterization of those collections of perfect matchings which correspond to triangulations of $\Delta_m \times \Delta_{\overline{n}}$, hence discovering a novel *matching ensemble representation* for triangulations of $\Delta_m \times \Delta_{\overline{n}}$. This combinatorial description turns out to be very practical to describe triangulations and determine their extendability.

6.3.B. The Dyck path triangulation (and some relatives). The Dyck path triangulation $\mathfrak{D}_n$ can be described in terms of *Dyck paths* in the grid representation $\boxplus_{n\times n}$, that is, monotonically increasing paths from the square $(1, \overline{1})$ to the square $(n, \overline{n})$ of $\boxplus_{n\times n}$, in which every square $(i, \overline{j})$ satisfies $i \leq \overline{j}$. The maximal simplices of $\mathfrak{D}_n$ are the Dyck paths in $\boxplus_{n\times n}$, together with the orbit of simplices they generate under an action that cyclically shifts the indices in both factors of $\Delta_{n-1} \times \Delta_{\overline{n-1}}$ simultaneously. One example is depicted in Figure 6(a).



(a) The triangulation $\mathfrak{D}_3$ of $\Delta_2 \times \Delta_{\overline{2}}$. **(b)** The triangulation $\mathfrak{Q}_3$ of $\Delta_2 \times \Delta_{\overline{2}}$.

Figure 6. The Dyck path triangulation of $\Delta_2 \times \Delta_{\overline{2}}$ and its flipped version in the grid and mixed subdivisions representations.

For us, the crucial property of $\mathfrak{D}_n$ that underlies the construction for Theorem 6.5 is that it admits a *geometric bistellar flip* supported on the central circuit. Performing this flip does not alter the restriction of $\mathfrak{D}_n$ to the boundary of $\Delta_{n-1} \times \Delta_{\overline{n-1}}$. The resulting triangulation is the *flipped Dyck path triangulation* $\mathfrak{Q}_n$; illustrated for $n = 2$ in Figure 6(b).

The next ingredient is a natural extension of $\mathfrak{D}_n$ to a triangulation of $\Delta_n \times \Delta_{\overline{n-1}}$ called the *extended Dyck path triangulation* and denoted by $\mathfrak{D}_n^{ext}$. The simplices of the extended Dyck path triangulation for $n = 3$ are shown in Figure 7. The restriction of $\mathfrak{D}_n^{\text{ext}}$ to $\partial(\Delta_n) \times \Delta_{\overline{n-1}}$ gives a partial triangulation whose restriction to a facet coincides with $\mathfrak{D}_n$. Replacing this instance of $\mathfrak{D}_n$ by $\mathfrak{Q}_n$, without modifying the triangulation of the remaining facets of $\partial(\Delta_n) \times \Delta_{\overline{n-1}}$ provides our non-extendable partial triangulation.

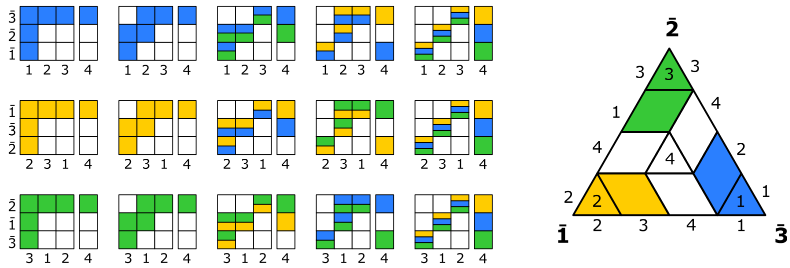


Figure 7. The triangulation $\mathfrak{D}_3^{\text{ext}}$ of $\Delta_3 \times \Delta_{\overline{2}}$ in the grid and mixed subdivision representations. (The grid of a simplex is colored with more than one color if it appears in more than one cyclic shift.)

6.4. Open problems and perspectives

In the original applications of tropical geometry, one tropicalizes an algebraic variety to get a piecewise linear object. Algebraic properties of the original variety become combinatorial features of a polyhedral complex, which are hopefully more approachable. Tropicalization has also been succesfully used to simplify polyhedra and answer computational complexity questions [All+18]. In [16] we use a somehow reverse strategy. We answer a question from combinatorial geometry by means of tropical geometry. Of course, many other problems on polytopal realizability can be approached with this technique; specially if one considers general tropical varieties and not only tropical hyperplane arrangements. A first challenge would be to find polytopal realizations for the *s-permutahedra* introduced in [CP19], which are a kind of permutahedral analogues of ν-Tamari complexes.

The generalization of triangulations to multitriangulations naturally leads to the definition of the *(simplicial) multiassociahedron*, which is the simplicial complex of $(k+1)$-crossing-free subsets of diagonals of a convex $(n+2)$-gon. It is conjectured that this complex should be realizable as the boundary complex of a simplicial polytope [Jon05], whose dual would be a simple polytope $\Delta^*_{n+2,k}$ known as the *simple multiassociahedron*. Surprisingly, despite the multiple different known realizations of the associahedron, only for very few cases this is known to hold (see the introductions of [BCL15] and [Man17], and the references therein, for the current knowledge on the existence of these polytopes). Proving the polytopality of general multiassociahedra is wide open.

In [18], we give new interpretations of the ν-Tamari lattice based on the notion of *ν-tree*. In particular, we show that the Hasse diagram of the ν-Tamari lattice can be obtained as the facet adjacency graph of certain subword complex. Subword complexes are simplicial complexes introduced by Knutson and Miller in their study of the Gröbner geometry of Schubert varieties [KM04; KM05]. Thanks to this interpretation, the definition of the multiassociahedron can be naturally generalized to ν-trees, giving rise to the *(k, ν)-Tamari complex*, which is also a subword complex (these are the complexes of k-north-east fillings considered in [SS12]). It specializes to the ν-Tamari complex when $k = 1$, and for $\nu = (NE)^n$ it is the classical k-multiassociahedron.

When $k = 1$, the $(1, \nu)$-Tamari complex is realized by the ν-associahedron that we just presented. And when $\nu = (NE^m)^{k+1}$, we show in [18] that the facet adjacency graph of the (k, ν)-Tamari complex can be realized as the edge graph of a polytopal subdivision of one of the few simple multiassociahedra that are known to be polytopal. It is very tempting to ask whether a similar result might hold for more general k and ν, generalizing our results from Section 6.1 to other multiassociahedra.

Bibliography

[00] Arnau Padrol. “Neighborly and almost neighborly configurations, and their duals”. Ph.D. Thesis. Advisor: Julian Pfeifle. Universitat Politècnica de Catalunya, Mar. 2013.

[01] Arnau Padrol. “Many neighborly polytopes and oriented matroids”. In: *Discrete Comput. Geom.* 50.4 (2013), pp. 865–902.

[02] Karim Adiprasito, Arnau Padrol, and Louis Theran. “Universality theorems for inscribed polytopes and Delaunay triangulations”. In: *Discrete Comput. Geom.* 54.2 (2015), pp. 412–431.

[03] Cesar Ceballos, Arnau Padrol, and Camilo Sarmiento. “Dyck path triangulations and extendability”. In: *J. Combin. Theory Ser. A* 131.0 (2015), pp. 187–208.

[04] Hiroyuki Miyata and Arnau Padrol. “Enumeration of neighborly polytopes and oriented matroids”. In: *Exp. Math.* 24.4 (2015), pp. 489–505.

[05] Benjamin Nill and Arnau Padrol. “The degree of point configurations: Ehrhart theory, Tverberg points and almost neighborly polytopes”. In: *European J. Combin.* 50 (2015). Combinatorial Geometries: Matroids, Oriented Matroids and Applications. Special Issue in Memory of Michel Las Vergnas, pp. 159–179.

[06] Arnau Padrol and Julian Pfeifle. “Polygons as sections of higher-dimensional polytopes”. In: *Electron. J. Combin.* 22.1 (2015), Paper 1.24, 16 pp.

[07] Karim Adiprasito and Arnau Padrol. “A universality theorem for projectively unique polytopes and a conjecture of Shephard”. In: *Israel J. Math.* 211.1 (2016), pp. 239–255.

[08] Bernd Gonska and Arnau Padrol. “Neighborly inscribed polytopes and Delaunay triangulations”. In: *Adv. Geom.* 16.3 (2016), pp. 349–360.

[09] Arnau Padrol. “Extension complexity of polytopes with few vertices or facets”. In: *SIAM J. Discrete Math.* 30.4 (2016), pp. 2162–217.

[10] Arnau Padrol. “Polytopes with few vertices and few facets”. In: *J. Combin. Theory Ser. A* 142 (2016), pp. 177–180.

[11] Karim A. Adiprasito and Arnau Padrol. “The universality theorem for neighborly polytopes”. In: *Combinatorica* 37.2 (2017), pp. 129–136.

[12] Hao Chen and Arnau Padrol. “Scribability problems for polytopes”. In: *European J. Combin.* 64 (2017), pp. 1–26.

[13] Francesco Grande, Arnau Padrol, and Raman Sanyal. “Extension complexity and realization spaces of hypersimplices”. In: *Discrete Comput. Geom.* 59.3 (2018), pp. 621–642.

[14] Jeffrey C. Lagarias, Yusheng Luo, and Arnau Padrol. “Moser’s shadow problem”. In: *Enseign. Math.* 64.3-4 (2018), pp. 477–496.

[15] Karim A. Adiprasito, Philip Brinkmann, Arnau Padrol, Pavel Paták, Zuzana Patáková, and Raman Sanyal. “Colorful simplicial depth, Minkowski sums, and generalized Gale transforms”. In: *Int. Math. Res. Not. IMRN* 6 (2019), pp. 1894–1919.

[16] Cesar Ceballos, Arnau Padrol, and Camilo Sarmiento. “Geometry of ν-Tamari lattices in types A and B”. In: *Trans. Amer. Math. Soc.* 371.4 (2019), pp. 2575–2622.

[17] Arnau Padrol and Günter M. Ziegler. “Six topics on inscribable polytopes”. In: *Advances in Discrete Differential Geometry*. Ed. by Alexander I. Bobenko. Berlin, Heidelberg: Springer, 2016, pp. 407–419.

[18] Cesar Ceballos, Arnau Padrol, and Camilo Sarmiento. “The ν-Tamari lattice as the rotation lattice of ν-trees”. Preprint, 22 pp., arXiv: 1805.03566, submitted for publication. May 2018.

[19] Arnau Padrol, Yann Palu, Vincent Pilaud, and Pierre-Guy Plamondon. “Associahedra for finite type cluster algebras and minimal relations between g-vectors”. Preprint, 67 pp., arXiv: 1906.06861, submitted for publication. July 2019.

[20] Leonardo Martínez-Sandoval and Arnau Padrol. “The convex dimension of hypergraphs and the hypersimplicial Van Kampen-Flores Theorem”. Preprint, 18 pp., arXiv: 1909.01189, submitted for publication. Sept. 2019.

[AB07] Federico Ardila and Sara Billey. “Flag arrangements and triangulations of products of simplices”. In: *Adv. Math.* 214.2 (2007), pp. 495–524.

[AC13] Federico Ardila and Cesar Ceballos. “Acyclic systems of permutations and fine mixed subdivisions of simplices”. In: *Discrete Comput. Geom.* 49.3 (2013), pp. 485–510.

[AD09] Federico Ardila and Mike Develin. “Tropical hyperplane arrangements and oriented matroids”. In: *Math. Z.* 262.4 (2009), pp. 795–816.

[AH+18] Nima Arkani-Hamed, Yuntao Bai, Song He, and Gongwang Yan. “Scattering forms and the positive geometry of kinematics, color and the worldsheet”. In: *Journal of High Energy Physics* 2018.5 (2018), p. 96.

[All+18] Xavier Allamigeon, Pascal Benchimol, Stéphane Gaubert, and Michael Joswig. “Log-barrier interior point methods are not strongly polynomial”. In: *SIAM J. Appl. Algebra Geom.* 2.1 (2018), pp. 140–178.

[Alo86] Noga Alon. “The number of polytopes, configurations and real matroids”. In: *Mathematika* 33.1 (1986), pp. 62–71.

[And71a] Evgeny M. Andreev. “On convex polyhedra in Lobačevskiĭ spaces”. In: *Math. USSR, Sb.* 10 (1971). Translation from Math. Sbornik (N.S.) 81 (123) (1970), pp. 445–478., pp. 413–440.

[And71b] Evgeny M. Andreev. “On convex polyhedra of finite volume in Lobačevskiĭ space”. In: *Math. USSR, Sb.* 12 (1971). Translation from Math. Sbornik (N.S.) 83 (125) (1970), pp. 256–260., pp. 255–259.

[Aro+09] Jorge L. Arocha, Imre Bárány, Javier Bracho, Ruy Fabila, and Luis Montejano. “Very colorful theorems”. In: *Discrete Comput. Geom.* 42.2 (2009), pp. 142–154.

[ARW13] Drew Armstrong, Brendon Rhoades, and Nathan Williams. “Rational associahedra and noncrossing partitions”. In: *Electron. J. Combin.* 20.3 (2013), Paper 54, 27.

[AS16] Karim A. Adiprasito and Raman Sanyal. “Relative Stanley–Reisner theory and Upper Bound Theorems for Minkowski sums”. In: *Publications mathématiques de l’IHÉS* (2016), pp. 1–65.

[AS73] Amos Altshuler and Leon Steinberg. “Neighborly 4-polytopes with 9 vertices”. In: *J. Combinatorial Theory Ser. A* 15 (1973), pp. 270–287.

[AT06] Christos A. Athanasiadis and Eleni Tzanaki. “On the enumeration of positive cells in generalized cluster complexes and Catalan hyperplane arrangements”. In: *J. Algebraic Combin.* 23.4 (2006), pp. 355–375.

[Ath05] Christos A. Athanasiadis. “On a refinement of the generalized Catalan numbers for Weyl groups”. In: *Trans. Amer. Math. Soc.* 357.1 (2005), 179–196 (electronic).

[AZ15] Karim A. Adiprasito and Günter M. Ziegler. “Many projectively unique polytopes”. In: *Invent. Math.* 199.3 (2015), pp. 581–652.

[Bar71] David W. Barnette. “The minimum number of vertices of a simple polytope”. In: *Israel J. Math.* 10 (1971), pp. 121–125.

[Bar73] David W. Barnette. “A proof of the lower bound conjecture for convex polytopes”. In: *Pacific J. Math.* 46 (1973), pp. 349–354.

[Bar81] David W. Barnette. “A family of neighborly polytopes.” In: *Isr. J. Math.* 39 (1981), pp. 127–140.

[BB02] Xiliang Bao and Francis Bonahon. “Hyperideal polyhedra in hyperbolic 3-space”. In: *Bull. Soc. Math. France* 130.3 (2002), pp. 457–491.

[BB98] Eric K. Babson and Louis J. Billera. “The geometry of products of minors”. In: *Discrete Comput. Geom.* 20.2 (1998), pp. 231–249.

[BCL15] Nantel Bergeron, Cesar Ceballos, and Jean-Philippe Labbé. “Fan Realizations of Type A Subword Complexes and Multi-associahedra of Rank 3”. In: *Discrete & Computational Geometry* 54.1 (2015), pp. 195–231.

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real algebraic geometry*. Vol. 36. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1998, pp. x+430.

[Bea+13] LeRoy B. Beasley, Hartmut Klauck, Troy Lee, and Dirk Oliver Theis. “Communication Complexity, Linear Optimization, and lower bounds for the nonnegative rank of matrices (Dagstuhl Seminar 13082)”. In: *Dagstuhl Reports* 3.2 (2013). Ed. by LeRoy B. Beasley, Hartmut Klauck, Troy Lee, and Dirk Oliver Theis, pp. 127–143.

[Bel02] Alexander Below. “Complexity of triangulation”. PhD thesis. Zürich, CH: ETH Zürich, 2002.

[Ber+07] Michael W. Berry, Murray Browne, Amy N. Langville, V. Paul Pauca, and Robert J. Plemmons. “Algorithms and applications for approximate nonnegative matrix factorization”. In: *Computational Statistics & Data Analysis* 52.1 (2007), pp. 155 –173.

[Ber12] François Bergeron. “Combinatorics of r-Dyck paths, r-Parking functions, and the r-Tamari lattices”. Preprint, 36 pp., arXiv: 1202.6269. Feb. 2012.

[Ber73] Abraham Berman. “Rank factorization of nonnegative matrices”. In: *SIAM Review* 15.3 (1973), p. 655.

[BFS90] Louis J. Billera, Paul Filliman, and Bernd Sturmfels. “Constructions and complexity of secondary polytopes”. In: *Adv. Math.* 83.2 (1990), pp. 155–179.

[BG70] David W. Barnette and Branko Grünbaum. “Preassigning the shape of a face”. In: *Pacific J. Math.* 32 (1970), pp. 299–306.

[BGR92] Edward A. Bender, Zhicheng Gao, and L. Bruce Richmond. “Submaps of maps. I. General 0-1 laws”. In: *J. Combin. Theory Ser. B* 55.1 (1992), pp. 104–117.

[Bjö+99] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. *Oriented matroids*. Second. Vol. 46. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1999, pp. xii+548.

[BL09] LeRoy B. Beasley and Thomas J. Laffey. "Real rank versus nonnegative rank". In: *Linear Algebra Appl.* 431.12 (2009), pp. 2330–2335.

[BM+18] Véronique Bazier-Matte, Guillaume Douville, Kaveh Mousavand, Hugh Thomas, and Emine Yıldırım. "ABHY Associahedra and Newton polytopes of F-polynomials for finite type cluster algebras". Preprint, 21 pp., arXiv:1808.09986. Aug. 2018.

[BM07] Imre Bárány and Jiří Matoušek. "Quadratically many colorful simplices". In: *SIAM J. Discrete Math.* 21.1 (2007), pp. 191–198.

[BM88] Edward Bierstone and Pierre D. Milman. "Semianalytic and subanalytic sets". In: *Inst. Hautes Études Sci. Publ. Math.* (1988), pp. 5–42.

[BMCPR13] Mireille Bousquet-Mélou, Guillaume Chapuy, and Louis-François Préville-Ratelle. "The representation of the symmetric group on m-Tamari intervals". In: *Adv. Math.* 247 (2013), pp. 309–342.

[BMFPR11] Mireille Bousquet-Mélou, Éric Fusy, and Louis-François Préville-Ratelle. "The number of intervals in the m-Tamari lattices". In: *Electron. J. Combin.* 18.2 (2011), Paper 31, 26.

[BML87] Roswitha Blind and Peter Mani-Levitska. "Puzzles and polytope isomorphisms". In: *aequationes mathematicae* 34.2 (1987), pp. 287–297.

[BO90] Jürgen Bokowski and António Guedes de Oliveira. "Simplicial convex 4-polytopes do not have the isotopy property". In: *Portugal. Math.* 47.3 (1990), pp. 309–318.

[BPR12] François Bergeron and Louis-François Préville-Ratelle. "Higher trivariate diagonal harmonics via generalized Tamari posets". In: *J. Comb.* 3.3 (2012), pp. 317–341.

[Bro79] Kevin Q. Brown. "Voronoi diagrams from convex hulls." In: *Inf. Process. Lett.* 9 (1979), pp. 223–228.

[BS86] Jürgen Bokowski and Bernd Sturmfels. "On the coordinatization of oriented matroids". In: *Discrete Comput. Geom.* 1.4 (1986), pp. 293–306.

[BS87] Jürgen Bokowski and Ido Shemer. "Neighborly 6-polytopes with 10 vertices". In: *Israel J. Math.* 58.1 (1987), pp. 103–124.

[BS92] Louis J. Billera and Bernd Sturmfels. "Fiber polytopes". In: *Ann. of Math. (2)* 135.3 (1992), pp. 527–549.

[BT94] Raoul Bott and Clifford Taubes. "On the self-linking of knots". In: *Journal of Mathematical Physics* 35.10 (Oct. 1994), pp. 5247–5287.

[BTN01] Aharon Ben-Tal and Arkadi Nemirovski. "On polyhedral approximations of the second-order cone". In: *Math. Oper. Res.* 26.2 (2001), pp. 193–205.

[Bur03] Benjamin Burton. "Minimal triangulations and normal surfaces". PhD thesis. The University of Melbourne, 2003.

[BW88] Edward A. Bender and Nicholas C. Wormald. "The number of rooted convex polyhedra". In: *Canad. Math. Bull.* 31.1 (1988), pp. 99–102.

[Bár82] Imre Bárány. "A generalization of Carathéodory's theorem". In: *Discrete Mathematics* 40.2-3 (1982), pp. 141–152.

[Car11] Constantin Carathéodory. "Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen." In: *Rendiconto del Circolo Matematico di Palermo* 32 (1911), pp. 193–217.

[CCZ10] Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. "Extended formulations in combinatorial optimization". In: *4OR* 8.1 (2010), pp. 1–48.

[CFZ02] Frédéric Chapoton, Sergey Fomin, and Andrei Zelevinsky. "Polytopal realizations of generalized associahedra". In: *Canad. Math. Bull.* 45 (2002), pp. 537–566.

[Cha05] F. Chapoton. "Sur le nombre d'intervalles dans les treillis de Tamari". In: *Sém. Lothar. Combin.* 55 (2005/07), Art. B55f, 18pp.

[Che16] Hao Chen. "Apollonian Ball Packings and Stacked Polytopes". In: *Discrete Comput. Geom.* 55.4 (2016), pp. 801–826.

[Chi+17] Dmitry Chistikov, Stefan Kiefer, Ines Marušić, Mahsa Shirmohammadi, and James Worrell. "Nonnegative matrix factorization requires irrationality". 2017.

[CHT07] Aldo Conca, Serkan Hosten, and Rekha R. Thomas. "Nice Initial Complexes of Some Classical Ideals". In: *Algebraic and geometric combinatorics, Contemp. Math.* 423 (2007), pp. 11–42.

[Cos00] Michel Coste. *An Introduction to Semialgebraic Geometry.* Dottorato di Ricerca in Matematica del Dipartimento di Matematica dell'Università di Pisa. Pisa: Istituti Editoriali e Poligrafici Internazionali, 2000.

[CP19] Cesar Ceballos and Viviane Pons. "Thes-weak order ands-permutahedra". In: *31th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2019).* Sém. Lothar. Combin., to appear. 2019.

[CR93] Joel E. Cohen and Uriel G. Rothblum. "Nonnegative ranks, decompositions, and factorizations of nonnegative matrices". In: *Linear Algebra Appl.* 190 (1993), pp. 149–168.

[CSZ15] Cesar Ceballos, Francisco Santos, and Günter M. Ziegler. "Many non-equivalent realizations of the associahedron". In: *Combinatorica* 35.5 (2015), pp. 513–551.

[Dez+06] Antoine Deza, Sui Huang, Tamon Stephen, and Tamás Terlaky. "Colourful Simplicial Depth". In: *Discrete & Computational Geometry* 35.4 (2006), pp. 597–615.

[DHM19] Michael G. Dobbins, Andreas Holmsen, and Tillmann Miltzow. "A Universality Theorem for Nested Polytopes". Preprint, 20 pp., arXiv: 1908.02213. Aug. 2019.

[DL96] Jesús A. De Loera. “Nonregular triangulations of products of simplices”. In: *Discrete Comput. Geom.* 15.3 (1996), pp. 253–264.

[DLRS10] Jesús A. De Loera, Jörg Rambau, and Francisco Santos. *Triangulations: Structures for Algorithms and Applications.* Vol. 25. Algorithms and Computation in Mathematics. Springer-Verlag, 2010, pp. xiv+535.

[DMS14] Antoine Deza, Frédéric Meunier, and Pauline Sarrabezolles. “A combinatorial approach to colourful simplicial depth”. In: *SIAM J. Discrete Math.* 28.1 (2014), pp. 306–322.

[Dob11] Michael G. Dobbins. “Representations of Polytopes”. PhD thesis. Philadelphia, US: Temple University, 2011.

[Doo+19] Joseph Doolittle, Jean-Philippe Labbé, Carsten E. M. C. Lange, Rainer Sinn, Jonathan Spreer, and Günter M. Ziegler. “Combinatorial inscribability obstructions for higher-dimensional polytopes”. Preprint, 27 pp., arXiv: 1910.05241. Oct. 2019.

[DS04] Mike Develin and Bernd Sturmfels. “Tropical convexity”. In: *Doc. Math.* 9 (2004), 1–27 (electronic).

[DS96] Michael B. Dillencourt and Warren D. Smith. “Graph-theoretical conditions for inscribability and Delaunay realizability”. In: *Discrete Mathematics* 161.1–3 (1996), pp. 63–77.

[DSX11] Antoine Deza, Tamon Stephen, and Feng Xie. “More Colourful Simplices”. In: *Discrete & Computational Geometry* 45.2 (2011), pp. 272–278.

[DT05] David L. Donoho and Jared Tanner. “Neighborliness of randomly projected simplices in high dimensions”. In: *Proc. Natl. Acad. Sci. USA* 102.27 (2005), 9452–9457 (electronic).

[Ede06] Herbert Edelsbrunner. *Geometry and topology for mesh generation.* Cambridge: Cambridge University Press, 2006.

[EKZ03] David Eppstein, Greg Kuperberg, and Günter M. Ziegler. “Fat 4-polytopes and fatter 3-spheres”. In: *Discrete Geometry: In honor of W. Kuperberg’s 60th birthday.* Ed. by A. Bezdek. Vol. 253. Monogr. Textbooks Pure Appl. Math. New York: Marcel Dekker Inc., 2003, pp. 239–265.

[Fio+13] Samuel Fiorini, Volker Kaibel, Kanstantsin Pashkovich, and Dirk Oliver Theis. “Combinatorial bounds on nonnegative rank and extended formulations”. In: *Discrete Mathematics* 313.1 (2013), pp. 67–83.

[Fio+15] Samuel Fiorini, Serge Massar, Sebastian Pokutta, Hans Raj Tiwary, and Ronald de Wolf. “Exponential lower bounds for polytopes in combinatorial optimization”. In: *J. ACM* 62.2 (2015), Art. 17, 23.

[Fir17] Moritz Firsching. “Realizability and inscribability for simplicial polytopes via nonlinear optimization”. In: *Mathematical Programming* 166.1–2 (2017), pp. 273 –295.

[FPR17] Wenjie Fang and Louis-François Préville-Ratelle. “The enumeration of generalized Tamari intervals”. In: *European Journal of Combinatorics* 61 (Mar. 2017), pp. 69–84.

[FR05] Sergey Fomin and Nathan Reading. “Generalized cluster complexes and Coxeter combinatorics”. In: *Int. Math. Res. Not.* 44 (2005), pp. 2709–2757.

[FR15] Alex Fink and Felipe Rincón. “Stiefel tropical linear spaces”. In: *J. Combin. Theory Ser. A* 135 (2015), pp. 291–331.

[FRT12] Samuel Fiorini, Thomas Rothvoß, and Hans Raj Tiwary. “Extended formulations for polygons”. In: *Discrete Comput. Geom.* 48.3 (2012), pp. 658–668.

[Fus06] Éric Fusy. “Counting d-polytopes with $d+3$ vertices”. In: *Electron. J. Combin.* 13.1 (2006), Research Paper 23, 25 pp.

[FZ02] Sergey Fomin and Andrei Zelevinsky. “Cluster algebras. I. Foundations”. In: *J. Amer. Math. Soc.* 15.2 (2002), 497–529 (electronic).

[FZ03a] Sergey Fomin and Andrei Zelevinsky. “Cluster algebras. II. Finite type classification”. In: *Invent. Math.* 154.1 (2003), pp. 63–121.

[FZ03b] Sergey Fomin and Andrei Zelevinsky. “Y-systems and generalized associahedra”. In: *Ann. of Math. (2)* 158 (2003), pp. 977–1018.

[Gel+87] Israel M. Gel′fand, R. Mark Goresky, Robert D. MacPherson, and Vera V. Serganova. “Combinatorial geometries, convex polyhedra, and Schubert cells”. In: *Adv. in Math.* 63.3 (1987), pp. 301–316.

[GG12] Nicolas Gillis and François Glineur. “On the geometric interpretation of the nonnegative rank”. In: *Linear Algebra Appl.* 437.11 (2012), pp. 2685–2712.

[GGP97] Israel M. Gel′fand, Mark I. Graev, and Alexander Postnikov. “Combinatorics of hypergeometric functions associated with positive roots”. In: *The Arnold-Gelfand mathematical seminars*. Birkhäuser Boston, Boston, MA, 1997, pp. 205–221.

[GKZ94] Israel M. Gel′fand, Mikhail M. Kapranov, and Andrey V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants.* Mathematics: Theory & Applications. Boston, MA: Birkhäuser Boston Inc., 1994, pp. x+523.

[Gon13] Bernd Gonska. “Inscribable polytopes via Delaunay triangulations”. Ph.D. Thesis. Freie Universität Berlin, 2013.

[GP86] Jacob E. Goodman and Richard Pollack. “Upper bounds for configurations and polytopes in $\mathbf{R}^d$.” In: *Discrete Comput. Geom.* 1 (1986), pp. 219–227.

[GPT13] João Gouveia, Pablo A. Parrilo, and Rekha R. Thomas. “Lifts of convex sets and cone factorizations”. In: *Math. Oper. Res.* 38.2 (2013), pp. 248–264.

[Grü03] Branko Grünbaum. *Convex polytopes*. Second. Vol. 221. Graduate Texts in Mathematics. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler. New York: Springer-Verlag, New York, 2003, pp. xvi+468.

[GS17] Francesco Grande and Raman Sanyal. “Theta rank, levelness, and matroid minors”. In: *Journal of Combinatorial Theory, Series B* 123 (2017), pp. 1 –31.

[GS87] Branko Grünbaum and Geoffrey C. Shephard. “Some problems on polyhedra”. In: *J. Geom.* 29.2 (1987), pp. 182–190.

[GS93] Peter Gritzmann and Bernd Sturmfels. “Minkowski addition of polytopes: computational complexity and applications to Gröbner bases”. In: *SIAM J. Discrete Math.* 6.2 (1993), pp. 246–269.

[GZ13] Bernd Gonska and Günter M. Ziegler. “Inscribable stacked polytopes.” In: *Adv. Geom.* 13.4 (2013), pp. 723–740.

[GZK90] Izrail M. Gel’fand, Andrei V. Zelevinskiĭ, and Mikhail M. Kapranov. “Newton polytopes of principal A-determinants”. In: *Soviet Math. Doklady* 40 (1990), pp. 278–281.

[GZK91] Izrail M. Gel’fand, Andrei V. Zelevinskiĭ, and Mikhail M. Kapranov. “Discriminants of polynomials in several variables and triangulations of Newton polyhedra”. In: *Leningrad Math. J.* 2 (1991), pp. 449–505.

[Gün96] Harald Günzel. “The universal partition theorem for oriented matroids”. In: *Discrete Comput. Geom.* 15 (1996), pp. 121–145.

[Hai84] Mark Haiman. “Constructing the associahedron”. Unpublished manuscript, 11 pages, available at http://www.math.berkeley.edu/~mhaiman/ftp/assoc/manuscript.pdf. 1984.

[Hai91] Mark Haiman. “A simple and relatively efficient triangulation of the n-cube”. In: *Discrete Comput. Geom.* 6.4 (1991), pp. 287–289.

[HL07] Christophe Hohlweg and Carsten E. M. C. Lange. “Realizations of the Associahedron and Cyclohedron”. In: *Discrete Comput. Geometry* 37 (2007), pp. 517–543.

[HLT11] Christophe Hohlweg, Carsten E. M. C. Lange, and Hugh Thomas. “Permutahedra and generalized associahedra”. In: *Advances in Math.* 226 (2011), pp. 608–640.

[HOR07] Nir Halman, Shmuel Onn, and Uriel G. Rothblum. “The convex dimension of a graph”. In: *Discrete Appl. Math.* 155.11 (2007), pp. 1373 –1383.

[Hor12] Silke Horn. “A topological representation theorem for tropical oriented matroids”. In: *24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012)*. Discrete Math.

Theor. Comput. Sci. Proc., AR. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2012, pp. 135–146.

[HPT08] Andreas Holmsen, János Pach, and Helge Tverberg. "Points surrounding the origin." In: *Combinatorica* 28.6 (2008), pp. 633–644.

[HRGZ04] Martin Henk, Jürgen Richter-Gebert, and Günter M. Ziegler. "Basic properties of convex polytopes". In: *Handbook of discrete and computational geometry*. Second. Discrete Mathematics and its Applications. Chapman & Hall/CRC, 2004, pp. 355–382.

[HRS00] Birkett Huber, Jörg Rambau, and Francisco Santos. "The Cayley trick, lifting subdivisions and the Bohne-Dress theorem on zonotopal tilings". In: *J. Eur. Math. Soc. (JEMS)* 2.2 (2000), pp. 179–198.

[HRS92] Craig D. Hodgson, Igor Rivin, and Warren D. Smith. "A characterization of convex hyperbolic polyhedra and of convex polyhedra inscribed in the sphere". In: *Bull. Amer. Math. Soc. (N.S.)* 27.2 (1992), pp. 246–251.

[HS95] Birkett Huber and Bernd Sturmfels. "A Polyhedral Method for Solving Sparse Polynomial Systems". In: *Mathematics of Computation* 64.212 (1995), pp. 1541–1555.

[Iva12] Sergei Ivanov. *Can all convex polytopes be realized with vertices on surface of convex body?* MathOverflow. URL: http://mathoverflow.net/q/107113 (version: 2012-09-13). 2012.

[Jag+89] Beat Jaggi, Peter Mani-Levitska, Bernd Sturmfels, and Neil White. "Uniform oriented matroids without the isotopy property". In: *Discrete Comput. Geom.* 4.2 (1989), pp. 97–100.

[JK10] Michael Joswig and Katja Kulas. "Tropical and ordinary convexity combined". In: *Adv. Geom.* 10.2 (2010), pp. 333–352.

[Jon05] Jakob Jonsson. "Generalized triangulations and diagonal-free subsets of stack polyominoes". In: *Journal of Combinatorial Theory, Series A* 112.1 (2005), pp. 117 –142.

[JZ00] M. Joswig and G. M. Ziegler. "Neighborly cubical polytopes". In: vol. 24. 2-3. The Branko Grünbaum birthday issue. 2000, pp. 325–344.

[Kai11] Volker Kaibel. "Extended Formulations in Combinatorial Optimization". In: *Optima* 85 (2011), 14 pp.

[Kal04] Gil Kalai. "Polytope skeletons and paths". In: *in "Handbook of Discrete and Computational Geometry"*. Ed. by J. E. Goodman and J. O'Rourke. 2nd. Chapman & Hall/CRC, Boca Raton, FL, 2004, pp. 455–476.

[Kal12] Gil Kalai. *Open problems for convex polytopes I'd love to see solved.* Talk on Workshop "Convex Polytopes" at RIMS Kyoto, slides available on gilkalai.files.wordpress.com/2012/08/kyoto-3.pdf. July 2012.

[Kal94] Gil Kalai. "Some aspects of the combinatorial theory of convex polytopes". In: *Polytopes: abstract, convex and computational (Scarborough, ON, 1993)*. Vol. 440. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. Kluwer Acad. Publ., Dordrecht, 1994, pp. 205–229.

[Kla+15] Hartmut Klauck, Troy Lee, Dirk Oliver Theis, and Rekha R. Thomas. "Limitations of Convex Programming: Lower Bounds on Extended Formulations and Factorization Ranks (Dagstuhl Seminar 15082)". In: *Dagstuhl Reports* 5.2 (2015). Ed. by Hartmut Klauck, Troy Lee, Dirk Oliver Theis, and Rekha R. Thomas, pp. 109–127.

[KM04] Allen Knutson and Ezra Miller. "Subword complexes in Coxeter groups". In: *Adv. Math.* 184.1 (May 2004), pp. 161–176.

[KM05] Allen Knutson and Ezra Miller. "Gröbner geometry of Schubert polynomials". In: *Ann. Math. (2)* 161.3 (2005), pp. 1245–1318.

[KM99] Michael Kapovich and John J. Millson. "Moduli spaces of linkages and arrangements". In: *Advances in Geometry*. Vol. 172. Progr. Math. Boston, MA: Birkhäuser Boston, 1999, pp. 237–270.

[Koe36] Paul Koebe. "Kontaktprobleme der konformen Abbildung". In: *Berichte Verh. Sächs. Akademie der Wissenschaften Leipzig, Math.-Phys. Klasse* 88 (1936), pp. 141–164.

[Kor97] Ulirch H. Kortenkamp. "Every simplicial polytope with at most $d+4$ vertices is a quotient of a neighborly polytope". In: *Discrete Comput. Geom.* 18 (1997), pp. 455–462.

[KRS15] Kaie Kubjas, Elina Robeva, and Bernd Sturmfels. "Fixed points EM algorithm and nonnegative rank boundaries". In: *Ann. Statist.* 43.1 (2015), pp. 422–461.

[Köm80] Magelone Kömhoff. "On a combinatorial problem concerning subpolytopes of stack polytopes". In: *Geometriae Dedicata* 9 (1 1980), pp. 73–76.

[Lee89] Carl W. Lee. "The associahedron and triangulations of the n-gon". In: *European J. Combin.* 10.6 (1989), pp. 551–560.

[Lin71] Bernt Lindström. "On the realization of convex polytopes, Euler's formula and Möbius functions". In: *Aequationes Math.* 6 (1971), pp. 235–240.

[Liu90] Regina Y. Liu. "On a notion of data depth based on random simplices". In: *Ann. Statist.* 18.1 (1990), pp. 405–414.

[Lod04] Jean L. Loday. "Realization of the Stasheff polytope". In: *Arch. Math.* 83 (2004), pp. 267–278.

[LS99] Daniel D. Lee and H. Sebastian Seung. "Learning the parts of objects by nonnegative matrix factorization". In: *Nature* 401 (1999), pp. 788–791.

[Man17] Thibault Manneville. "Fan Realizations for Some 2-Associahedra". In: *Experimental Mathematics* 0.0 (2017), pp. 1–18.

[Man74] Peter Mani. “Inner illumination of convex polytopes”. In: *Comment. Math. Helv.* 49.1 (1974), pp. 65–73.

[Mar81] Daniel A. Marcus. “Minimal positive 2-spanning sets of vectors”. In: *Proc. Am. Math. Soc.* 82 (1981), pp. 165–172.

[Mar84] Daniel A. Marcus. “Gale diagrams of convex polytopes and positive spanning sets of vectors”. In: *Discrete Appl. Math.* 9 (1984), pp. 47–67.

[Mar99] Martin Markl. “Simplex, associahedron, and cyclohedron”. In: *Higher homotopy structures in topology and mathematical physics (Poughkeepsie, NY, 1996).* Vol. 227. Contemp. Math. Amer. Math. Soc., Providence, RI, 1999, pp. 235–265.

[Mat02] Jiří Matoušek. *Lectures on discrete geometry.* Vol. 212. Graduate Texts in Mathematics. Springer-Verlag, New York, 2002, pp. xvi+481.

[McM70] Peter McMullen. “The maximum numbers of faces of a convex polytope.” In: *Mathematika, Lond.* 17 (1970), pp. 179–184.

[McM71] Peter McMullen. “On zonotopes”. In: *Trans. Amer. Math. Soc.* 159 (1971), pp. 91–109.

[McM73] Peter McMullen. “Representations of polytopes and polyhedral sets”. In: *Geometriae Dedicata* 2 (1973), pp. 83–99.

[McM76] Peter McMullen. “Constructions for projectively unique polytopes”. In: *Discrete Mathematics* 14.4 (1976), pp. 347 –358.

[McM79] Peter McMullen. “Transforms, diagrams and representations”. In: *Contributions to Geometry (Proc. Geom. Sympos., Siegen, 1978).* Ed. by Jürgen Tölke and Jörg M. Wills. Birkhäuser, Basel-Boston, Mass., 1979, pp. 92–130.

[MD13] Frédéric Meunier and Antoine Deza. “A further generalization of the colourful Carathéodory theorem”. In: *Discrete geometry and optimization. Selected papers based on the presentations at the conference and workshop, Toronto, Canada, September 19–23, 2011.* Vol. 69. Fields Institute Communications. New York: Springer, 2013, pp. 179–190.

[MHPS12] Folkert Müller-Hoissen, Jean Marcel Pallo, and Jim Stasheff, eds. *Associahedra, Tamari Lattices and Related Structures. Tamari Memorial Festschrift.* Vol. 299. Progress in Mathematics. Birkhäuser Basel, 2012, pp. xx+433.

[Mil+97] Gary L. Miller, Shang-Hua Teng, William Thurston, and Stephen A. Vavasis. “Separators for sphere-packings and nearest neighbor graphs”. In: *J. ACM* 44.1 (1997), pp. 1–29.

[Mnë85] Nikolai E. Mnëv. “On manifolds of combinatorial types of projective configurations and convex polyhedra”. In: *Sov. Math., Dokl.* 32 (1985), pp. 335–337.

[Mnë86] Nikolai E. Mnëv. "The topology of configuration varieties and convex polytopes varieties". Russian. 116 pages, pdmi.ras.ru/~mnev/mnev_phd1.pdf. PhD thesis. St. Petersburg, RU: St. Petersburg State University, 1986.

[Mnë88] Nikolai E. Mnëv. "The universality theorems on the classification problem of configuration varieties and convex polytopes varieties". In: *Topology and Geometry—Rohlin Seminar*. Vol. 1346. Lecture Notes in Math. Berlin Heidelberg: Springer-Verlag, 1988, pp. 527–544.

[Mos66] Leo Moser. "Poorly formulated unsolved problems of combinatorial geometry". Mimeographed notes. (East Lansing conference.) 1966.

[Mos91] William O. J. Moser. "Problems, problems, problems". In: vol. 31. 2. First Canadian Conference on Computational Geometry (Montreal, PQ, 1989). 1991, pp. 201–225.

[MPP11] Benjamin Matschke, Julian Pfeifle, and Vincent Pilaud. "Prodsimplicial-neighborly polytopes". In: *Discrete Comput. Geom.* 46.1 (2011), pp. 100–131.

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*. Vol. 161. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2015, pp. xii+363.

[MS68] Peter McMullen and Geoffrey C. Shephard. "Diagrams for centrally symmetric polytopes". In: *Mathematika* 15 (1968), pp. 123–138.

[Més11] Karola Mészáros. "Root polytopes, triangulations, and the subdivision algebra. I". In: *Trans. Amer. Math. Soc.* 363.8 (2011), pp. 4359–4382.

[Nov14] Jean-Christophe Novelli. "m-Dendriform algebras". Preprint, 21 pp., arXiv:1406.1616. June 2014.

[NT14] Jean-Christophe Novelli and Jean-Yves Thibon. "Hopf algebras of m-permutations,(m+ 1)-ary trees, and m-parking functions". Preprint, 51 pp., arXiv:1403.5962. June 2014.

[OS03] David Orden and Francisco Santos. "Asymptotically efficient triangulations of the d-cube". In: *Discrete Comput. Geom.* 30.4 (2003), pp. 509–528.

[Oxl11] James G. Oxley. *Matroid theory*. Second. Vol. 21. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2011, pp. xiv+684.

[OY11] Suho Oh and Hwanchul Yoo. "Triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ and tropical oriented matroids". In: *23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011)*. Discrete Math. Theor. Comput. Sci. Proc., AO. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2011, pp. 717–728.

[OY13] Suho Oh and Hwanchul Yoo. “Triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ and Matching Ensembles”. Nov. 2013.

[Pil20] Vincent Pilaud. “Cambrian triangulations and their tropical realizations”. In: *European J. Combin.* 83 (2020), pp. 102997, 19.

[Pos09] Alexander Postnikov. “Permutohedra, associahedra, and beyond”. In: *Int. Math. Res. Not. IMRN* 6 (2009), pp. 1026–1106.

[PPS10] T. Kyle Petersen, Pavlo Pylyavskyy, and David E. Speyer. “A noncrossing standard monomial theory”. In: *J. Algebra* 324.5 (2010), pp. 951–969.

[PRV17] Louis-François Préville-Ratelle and Xavier Viennot. “An extension of Tamari lattices”. In: *Trans. Amer. Math. Soc.* 369.7 (2017), pp. 5219–5239.

[PS15] Vincent Pilaud and Christian Stump. “Brick polytopes of spherical subword complexes and generalized associahedra”. In: *Adv. Math.* 276 (2015), pp. 1–61.

[PS19] Vincent Pilaud and Francisco Santos. “Quotientopes”. In: *Bull. Lond. Math. Soc.* 51.3 (2019), pp. 406–420.

[Rea06] Nathan Reading. “Cambrian lattices”. In: *Adv. Math.* 205.2 (2006), pp. 313–353.

[RG11] Jürgen Richter-Gebert. *Perspectives on Projective Geometry*. Heidelberg: Springer, 2011, pp. xxii+571.

[RG95] Jürgen Richter-Gebert. “Mnëv’s universality theorem revisited”. In: *Sém. Lothar. Combin.* 34 (1995), Art. B34h, approx. 15 pp. (electronic).

[RG96] Jürgen Richter-Gebert. *Realization Spaces of Polytopes*. Vol. 1643. Lecture Notes in Mathematics. Berlin: Springer, 1996, pp. xii+187.

[RG99] Jürgen Richter-Gebert. “The universality theorems for oriented matroids and polytopes”. In: *Advances in discrete and computational geometry (South Hadley, MA, 1996)*. Vol. 223. Contemp. Math. Amer. Math. Soc., Providence, RI, 1999, pp. 269–292.

[RGZ95] Jürgen Richter-Gebert and Günter M. Ziegler. “Realization spaces of 4-polytopes are universal”. In: *Bulletin Amer. Math. Soc.* 32 (1995), pp. 403–412.

[Riv03] Igor Rivin. “Combinatorial optimization in geometry.” In: *Adv. Appl. Math.* 31.1 (2003), pp. 242–271.

[Riv94] Igor Rivin. “Euclidean structures on simplicial surfaces and hyperbolic volume.” In: *Ann. Math. (2)* 139.3 (1994), pp. 553–580.

[Riv96] Igor Rivin. “A characterization of ideal polyhedra in hyperbolic 3-space”. In: *Ann. of Math. (2)* 143.1 (1996), pp. 51–70.

[Rod03] Simion Rodica. “A type-B associahedron”. In: *Adv. in Appl. Math.* 30.1-2 (2003). Formal power series and algebraic combinatorics (Scottsdale, AZ, 2001), pp. 2–25.

[Rot13] Thomas Rothvoß. “Some 0/1 polytopes need exponential size extended formulations”. In: *Math. Program.* 142.1-2, Ser. A (2013), pp. 255–268.

[Rot14] Thomas Rothvoß. “The Matching Polytope Has Exponential Extension Complexity”. In: *Proceedings of the 46th Annual ACM Symposium on Theory of Computing.* STOC ’14. New York, New York: ACM, 2014, pp. 263–272.

[RS09] Nathan Reading and David E. Speyer. “Cambrian fans”. In: *J. Eur. Math. Soc. (JEMS)* 11.2 (2009), pp. 407–447.

[RS12] Thilo Rörig and Raman Sanyal. “Non-projectability of polytope skeleta”. In: *Adv. Math.* 229.1 (2012), pp. 79–101.

[RSS03] Günter Rote, Francisco Santos, and Ileana Streinu. “Expansive motions and the polytope of pointed pseudo-triangulations”. In: *Discrete and computational geometry.* Vol. 25. Algorithms Combin. Springer, Berlin, 2003, pp. 699–736.

[RW82] L. Bruce Richmond and Nicholas C. Wormald. “The asymptotic number of convex polyhedra”. In: *Trans. Amer. Math. Soc.* 273.2 (1982), pp. 721–735.

[San00] Francisco Santos. “A point set whose space of triangulations is disconnected”. In: *J. Amer. Math. Soc.* 13.3 (2000), pp. 611–637.

[San05] Francisco Santos. “The Cayley trick and triangulations of products of simplices”. In: *Integer Points in Polyhedra – Geometry, Number Theory, Algebra, Optimization, Contemp. Math.* Contemp. Math. 374 (2005), pp. 151–177.

[San13] Francisco Santos. “Some acyclic systems of permutations are not realizable by triangulations of a product of simplices”. In: *Algebraic and Combinatorial Aspects of Tropical Geometry, Contemp. Math.* 589 (2013), pp. 317–328.

[Sar15] Pauline Sarrabezolles. “The colourful simplicial depth conjecture”. In: *Journal of Combinatorial Theory, Series A* 130.0 (2015), pp. 119 –128.

[Sch05] Jean-Marc Schlenker. “Hyperideal circle patterns”. In: *Math. Res. Lett.* 12.1 (2005), pp. 85–102.

[Sch87] Egon Schulte. “Analogues of Steinitz’s theorem about non-inscribable polytopes”. In: *Intuitive geometry (Siófok, 1985).* Vol. 48. Colloq. Math. Soc. János Bolyai. North-Holland, Amsterdam, 1987, pp. 503–516.

[Sch92] Oded Schramm. “How to cage an egg”. In: *Invent. Math.* 107.3 (1992), pp. 543–560.

[Sch93] Rolf Schneider. *Convex bodies: the Brunn-Minkowski theory.* Vol. 44. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1993, pp. xiv+490.

[Sei87] Raimund Seidel. "On the number of faces in higher-dimensional Voronoi diagrams". In: *Proceedings of the third annual symposium on Computational geometry*. SCG '87. Waterloo, Ontario, Canada: ACM, 1987, pp. 181–185.

[Sei91] Raimund Seidel. "Exact upper bounds for the number of faces in d-dimensional Voronoi diagrams." In: *Applied Geometry and Discrete Mathematics: The Victor Klee Festschrift* 4 (1991), pp. 517–530.

[She74] Geoffrey C. Shephard. "Subpolytopes of stack polytopes". In: *Israel Journal of Mathematics* 19 (3 1974), pp. 292–296.

[She82] Ido Shemer. "Neighborly polytopes". In: *Israel J. Math.* 43 (1982), pp. 291–314.

[Shi14a] Yaroslav Shitov. "An upper bound for nonnegative rank". In: *J. Combin. Theory Ser. A* 122 (2014), pp. 126–132.

[Shi14b] Yaroslav Shitov. "Sublinear extensions of polygons". Preprint, 10 pp., arXiv: 1412.0728. Dec. 2014.

[Shi16a] Yaroslav Shitov. "A universality theorem for nonnegative matrix factorizations". Preprint, 8 pp., arXiv: 1606.09068. June 2016.

[Shi16b] Yaroslav Shitov. "Nonnegative rank depends on the field II". Preprint, 3 pp., arXiv: 1605.07173. May 2016.

[Sho91] Peter W. Shor. "Stretchability of pseudolines is *NP*-hard". In: *Applied Geometry and Discrete Mathematics — The Victor Klee Festschrift (P. Gritzmann and B. Sturmfels, eds.)* Vol. 4. DIMACS Series in Discrete Mathematics and Theoretical Computer Science. Providence RI: Amer. Math. Soc., 1991, pp. 531–554.

[Smi91] Warren D. Smith. "On the Enumeration of Inscribable Graphs". Manuscript, 7 pages, NEC Research Institute; doi:10.1.1.29.4543. 1991.

[SP02] Richard P. Stanley and Jim Pitman. "A polytope related to empirical distributions, plane trees, parking functions, and the associahedron". In: *Discrete Comput. Geom.* 27.4 (2002), pp. 603–634.

[SS12] Luis Serrano and Christian Stump. "Maximal Fillings of Moon Polyominoes, Simplicial Complexes, and Schubert Polynomials". In: *Electron. J. Combin.* 19.1 (Jan. 2012), P16.

[SS93] Steven Shnider and Shlomo Sternberg. *Quantum groups. From coalgebras to Drinfel'd algebras: A guided tour.* Graduate Texts in Math. Physics, II. Cambridge, MA: International Press, 1993.

[SSW17] Francisco Santos, Christian Stump, and Volkmar Welker. "Noncrossing sets and a Grassmann associahedron". 2017.

[ST08] Tamon Stephen and Hugh Thomas. "A quadratic lower bound for colourful simplicial depth". In: *Journal of Combinatorial Optimization* 16.4 (2008), pp. 324–327.

[Sta57] Karl G. C. von Staudt. *Beiträge zur Geometrie der Lage*. 2. Nürnberg: Baur und Raspe, 1857.

[Sta63] James D. Stasheff. “Homotopy associativity of H-spaces”. In: *Transactions Amer. Math. Soc.* 108 (1963), pp. 275–292.

[Sta86] Richard P. Stanley. “Two poset polytopes”. In: *Discrete Comput. Geom.* 1.1 (1986), pp. 9–23.

[Ste12] Salvatore Stella. “Polyhedral models for generalized associahedra via Coxeter elements”. In: *Journal of Algebraic Combinatorics* (2012), pp. 1–38.

[Ste28] Ernst Steinitz. “Über isoperimetrische Probleme bei konvexen Polyedern”. German. In: *J. Reine Angew. Math.* 159 (1928), pp. 133–143.

[Ste32] Jacob Steiner. Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander. Also in: Gesammelte Werke, Vol. 1, Reimer, Berlin 1881, pp. 229–458. Fincke, Berlin, 1832.

[Stu88a] Bernd Sturmfels. “Neighborly polytopes and oriented matroids”. In: *European J. Combin.* 9 (1988), pp. 537–546.

[Stu88b] Bernd Sturmfels. “Some applications of affine Gale diagrams to polytopes with few vertices”. In: *SIAM J. Discrete Math.* 1.1 (1988), pp. 121–133.

[Stu96] Bernd Sturmfels. *Gröbner bases and convex polytopes*. Vol. 8. University Lecture Series. Providence, RI: American Mathematical Society, 1996, pp. xii+162.

[Suk14] Andrew Suk. “A note on order-type homogeneous point sets”. In: *Mathematika* 60.1 (2014), pp. 37–42.

[Suv88] P. Suvorov. “Isotopic but not rigidly isotopic plane systems of straight lines”. In: *Topology and geometry—Rohlin Seminar*. Vol. 1346. Lecture Notes in Math. Berlin Heidelberg: Springer-Verlag, 1988, pp. 545–556.

[SZ10] Raman Sanyal and Günter M. Ziegler. “Construction and analysis of projected deformed products”. In: *Discrete Comput. Geom.* 43.2 (2010), pp. 412–435.

[Tam51] Dov Tamari. “Monoïdes préordonnés et chaînes de Malcev”. PhD thesis. Sorbonne Paris, 1951, 81 pages.

[Tho06] Hugh Thomas. “Tamari lattices and noncrossing partitions in type B”. In: *Discrete Math.* 306.21 (2006), pp. 2711–2723.

[Thu79] William P. Thurston. “Geometry and Topology of 3-Manifolds”. Lecture Notes, Princeton University; http://library.msri.org/books/gt3m/. 1979.

[Tsu13] Yasuyuki Tsukamoto. “New examples of oriented matroids with disconnected realization spaces.” In: *Discrete Comput. Geom.* 49.2 (2013), pp. 287–295.

[TWZ17] Hans Raj Tiwary, Stefan Weltge, and Rico Zenklusen. "Extension complexities of Cartesian products involving a pyramid". In: *Information Processing Letters* 128 (2017), pp. 11–13.

[Tza06] Eleni Tzanaki. "Polygon dissections and some generalizations of cluster complexes". In: *J. Combin. Theory Ser. A* 113.6 (2006), pp. 1189–1198.

[Vak06] Ravi Vakil. "Murphy's law in algebraic geometry: badly-behaved deformation spaces". In: *Invent. Math.* 164.3 (2006), pp. 569–590.

[Van+15] Arnaud Vandaele, Nicolas Gillis, François Glineur, and Daniel Tuyttens. "Heuristics for exact nonnegative matrix factorization". In: *Journal of Global Optimization* (2015), pp. 1–32.

[Ver88] Anatoly M. Vershik. "Topology of the convex polytopes' manifolds, the manifold of the projective configurations of a given combinatorial type and representations of lattices". In: *Topology and geometry—Rohlin Seminar.* Vol. 1346. Lecture Notes in Math. Berlin Heidelberg: Springer-Verlag, 1988, pp. 557–581.

[Whe15] Ashley K. Wheeler. "Ideals generated by principal minors". In: *Illinois J. Math.* 59.3 (2015), pp. 675–689.

[Whi89] Neil L. White. "A nonuniform matroid which violates the isotopy conjecture". In: *Discrete Comput. Geom.* 4.1 (1989), pp. 1–2.

[Wot09] Ronald F. Wotzlaw. "Incidence graphs and unneighborly polytopes". Published online at http://dx.doi.org/10.14279/depositonce-2146. PhD thesis. Technische Universität Berlin, 2009.

[WZ11] Ronald F. Wotzlaw and Günter M. Ziegler. "A lost counterexample and a problem on illuminated polytopes". In: *Am. Math. Mon.* 118.6 (2011), pp. 534–543.

[Yan91] Mihalis Yannakakis. "Expressing combinatorial optimization problems by linear programs". In: *J. Comput. System Sci.* 43.3 (1991), pp. 441–466.

[Zie04] Günter M. Ziegler. "Projected products of polygons". In: *Electron. Res. Announc. Amer. Math. Soc.* 10 (2004), 122–134 (electronic).

[Zie07] Günter M. Ziegler. "Convex polytopes: extremal constructions and f-vector shapes". In: *Geometric combinatorics.* Vol. 13. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, 2007, pp. 617–691.

[Zie95] Günter M. Ziegler. *Lectures on polytopes.* Vol. 152. Graduate Texts in Mathematics. Revised edition, 1998; seventh updated printing 2007. New York: Springer-Verlag, 1995, pp. x+370.